Day: Wednesdays
Dates: Sep 10 – Dec 10, excepting Nov 26
Time: 10:00am–11:00am.
Place: MCS B31.

INTRODUCTION

Summary. The goal of this year-long seminar is to study $p$-adic Hodge theory, a series of results and techniques which lie at the heart of modern number theory. The first semester will be spent learning the origins, focusing on the classical results of Tate, Sen, Fontaine and many (many) others. The second semester will be focused on a more advanced topic, yet to be determined (further directions in $p$-divisible groups, $(\varphi, \Gamma)$-modules, $p$-adic Langlands etc.).

The fall semester will be split up into two main courses, with room for dessert. Our first goal will be to focus on reading Tate’s seminal paper on $p$-divisible groups [7] along with the necessary background. The second half of the semester will focus on period rings and the development of $p$-adic Hodge theory following Fontaine. We have two talks planned for the end which will be more expository in nature; either will require significant prior work or experience.

Talks. Almost all of the talks will be given by graduate students. They should be accessible to the fellow graduate student in the audience. Emphasis should be given on a precise understanding, by the speaker and for the audience, of the statements and results presented. Proofs should be included wherever it aids the communication of mathematics.

Prerequisites. The prerequisites for the seminar will vary from place-to-place. One could get away in the beginning with just a knowledge of local fields. For the first half, it will be helpful to have some prior knowledge of abelian varieties (even elliptic curves would be enough). Beginning with talk 6, much of what we do is motivated by global considerations in the Langlands program; a cursory knowledge of how the local picture fits with the global one will be helpful, though not necessary, for an intuitive kickstart. At least two talks (talk 7 and talk 12) are essentially discussions of linear algebra.

References. References for the material to be discussed will be included in the explanation of each talk.

Notes. I will be doing something slightly different this semester and attempting to “live \TeX” notes. Speakers will be given a copy of the notes afterwards and asked to make revisions and/or contributions beyond the material given in the talk. The edited notes will then appear on the website promptly.

Schedule: Fall 2014

Each of the following is considered one week. More references will be given.
**Sept. 10 – Introduction: the cohomology of abelian varieties.** The goal of this talk is to introduce the topic for the semester. It will happen in two parts. First, we want to remember two separate results about abelian varieties: the Hodge decomposition over \( \mathbb{C} \) \([5, \S 1]\) and the Néron-Ogg-Shafarevich criterion for the good reduction of an abelian variety over a \( p \)-adic field in terms of \( \ell \)-adic cohomology. The second half should focus on a brief reminder of the basic theory of \( p \)-adic Galois representations over \( p \)-adic fields.

**Sept. 17 – Finite flat group schemes and Dieudonné theory.** This talk will have two parts, one on generalities of finite flat group schemes and the other on Dieudonné theory over perfect fields in characteristic \( p \). For the first half, define finite flat groups schemes and discuss examples of torsion in (usual) group schemes. The level of generality should fit with \([1, \S 7.1]\) and \([7, \S 1]\). A good guide is the material discussed in \([7, \S 1]\), for which \([8]\) is a useful reference. In the second half of the talk we want to explain how finite flat group schemes can be understood in terms of their Dieudonné modules \([1, \text{Theorem 7.2.4}]\). A typical reference is the (long) paper \([4]\), one might also benefit from Fontaine’s book \([2, \text{Ch. III}]\) and \([3, \text{Part V}]\) (these reference comes recommended from J. Weinstein himself).

**Sept. 24 – \( p \)-divisible groups: introduction.** This talk will be an introduction to \( p \)-divisible groups. From Tate’s paper we want to cover \([7, \S 2]\), except \( \S 2.2 \). You should explain the analogs of duality (\( \S 2.3 \)), the connected-étale sequence and Dieudonné modules. Include the important notions of *height* and *dimension*. Work out the examples of elliptic curves (or abelian varieties). If time permits, it would be nice to see a statement of the main theorems of Tate’s paper in the introductory talk.

**Oct. 1 – Formal groups and Serre-Tate theory.** This interlude will go through \([7, \S 2.2]\) and describe the equivalence between connected \( p \)-divisible groups and divisible commutative formal groups using references to be given (and found).

**Oct. 8 – \( p \)-divisible groups: results.** Here we will assemble the previous talks and explain the main results of \([7]\). We will state, but omit the proofs for now, the cohomological result \([7, \text{Theorem 1}]\). But use this result to prove the results of \([7, \S 4]\) on the Hodge-Tate decomposition of the \( p \)-adic étale cohomology of an abelian variety and Tate’s theorem on the full faithfulness of the generic fiber \([1, \text{Theorem 7.2.8}]\).

**Oct. 15 – A pivot: Tate-Sen theory.** This talk marks a pivot point towards \( p \)-adic Hodge theory via period rings. We will describe the Hodge-Tate-Sen decomposition of \( p \)-adic linear representations of \( p \)-adic Galois groups. Begin by recalling the definition of a \( p \)-adic Galois representation and the example of the cyclotomic character. What we want to know about is the construction of the functor \( \mathcal{D}_{\text{Sen}} \). The article \([6]\) is beautifully written and contains the major results. As an application we should explain the classification of representations with finite image on inertia via Sen’s operator \([6, \S 3, \text{Corollary}]\). (This last point will arise again.)

**Oct. 22 – Fontaine’s formalism.** Fontaine introduced the notation of admissible representations (depending on certain data \( B, G \), etc.) This talk will focus on exposing this material via \([1, \S 5]\). The most important formal result is \([1, \text{Theorem 5.2.1}(1)]\); the proof (which is easy) should be explained. We can then discuss rephrasing the previous talk in terms of the Hodge-Tate period ring \( B_{\text{HT}} \).
Oct. 29 – The period ring \( B_{\text{dR}} \). The goal of this talk will be to go through the construction of the period ring \( B_{\text{dR}} \) following [1, §4]. Here you may find it useful to review the rings over Witt vectors (which will have been used tacitly so far). Make sure to explain how the cohomological results of Tate give [1, Theorem 4.4.14] and thus how \( B_{\text{dR}} \) can be used as a period ring in Fontaine’s formalism.

Nov. 5 – de Rham representations. We’d like to go through in detail [1, §6]. That means explaining the Hodge-Tate filtration on \( D_{\text{dR}}(\cdot) \) and its relation to the Hodge-Tate decomposition of Tate-Sen theory. Classify the one-dimensional de Rham representations in terms of Sen weights [1, Example 6.3.9] (needs the potentially unramified result of Tate-Sen from Talk 6).

Nov. 12 – Crystalline and semi-stable period rings. We will follow [1, §9] to give the construction of the rings \( B_{\text{cris}} \) and \( B_{\text{st}} \). Once again, classify the one-dimensional crystalline and semi-stable representations.

Nov. 19 – Filtered \((\phi,N)\)-modules. In the previous talk we learned about two new period rings. In this talk we will study the category in which the corresponding admissible representations live. The main result we want to cover is the correspondence between semi-stable Galois representations and weakly admissible filtered \((\phi,N)\)-modules (we will only show one implication) with the crystalline subcategory corresponding to the modules where \( N = 0 \). We could finish by giving the construction of a Weil-Deligne representation associated to potentially semi-stable Galois representations.

Dec. 3 – Wrapping up with examples. Granting the previous results, we classify all two-dimensional semi-stable representations of \( G_{\Q_p} \) following [1, §8.3].

Date TBD – Mini-workshop. \( p \)-adic Hodge theory has had many successes. If time permits at the end of the semester, I would like to have a mini-workshop with multiple shorter talks. This will be dealt with during the semester, on the basis of audience interest. Some possible topics include:

- Fontaine’s proof that there are no abelian schemes over \( \Z \).
- The extension of Néron-Ogg-Shafarevich to \( \ell = p \) (combining Grothendieck’s formulation in terms of \( p \)-divisible groups and the work of Breuil and Kisin on \( p \)-divisible groups over rings of integers).
- \( p \)-adic Hodge theoretic deformation conditions.
- Kisin’s analytic continuation of crystalline periods (and the Fontaine-Mazur conjecture over the eigencurve) and its generalizations.
- Towards \((\varphi,\Gamma)\)-modules.
- The list could go on...

REFERENCES
