Accessible Mandelbrot Sets in the Family

\[ z^n + \lambda/z^n \]

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0 Introduction

In recent years, there have been a number of papers [1], [14] dealing with the topology of Julia sets that arise from parameters lying in certain Mandelbrot sets that are subsets of the parameter plane for the family of rational maps given by

\[ F_\lambda(z) = z^n + \frac{\lambda}{z^n} \]

where \( n \geq 2 \). In Figure 1 we display the parameter plane for this family when \( n = 3 \) and 4. Note that there are two large copies of the Mandelbrot set visible in this picture when \( n = 3 \) and three when \( n = 4 \); these are known as the “principal” Mandelbrot sets. But there are many other smaller copies of “baby” Mandelbrot sets in these parameter planes as well, as it is known [16] that they are dense in the bifurcation locus for a holomorphic family of rational maps.

![Figure 1](image.png)

Figure 1: The parameter planes when \( n = 3 \) (left) and \( n = 4 \) with the principal Mandelbrot sets in grey.
In these parameter planes, the external region consists of parameters for which the corresponding Julia sets are Cantor sets. There is also a small disk centered at the origin in which the corresponding Julia sets are Cantor sets of simple closed curves [15]. This is the McMullen domain. In the remaining region, the Julia sets are always connected sets. This region is therefore called the connectedness locus; it is known that this set is a connected set in the parameter plane [10].

In Figure 2 we display the parameter plane for this family when \( n = 2 \) together with several magnifications. In this image, there is a large main cardioid in what appears to be a principal Mandelbrot set in this image that extends to the external boundary of this set. In the magnifications, there appear to be other baby Mandelbrot sets that extend out to the boundary (we call these “accessible” Mandelbrot sets), as well as other baby Mandelbrot sets that do not extend to the boundary (these are “buried” Mandelbrot sets).

The topology of the Julia sets drawn from these Mandelbrot sets is quite interesting. It is known that, for a given value of \( n \), all of the Julia sets drawn from the main cardioids of the principal Mandelbrot sets are the same topologically, i.e., they are all homeomorphic to one another [1]. These sets are called “checkerboard” Julia sets. But for other accessible main cardioids, the topology of the Julia sets is quite different and only those cardioids that are symmetric under certain rotations or complex conjugation are homeomorphic [14].

For the buried Mandelbrot sets, the situation is again very different: all Julia sets drawn from such a main cardioid are Sierpiński curves (i.e., they are homeomorphic to the well-known Sierpiński carpet fractal) and hence they all have the same topology [9].

It is known that there are exactly \( n - 1 \) principal Mandelbrot sets in
Figure 2: The parameter plane for the family $z^2 + \lambda/z^2$ together with two magnifications.
the parameter plane for the maps $z^n + \lambda/z^n$ [4]. These are the Mandelbrot sets which have base period one (defined below). It is also known that there are infinitely many other accessible Mandelbrot sets when $n \geq 3$ [5] and also when $n = 2$ [8]. And finally, infinitely many buried Mandelbrot sets have been shown to exist [11]. However, to date there has been no complete description of the arrangement of these Mandelbrot sets in these parameter planes.

Our goal in this paper is to describe the exact arrangement of all of the accessible Mandelbrot sets in these families. We shall show that there is a unique accessible Mandelbrot set whose cusp is at the landing point of each external ray in the parameter plane that is periodic under the map $\theta \mapsto n\theta$. This provides the complete classification of all of these accessible Mandelbrot sets.

1 Preliminaries

Consider the family of rational maps given by

$$F_{\lambda}(z) = z^n + \frac{\lambda}{z^n}$$

where $n \geq 2$. One checks easily that the point at $\infty$ is a superattracting fixed point for these maps. Hence there is an immediate basin of attraction of $\infty$ which we denote by $B_{\lambda}$. There is also a neighborhood of 0 that is mapped into this basin. This neighborhood may or may not be disjoint from $B_{\lambda}$. In this paper, we shall consider only the case where the preimage of $B_{\lambda}$ containing 0 is disjoint from $B_{\lambda}$. In this case we call this preimage the trap door and denote it by $T_{\lambda}$. The full basin of $\infty$ is then the union of the infinitely many distinct preimages of $B_{\lambda}$.

The Julia set of $F_{\lambda}$, denoted by $J(F_{\lambda})$, has several definitions. First, $J(F_{\lambda})$ is the closure of the set of repelling periodic points of $F_{\lambda}$. Second,
$J(F_\lambda)$ is also the set of points at which the family of iterates of $F_\lambda$ fails to be a normal family in the sense of Montel. And third, just as in the case of polynomials, $J(F_\lambda)$ is also the boundary of the full basin of $\infty$.

There are several symmetries associated with this family of maps. First, let $\omega$ be a primitive $2n^{th}$ root of unity. Then we have $F_\lambda(\omega z) = -F_\lambda(z)$. So, if $n$ is even, $F_\lambda^2(z) = F_\lambda^2(\omega^j z)$ for each $j$, so the orbits of these points are eventually the same. If $n$ is odd, we have $F_\lambda(-z) = -F_\lambda(z)$ and so the orbits of $z$ and $\omega z$ behave symmetrically under $z \mapsto -z$. In either case, it follows that $J(F_\lambda)$, $B_\lambda$, and $T_\lambda$ are all symmetric under $z \mapsto \omega z$, i.e., we have $2n$-fold symmetry in the dynamical plane.

Let $H_\lambda$ be the involution $H_\lambda(z) = \lambda^{1/n}/z$. Then $F_\lambda(H_\lambda(z)) = F_\lambda(z)$ so $J(F_\lambda)$ is also symmetric under $H_\lambda$. Note that $H_\lambda$ interchanges $B_\lambda$ and $T_\lambda$.

There are $2n$ “free” critical points for $F_\lambda$ given by $\lambda^{1/2n}$. We denote these critical points by $c_0, \ldots, c_{2n-1}$, where $0 \leq \text{Arg} c_0 < \pi/n$ and the other $c_j$ are ordered in the counterclockwise direction. We call these critical points free because there are two other critical points: one at $\infty$, which is fixed, and the other at 0, which is mapped directly to $\infty$. So these two critical points are not “free.” There are only 2 critical values corresponding to the free critical points of $F_\lambda$ given by $v_\lambda = \pm 2\sqrt{\lambda}$; $n$ of the free critical points are mapped to $+v_\lambda$ and the other $n$ are mapped to $-v_\lambda$. However, there really is only one critical orbit up to symmetry, since, when $n$ is even, both $\pm v_\lambda$ are mapped onto the same point, whereas, when $n$ is odd, the orbits of $\pm v_\lambda$ behave symmetrically under $z \mapsto -z$. There are also $2n$ prepoles for $F_\lambda$ given by $(-\lambda)^{1/2n}$. Note that all of the critical points and prepoles lie on the same circle which we call the critical circle. Also, each involution $H_\lambda$ fixes two critical points $c_j$ and $c_{j+n}$.

The difference between the behaviors of the critical orbits when $n$ is even or odd causes a slight difference in the periods associated to parameters
drawn from the main cardioids of the Mandelbrot sets that we will work with. Let \( \lambda \) be the center of the main cardioid of such a Mandelbrot set (i.e., \( \lambda \) is the unique parameter in that region for which a critical orbit is periodic). Suppose that \( c_j \) is a critical point that lies on this periodic orbit and that \( F^k_\lambda(c_j) \) is the first subsequent critical point on the orbit of \( c_j \). Then, if \( n \) is even, we must have that \( F^k_\lambda(c_j) = c_j \) since all the critical orbits agree at iteration two and beyond. However, if \( n \) is odd, there is a second possible scenario. It could be the case that, when the orbit of \( c_j \) lands on a critical point at iteration \( k \), this orbit now lands on \(-c_j\). This follows from the fact that the orbits of any pair of critical points \( c_k \) and \(-c_k \) are symmetric under the \( z \mapsto -z \) symmetry. So these orbits are either distinct or else they are the same. In either case, all of the other critical points must land on these orbits. So, if we have that \( F^k_\lambda(c_j) = -c_j \), then the period of \( c_j \) (and \(-c_j\)) is \( 2k \), not \( k \). In any event, we say in both cases that such a Mandelbrot set has base period \( k \) if some critical orbit first returns to the critical circle at iteration \( k \). For example, in the parameter plane for \( n = 3 \) displayed in Figure 1, both principal Mandelbrot sets have base period 1. The right-hand Mandelbrot set does have a parameter at the center of the main cardioid for which there is a superattracting fixed point. However, the left-hand principal Mandelbrot set has a parameter at the center of the main cardioid for which two critical points are interchanged as above by \( F_\lambda \), so we have an attracting cycle of period two. However, by definition, this set has base period one.

Throughout this paper we shall assume that the orbits of the critical points are all bounded. In this case it is known that \( J(F_\lambda) \) is always a connected set [10], [12].

The straight rays extending out from the origin and passing through the critical points are called the critical point rays. These are mapped by \( F_\lambda \) two-to-one onto the critical value rays given by \( tv_\lambda \) where \( t \geq 1 \). Let \( S_j \)
denote the open sector bounded by the two critical point rays through $c_{j-1}$ and $c_j$. Then each $S_j$ is mapped one-to-one onto the complement of the two critical value rays. It is known that the set of points in $J(F_\lambda)$ whose orbits remain for all iterations in the two sectors $S_0 \cup S_n$ is a Cantor set on which $F_\lambda$ is conjugate to the one-sided shift map on two symbols. Call this Cantor set $\Lambda$. Then $\Lambda$ meets $\partial B_\lambda$ in two points, both of which are fixed when $n$ is odd, and one is fixed and the other pre-fixed when $n$ is even. This follows since the external ray of angle 0 is clearly contained in the sector $S_0$, and this ray lands on a fixed point in $\partial B_\lambda$, while the external ray of angle 1/2 lies in $S_n$ and lands on a different fixed point if $n$ is odd or on a pre-fixed point if $n$ is even. We denote the fixed point that lies in $\Lambda$ and in the right half-plane by $p_\lambda$. By the $z \mapsto -z$ symmetry, the other point in $\partial B_\lambda \cap \Lambda$ is $-p_\lambda$. Similarly, there is a pair of points $\pm q_\lambda$ lying in $\partial T_\lambda \cap \Lambda$. We choose $q_\lambda$ to be the point that lies in the sector $S_0$ so that $-q_\lambda$ lies in the sector $S_n$. See [6].

The set of points that lie in $\Lambda$, $T_\lambda$, and all of the preimages of $T_\lambda$ whose boundaries meet $\Lambda$ in a pair of points forms what is known as a Cantor necklace. This set is defined as follows. Consider the Cantor middle-thirds set lying in the interval $[0,1]$ on the $x$-axis in the plane. Adjoin to this set an open disk in place of each removed open interval so that the boundary of this disk meets the Cantor set at the two endpoints of this removed open interval. This is the Cantor middle-thirds necklace. Then any planar set that is the image of this necklace under a map that is continuous, one-to-one, and onto is called a Cantor necklace.
2 Internal Rays in the Dynamical Plane

In order to prove the existence of the accessible Mandelbrot sets in the parameter plane, we first need to introduce the notion of internal rays in the dynamical plane. For simplicity, for most of this paper, we shall restrict to the case where \( n = 2 \), i.e., to the family of maps

\[
F_\lambda(z) = z^2 + \frac{\lambda}{z^2}.
\]

We describe the minor modifications necessary for the case \( n > 2 \) at the end of this paper.

As mentioned above, we shall assume throughout this paper that the critical values do not lie in \( B_\lambda \) or \( T_\lambda \) (and so \( J(F_\lambda) \) is connected). As is well known [10], in this case, \( B_\lambda \) is simply connected and there is a natural uniformization of \( B_\lambda \) that conjugates \( F_\lambda|B_\lambda \) to \( z \mapsto z^2 \) on the open unit disk. The external ray of angle \( \theta \), denoted by \( \xi_\theta = \xi_\theta \), is the curve that maps to the straight ray of angle \( \theta \) under this conjugacy. All rational rays are known to land at a unique point in \( \partial B_\lambda \) [18], so the external ray of angle 0 lands at the fixed point \( p_\lambda \) and \( \xi_{1/2} \) lands at \( -p_\lambda \). Since \( H_{\lambda}(T_\lambda) = B_\lambda \), this uniformization may then be extended to \( T_\lambda \). We then assign angles to the corresponding rays so that the trap door ray of angle 0 in \( T_\lambda \), namely \( \tau_0^\lambda = \tau_0 \), now extends from the origin to \( q_\lambda \), while \( \tau_{1/2} \) extends from 0 to \( -q_\lambda \). Therefore \( F_\lambda(\tau_0) = F_\lambda(\tau_{1/2}) = \xi_{1/2} \). We then have the corresponding trap door rays \( \tau_\theta \) lying in \( T_\lambda \), where the angles \( \theta \) are chosen so that they increase in the counterclockwise direction around the origin. (Note that \( H_{\lambda}(\xi_\theta) \neq \tau_\theta \) if \( \theta \neq 0, 1/2 \).

We now define the internal rays of angles 0 and 1/2 for \( F_\lambda \). These will be curves that pass through portions of the Julia set and certain preimages of \( B_\lambda \) and connect the endpoints of the corresponding external ray in \( \partial B_\lambda \) to the endpoints of the corresponding trap door ray in \( \partial T_\lambda \). We may successively
pull back the trap door rays $\tau_0$ and $\tau_{1/2}$ to each of the preimages of $T_\lambda$ that lie in the Cantor necklace in $S_0 \cup S_2$. These preimage curves each connect to two distinct points in the invariant Cantor set $\Lambda$. Then, as shown in [6], the union of these curves that lie in the sector $S_0$ together with the portion of $\Lambda$ in this region is a continuous curve that connects $p_\lambda$ to $q_\lambda$. This is the internal ray which we denote by $\nu_0$. The internal ray $\nu_{1/2}$ is defined in similar fashion in the opposite sector $-S_0$, so, by the $z \mapsto -z$ symmetry, $\nu_{1/2} = -\nu_0$.

Note that each of these internal rays contains a unique preimage of 0. The full ray of angle 0 (resp., 1/2) is then the union of the internal, external, and trap door rays of angles 0 (resp., 1/2). Then one checks easily that the full ray of angle 0 is mapped one-to-one onto the pair of full rays of angles 0 and 1/2 together with the origin. The full ray of angle 1/2 is mapped in similar fashion onto the same full rays and the origin.

We next define the internal rays of angle $j/2^k$ for $F_\lambda$ where $k \geq 2$. These will be extensions of the external rays $\xi_{j/2^k}$ into the complement of $B_\lambda \cup T_\lambda$. We will define these internal rays inductively by simply taking the particular preimages of the internal rays $j/2^{k-1}$ that meet $\partial B_\lambda$. To begin, we define the internal rays $\nu_{1/4}^\lambda = \nu_{1/4}$ and $\nu_{3/4}$ to be the two other curves contained in the complement of $B_\lambda \cup T_\lambda$ that are each mapped one-to-one over $\nu_0 \cup \nu_{1/2}$ as well as $\tau_0 \cup \tau_{1/2}$ and the origin. Specifically, $\nu_{1/4}$ is given by $i\nu_0$ and $\nu_{3/4} = -\nu_{1/4}$, so $\nu_{1/4}$ (resp., $\nu_{3/4}$) lies in the upper (resp., lower) half-plane. As above, both $\nu_{1/4}$ and $\nu_{3/4}$ meet $\partial B_\lambda$ and $\partial T_\lambda$ at unique points. See Figure 3. We then define the full rays of angles 1/4 and 3/4 to be the union of the corresponding internal, external, and trap door rays with this angle. This gives a continuous curve extending from the origin to $\infty$. These full rays are also mapped one-to-one over the union of full rays of angles 0 and 1/2 as well as the origin.

Now consider the open region in the complement of $\overline{B_\lambda} \cup \overline{T_\lambda}$. The four internal rays $\nu_0, \nu_{1/4}, \nu_{1/2},$ and $\nu_{3/4}$ divide this region into four open sectors.
Figure 3: The internal rays of angles 0, 1/4, 1/2, and 3/4.

$I_0, \ldots, I_3$ where $I_0$ is bounded by $\nu_0, \nu_{1/4}$, and portions of $\partial B_\lambda$ and $\partial T_\lambda$, and the other three sectors are arranged in the counterclockwise direction. Note that $F_\lambda$ maps $I_0$ and $I_2$ two-to-one over $I_0 \cup I_1 \cup \nu_{1/4}$ as well as the portion of $T_\lambda$ lying above $\tau_0 \cup \tau_{1/2}$ and, similarly, $I_1$ and $I_3$ are mapped two-to-one over $I_2 \cup I_3 \cup \nu_{3/4}$ plus the portion of $T_\lambda$ lying below $\tau_0 \cup \tau_{1/2}$. In addition, the critical point $c_j$ lies in $I_j$ for each $j$.

We may then define the itinerary $S(z)$ of any point $z$ whose orbit remains for all iterations in the union of the $I_j$ in the usual way: $S(z) = (s_0 s_1 s_2 \ldots)$ if $F_\lambda^j(z) \in I_{s_j}$. Note that, in such an itinerary, 0 and 2 can only be followed by 0 or 1 because of how the corresponding sectors are mapped. Similarly, 1 and 3 can only be followed by 2 or 3. Any finite or infinite sequence that follows this coding is called an allowable itinerary.
For the moment we assume that the critical orbits never land on the internal or trap door rays of angles 0 and 1/2. Hence each critical point \( c_j \) whose orbit is bounded in \( \mathbb{C} \) has an infinite itinerary of the form \((s_0 s_1 s_2 \ldots)\). Moreover, since all the critical points land on the same point after two iterations, the “tail” of the itinerary of each critical point, namely \((s_2 s_3 s_4 \ldots)\), is the same.

Since \( F_\lambda \) maps \( I_0 \) two-to-one over \( I_0 \cup I_1 \cup \nu_{1/4} \) plus a portion of \( T_\lambda \) and, by assumption, the critical value does not lie on the internal or trap door ray of angle 1/4, it follows that there are two preimages of \( \nu_{1/4} \) in \( I_0 \). We call these preimages the boundary curves. One of these boundary curves must extend from \( \partial B_\lambda \) to one of the two preimages of \( \partial T_\lambda \) that lie on the bounding internal rays of \( I_0 \) and the other necessarily extends from \( \partial T_\lambda \) to the opposite preimage of \( \partial T_\lambda \). This follows since these two curves are located symmetrically with respect to the involution \( H_\lambda(z) = \lambda^{1/2}/z \) that fixes the critical point \( c_0 \). This implies that there are then two possibilities for how these boundary curves are arranged as shown in Figure 4. These two different cases depend upon whether \( F_\lambda(c_0) \in I_0 \) or \( F_\lambda(c_0) \in I_1 \), both of which may occur.

We now append the preimages of the trap door ray \( \tau_{1/4} \) to these two curves so that each of these longer boundary curves now extends to one of the first preimages of 0. These two curves then divide \( I_0 \) into three separate regions. We call the unique boundary curve that meets the boundary of \( B_\lambda \) the internal ray of angle 1/8, so \( \nu_{1/8} \) extends from \( \partial B_\lambda \) to a first preimage of 0. If \( F_\lambda(c_0) \) lies in \( I_0 \), then there is an open region that connects \( \partial B_\lambda \) to \( \partial T_\lambda \) which is mapped two-to-one onto \( I_0 \). This is the “connecting” region \( I_{\infty} \). One checks easily that this occurs when \( \text{Im} \lambda > 0 \). There are also two other open subsets in \( I_0 \) that are mapped one-to-one onto \( I_1 \). Neither of these sets connects \( \partial B_\lambda \) to \( \partial T_\lambda \); rather, one abuts \( \partial B_\lambda \) (we call this the “non-
Figure 4: The two possible configurations for \( \nu_{1/8} \). The unmarked curves are also boundary curves that are mapped to \( \nu_{1/4} \).

connecting" region \( I_{01} \) and the other abuts \( \partial T_{\lambda} \). Note that, in this case, the internal ray \( \nu_{1/8} \) extends from \( \partial B_{\lambda} \) to the "middle" of \( \nu_{1/4} \), i.e., to the unique first preimage of 0 in this ray. On the other hand, if the image of \( c_0 \) lies in \( I_1 \) (which occurs if \( \text{Im} \lambda < 0 \)), there is now a connecting region \( I_{01} \) that is mapped two-to-one onto \( I_1 \) and a pair of other regions mapped one-to-one to \( I_0 \). The internal ray \( \nu_{1/8} \) now extends from \( \partial B_{\lambda} \) to the middle of \( \nu_0 \) and the non-connecting region bounded by this ray is called \( I_{00} \). In particular, we have that there is an open set of parameters for which a critical value lies in \( I_j \).

By the fourfold symmetry, we have a similar configuration in the other regions \( I_j \) and thus we can define the internal rays \( \nu_{3/8}, \nu_{5/8}, \) and \( \nu_{7/8} \). See Figure 5.

To summarize, at this first stage, each of the regions \( I_{s_0} \) may be divided into three separate subregions. Exactly two of these subregions abut \( \partial B_{\lambda} \).
Figure 5: The $j/8$ internal rays when $c_0$ lies in $I_{00}$ together with the connecting and non-connecting regions in this case.

We denote these special regions by $I_{s_0 s_1}$ where there are two possibilities for the digit $s_1$. One of these regions is a non-connecting region that is mapped one-to-one onto its image $I_{s_1}$ while the other is the connecting region that is mapped two-to-one onto its image. The corresponding internal ray $\nu_{j/8}$ provides a portion of the boundary of these two regions. And there is an open set of parameters for which the critical point in $I_{s_0}$ lies in either possible connecting region. These are the open sets that we shall use in a later section to define the polynomial-like maps that will generate the accessible Mandelbrot sets. We will not consider the third region in $I_{s_0}$ as this region will not play a role when we generate these Mandelbrot sets. See Figure 5 for a picture of the arrangement of the regions $I_{s_0 s_1}$ in the case where $F_\lambda(c_0) \in I_0$.

As a remark, if the critical value does lie in $\nu_{1/4}$, then the connecting
region becomes a pair of open disks whose boundaries meet at exactly one point, namely the critical point that lies on their boundary. One of these disks extends to \( \partial B_\lambda \), the other to \( \partial T_\lambda \).

We may now continue with this procedure. We begin by assuming that the critical values lie in one of the two regions that abut \( \partial B_\lambda \), not in the third region that does not extend out to \( \partial B_\lambda \). This is possible since, as \( \lambda \) rotates around a simple closed curve surrounding the origin, the critical values both rotate half way around a similar simple closed curve. Hence we may choose this curve in the parameter plane so that the critical values rotate around in a narrow annular neighborhood bounded on the outside by \( \partial B_\lambda \). Consequently a critical value enters each region \( I_{s_1s_2} \), and thus there is an open set of parameters for which this occurs.

First, in the non-connecting region \( I_{s_0s_1} \) that is mapped one-to-one onto \( I_{s_1} \), we can simply pull back the boundary curves that we previously constructed in \( I_{s_1} \) via \( F^{-1}_\lambda \). The preimage of \( \nu_{j/8} \) in \( I_{s_1} \) then gives a curve that connects \( \partial B_\lambda \) to one of the second preimages of 0 lying in the boundary of \( I_{s_0s_1} \). As before, there are two such second preimages of 0 in this region, but which one the preimage of \( \nu_{j/8} \) connects to is already determined since \( F_\lambda \) is one-to-one in this region. This defines the internal ray \( \nu_{j/16} \) or \( \nu_{(j+8)/16} \) that lies in \( I_{s_0s_1} \). Note that this now gives two smaller non-connecting regions \( I_{s_0s_1s_2} \) that abut \( \partial B_\lambda \).

In the second case, \( I_{s_0s_1} \) is now a connecting region and hence contains a critical point. Therefore this region now divides into five subregions. The reason for this is that, since \( F_\lambda \) maps \( I_{s_0s_1} \) two-to-one onto \( I_{s_1} \) and there are two preimages of \( \nu_{1/4} \) in \( I_{s_1} \), there must now be four preimages of these two curves in \( I_{s_0s_1} \) and hence this divides this region into five subregions. Four of these subregions are mapped one-to-one onto their images and one subregion (containing the critical point) is mapped two-to-one onto its image.
However, only two of these regions meet $\partial B_\lambda$. By assumption, one of these abutting regions must contain the critical point and this region therefore extends to $\partial T_\lambda$ (this is the connecting region at this stage); the other as before is mapped one-to-one onto its image (this is the non-connecting region) and the new internal ray is defined to be the curve that separates these two regions. There are also three additional preimage regions, each of which is mapped one-to-one onto its image in $I_{s_1s_2}$. However, none of these regions meet $\partial B_\lambda$, so these sets again will not play a role in the polynomial-like map construction in the next section. Note also that a critical value may now lie in any of the four regions abutting $\partial B_\lambda$.

We now continue inductively. At the $k^{th}$ stage, we simply pull back all of the internal rays of angle $j/2^{k+1}$ together with the other previously defined boundary curves. In the sector $I_{s_0}$, these curves then bound exactly $2^k$ open disks that meet $\partial B_\lambda$ and are given by $I_{s_0...s_k}$. Here, $s_0$ is fixed, and $s_1...s_k$ can be any of the allowable sequences of length $k$ that can follow $s_0$. Each of these is bounded by a pair of portions of internal rays of angles $j/2^{k+2}$ and $(j+1)/2^{k+2}$ that meet $\partial B_\lambda$, by the portion of $\partial B_\lambda$ between these two rays, and by other curves that are portions of the earlier defined boundary curves or their preimages (and, in one special case, by a portion of $\partial T_\lambda$). As earlier, $2^k - 1$ of these disks that abut $\partial B_\lambda$ do not extend down to $\partial T_\lambda$ and so are the non-connecting regions. None of these disks contain the critical point $c_{s_0}$, and so each is mapped one-to-one onto $I_{s_1...s_k}$. Therefore we can pull back the internal ray structure already constructed in the set $I_{s_1...s_k}$ to determine how the external rays of angle $j/2^{k+3}$ as well as the other boundary curves are configured in these preimages. More specifically, we have two regions in $I_{s_1...s_k}$ that meet $\partial B_\lambda$, namely $I_{s_1...s_k s_{k+1}}$ where there are two choices for the digit $s_{k+1}$. These regions are separated in an already determined fashion by the previously defined boundary curves, and so their preimages in $I_{s_0...s_k}$.
given by $I_{s_0\ldots s_{k+1}}$ have a similar configuration. As above, there are parameters where a critical value lies in any of these regions.

The one exception to this is again the open disk that is the connecting region. This disk contains the critical point and extends down to $\partial T_\lambda$. Suppose this region is $I_{s_0\ldots s_k}$. Then $F_\lambda$ maps this region two-to-one onto its image $I_{s_1\ldots s_k}$, so the two subregions $I_{s_1\ldots s_n s_{k+1}}$ touching $\partial B_\lambda$ may have one or two preimages in $I_{s_0\ldots s_k}$ depending upon the location of the critical value in these regions. As above, the critical value can lie in either of these regions. So, depending upon the location of this critical value, we can determine how the new external ray that separates these preimages in $I_{s_0\ldots s_k}$ is configured, i.e., to which of the $k$th preimages of 0 it connects. In any event, we have that $F_\lambda$ maps one of these preimages two-to-one onto its image and this gives the new connecting region at this stage. Thus we have the following results:

**Proposition 1.** Given an allowable sequence $s_0\ldots s_k$, there are parameter values for which there is an open set $I_{s_0\ldots s_k}$ that contains the critical point $c_{s_0}$ and is mapped two-to-one onto its image. This is the connecting region that extends from $\partial B_\lambda$ to $\partial T_\lambda$. As a consequence, there also exists an open set in the parameter plane for which a critical value lies in $I_{s_1\ldots s_k}$ for any allowable sequence of the form $s_1\ldots s_k$.

For the polynomial-like maps that we will consider in the next section, the connecting regions whose boundaries meet the internal ray of angle 0 or 1/2 will play an important role. As the following proposition shows, there are relatively few such connecting regions.

**Proposition 2.** The only connecting regions whose boundary meets the internal ray of angles 0 or 1/2 are $I_{0\ldots 0}$, $I_{3\ldots 3}$, and each of their three symmetrically located regions under the fourfold symmetry.

**Proof:** Consider the region $I_0$. As we showed above, each of the two con-
necting regions $I_{00}$ and $I_{01}$ in this region meet the internal ray of angle 0 (see Figure 4). Note that the connecting region $I_{01}$ is the rotation of another connecting region $I_{33}$ via $z \mapsto iz$. Using the inductive procedure described above, $I_{00}$ can be broken into two different connecting regions, $I_{000}$ and $I_{001}$. The internal ray 1/16 determines this structure. For $I_{000}$, this ray meets the internal ray of angle 1/8 and hence this region has a portion of its boundary on the internal ray of angle 0. In the other case, the 1/16 internal ray now connects to the boundary curve of a region that does not meet $B_\lambda$, and so the boundary of $I_{001}$ does not meet the internal ray of angle 0. A similar argument using the internal ray of angle 3/16 shows that the connecting region $I_{013}$ meets the internal ray of angle 0, while the connecting region $I_{012}$ does not. Hence any of the connecting regions of the form $I_{001^s_3...s_k}$ or $I_{012^s_3...s_k}$ do not meet this internal ray.

Continuing in this fashion, we use the internal rays of angles 1/32 and 7/32 to show that the boundaries of the connecting regions $I_{0000}$ and $I_{0133}$ meet the internal ray of angle 0. Continuing inductively shows that the only connecting regions in $I_0$ whose boundaries meet the internal ray of angle 0 are those of the form $I_{0^s_3...0}$ and $I_{01^s_3...3}$ (where $I_{01}$ is also included in this list). Then the $z \mapsto iz$ symmetry gives the result.

\[ \square \]

3 Polynomial-like Mappings

Our goal in this section is to prove the existence of infinitely many accessible Mandelbrot sets in the parameter plane for the family $F_\lambda$, one corresponding to each external ray whose angle is periodic under angle-doubling. To accomplish this, we make use of the concept of polynomial-like maps [2], [13]. Recall that a family of holomorphic maps $G_\lambda$ is said to be polynomial-like of
degree two if

1. The parameter $\lambda$ lies in a closed disk $\mathcal{O}$ in the complex plane and $G_\lambda$ depends analytically on $\lambda$;

2. For each $\lambda$ there are open disks $U_\lambda \subset V_\lambda$ that depend continuously on $\lambda$ and $G_\lambda$ maps $U_\lambda$ two-to-one onto $V_\lambda$, so there is a unique critical point $c_\lambda \in U_\lambda$ and critical value $v_\lambda \in V_\lambda$;

3. As $\lambda$ rotates once around the boundary of $\mathcal{O}$, the critical value lies in $V_\lambda - U_\lambda$ and rotates once around $U_\lambda$.

For such a family of maps, it is then known that there is a copy of the Mandelbrot set in $\mathcal{O}$ and, for $\lambda$-values in this set, there is a subset of $U_\lambda$ on which $G_\lambda$ is conjugate to the corresponding quadratic map.

The open disks $U_\lambda$ that we shall use for this construction will be slight variations of the connecting and non-connecting sets $I_{s_0 \ldots s_k}$ described in the previous section.

Consider the connecting region $I_{s_0 \ldots s_k}$. We may expand this region slightly by including a portion of $B_\lambda$ and $T_\lambda$ in this set. To accomplish this, first assume that the portion of the boundary of this region reaching out to $\partial B_\lambda$ is given by the two internal rays $j/2^{k+2}$ and $(j + 1)/2^{k+2}$. Then we may extend this region slightly into $B_\lambda$ by first choosing a specific level set given by the Böttcher coordinate on $B_\lambda$, and then adding to $I_{s_0 \ldots s_k}$ the adjacent “rectangular” region in $B_\lambda$ bounded by this level set and the two external rays with the above angles $j/2^{k+2}$ and $(j + 1)/2^{k+2}$. Via the $H_\lambda$ symmetry, we can add a similar rectangular region in $T_\lambda$ to this set. For simplicity, we also call this expanded region $I_{s_0 \ldots s_k}$.

Since $I_{s_0 \ldots s_k}$ is a connecting region, this set is mapped two-to-one onto an expanded region that we also call $I_{s_1 \ldots s_k}$. Here the outer boundary of this
set again includes a rectangular region in $B_{\lambda}$ (and possibly but not always in $T_{\lambda}$), this time extending a bit further toward $\infty$ since the level set of the Böttcher coordinate is mapped further outward. Continuing in this fashion, we let $I_{s_j \ldots s_k}$ be the images of $I_{s_{j-1} \ldots s_k}$.

We now show that there is an open disk of parameters for which a critical value lies in any given (connecting or non-connecting) region $I_{s_0 \ldots s_k}$. Because the parameter plane is symmetric under complex conjugation ($F_{\lambda}$ is conjugate to $F_{\bar{\lambda}}$ via $z \mapsto \bar{z}$), it suffices to consider parameters in the upper half-plane. Then one checks easily that this implies that $F_{\lambda}(c_0)$ lies in $I_0$, so the itinerary of $c_0$ begins with $00$. In fact, as shown in [3], it is precisely when $\lambda \in \mathbb{R}^-$ that the critical value $v_{\lambda}$ in the upper half-plane lies on the internal ray of angle $1/4$.

We shall also assume that the itinerary under consideration is not $00\ldots 0$. The reason for this is that it is well known that there is no accessible Mandelbrot set corresponding to the itinerary (0). Indeed, the tip of the “tail” of the set of parameters with this itinerary (i.e., the map that should be conjugate to $z^2 - 2$) now lies at the origin in the parameter plane, and the map $z^2$ does not correspond to the parameter that would lie at this point in a Mandelbrot set. On the other hand, for $n \geq 3$, it is known [4] that there is a “principal” Mandelbrot set lying along the positive real axis in the parameter plane corresponding to this itinerary.

**Proposition 3.** Suppose $I_{s_0 \ldots s_k}$ is a connecting region which therefore contains the critical point $c_{s_0}$. Let $\mathcal{O}$ be the set of parameters for which $F_{\lambda}(c_{s_0})$ lies in $I_{s_1 s_2 \ldots s_k}$. Then $\mathcal{O}$ is an open disk, and, as $\lambda$ winds around the boundary of $\mathcal{O}$, the critical value $F_{\lambda}(c_{s_0})$ winds once around the boundary of $I_{s_1 s_2 \ldots s_k}$.

**Proof:** As above, we may assume that this connecting region is $I_{00 s_2 \ldots s_k}$ which thus contains $c_0$. The result for the other connecting regions follows by symmetry. By our earlier results we know that there is a non-empty
open set of parameters for which $F_\lambda(c_0)$ lies in the open region $I_{0s_2...s_k}$. By construction, on the boundary of this set of parameters, the corresponding critical value lies in the boundary of $I_{0s_2...s_k}$.

To prove that $\mathcal{O}$ is an open disk, we first choose a natural parameterization $\beta_\lambda(\theta)$ of the portion of the boundary of $I_{0s_2...s_k}$ that lies in the chosen level set given by the Böttcher coordinate in $B_\lambda$. We may choose this parametrization so that, for a fixed $\theta$, the point $\beta_\lambda(\theta)$ varies analytically with $\lambda$. Then standard results from complex dynamics show that there is a unique parameter $\lambda_0$ for which $\nu_{\lambda_0}$ lies on the given point $\beta_{\lambda_0}(\theta)$, i.e., for this unique parameter, the critical value lies on that one special boundary point of $I_{0s_2...s_k}$. Hence we have a unique portion of $\partial \mathcal{O}$ for which the critical values of these maps lie in the portion of the boundary of $I_{0s_2...s_k}$ given by the Böttcher coordinate in $B_\lambda$. As $\lambda$ varies continuously along $\partial \mathcal{O}$, $\nu_{\lambda}$ also moves continuously around the boundary of the corresponding region in the dynamical plane. Hence there can be at most one boundary component of $\mathcal{O}$, and, because of the uniqueness of the parameters for which $\nu_{\lambda}$ lies along the level set in $B_\lambda$, it follows that as $\lambda$ winds around $\partial \mathcal{O}$, $\nu_{\lambda}$ winds once around $\partial I_{0s_2...s_k}$.

We next prove:

**Proposition 4.** Suppose $(s_0...s_k)$ is an allowable sequence with prime period $k+1$. Let $I_s = I_{s_0...s_k}$ be a connecting region. Then $F^{k+1}_\lambda$ maps $I_s$ two-to-one onto $I_{s_0...s_k}$, i.e., there are no critical points in $F^j_\lambda(I_s)$ for $j = 1, \ldots, k-1$.

**Proof:** Since $I_s$ is a connecting region, the critical point $c_{s_0}$ lies in $I_s$. Suppose that the critical point in $I_{s_{k-j}}$ lies in the region $I_{s_{k-j}...s_k}$ for some $j$ with $0 < j < k$. Since the itineraries of $c_{s_0}$ and $c_{s_{k-j}}$ must agree after the second digit, the itinerary of the critical point $c_{s_0}$ must be $s_0s_1s_{k-j+2}...s_{k}s_0...s_k$.  

20
But this itinerary is also $s_0 s_1 \ldots s_k s_0 \ldots s_k$. Therefore we must have

\[
\begin{align*}
    s_{k-j+2} &= s_2 \\
    s_{k-j+3} &= s_3 \\
    &\vdots \\
    s_k &= s_j \\
    s_0 &= s_{j+1} \\
    s_2 &= s_{j+2} \\
    &\vdots
\end{align*}
\]

It then follows easily that the itinerary $(s_0 \ldots s_k)$ actually has prime period less than $k + 1$, which gives a contradiction.

\[\square\]

In order to invoke the polynomial-like map argument which will produce the accessible Mandelbrot sets, we need to show that $F^{k+1}_\lambda$ maps the connecting region $I_s$ onto a region that properly contains $I_s$ in its interior. Unfortunately, there are certain allowable sequences for which this is not true. These are the exceptional sequences described in Proposition 2. For example, if $s = (0 \ldots 0)$ with $2(k + 1)$ zeroes, then it is easy to see that the boundary of $F^{k+1}_\lambda(I_s)$ meets the boundary of $(s_0 \ldots s_k) = (0 \ldots 0)$ along arcs that lie in the internal rays of angles $0$ and $1/4$. Similarly, if $s = (3 \ldots 3)$ the boundaries again meet, this time along the internal rays of angles $0$ and $3/4$. However, we do not care about these particular sequences, because they do not have prime period $k + 1$.

On the other hand, the two collections of symmetrically located connecting regions where $s = (120 \ldots 0120 \ldots 0)$ and $s = (213 \ldots 3213 \ldots 3)$ also have this property. We shall deal with these exceptional cases at the end of this section.

First, however, we deal with the non-exceptional cases.
Proposition 5. Suppose $(s_0 \ldots s_k)$ has prime period $k + 1$ (and is not equal to either $(120 \ldots 0)$ or $(213 \ldots 3)$). Then the connecting regions $I_{s_0 \ldots s_k}$ and $I_{s_0 \ldots s_k}$ have disjoint boundaries.

Proof: Suppose these two boundaries meet. Applying $F_{\lambda}^{k+1}$ to $I_{s_0 \ldots s_k}$ yields the set $I_{s_0 \ldots s_k} \cap F_{\lambda}(I_{s_k})$. But $F_{\lambda}(I_{s_k})$ is either $I_0 \cup I_1$ or $I_2 \cup I_3$ depending on $s_k$, and so the only place where the boundaries could meet would be along the internal rays of angles 0 or 1/2. However, by Proposition 2, the only connecting regions where this would occur would be $I_{0,0}$ and $I_{3,3}$ and their symmetrically located images. But we have excluded these cases.

It therefore follows from this Proposition that, assuming $(s_0 \ldots s_k)$ is an allowable sequence with prime period $k + 1$ (and not one of the exceptional cases) and $I_{s_0 \ldots s_k}$ is a connecting region, then $F_{\lambda}^{k+1}$ is a polynomial-like map of degree two on this disk. By Proposition 3, there exists an open disk $O$ of parameters for which $F_{\lambda}(c_{s_0})$ lies in $I_{s_1 \ldots s_k}$. When $\lambda \in \partial O$, as shown above, $F_{\lambda}(c_{s_0})$ lies in the boundary of $I_{s_1 \ldots s_k}$, and, as $\lambda$ winds once around $\partial O$, $F_{\lambda}(c_{s_0})$ winds once around the boundary of $I_{s_1 \ldots s_k}$. Moreover, since $F_{\lambda}^j$ maps $I_{s_0 \ldots s_k}$ one-to-one onto $I_{s_j \ldots s_k}$, we have that $F_{\lambda}^j(v_{\lambda})$ also winds once around $\partial I_{s_j \ldots s_k}$ for each $j \leq k$. In particular, $F_{\lambda}^{k+1}(v_{\lambda})$ winds once around the boundary of $I_{s_0 \ldots s_k}$ as $\lambda$ winds around the boundary of $O$. Thus we have shown modulo the exceptional sequences:

Theorem. Given any allowable sequence $(s_0 \ldots s_k)$, there is an open disk of parameters for which $F_{\lambda}^{k+1}$ is a family of polynomial-like maps of degree two on the connecting region $I_{s_0 \ldots s_k}$.

By this result, we therefore know that there exists a baby Mandelbrot set in the parameter plane for which the itinerary of the critical point is always
$(s_0 \ldots s_k)$. The question is: is this one of the accessible Mandelbrot sets that extends to the boundary of the connectedness locus in the parameter plane? Since $F^{k+1}_\lambda$ maps $I_{s_0 \ldots s_k}$ two-to-one over itself, there must be two fixed points (up to multiplicity) for $F^{k+1}_\lambda$ in this region. One of these fixed points then lies in $\partial B_\lambda$ since the external ray that corresponds to the itinerary $(s_0 \ldots s_k)$ lands at a periodic point of period $k + 1$ in $\partial B_\lambda$ which then must lie in $I_{s_0 \ldots s_k}$. Consequently, the corresponding quadratic-like Julia sets must extend to the boundary of $B_\lambda$. Therefore the corresponding Mandelbrot set is indeed accessible. We thus have:

**Corollary.** Given the allowable sequence $(s_0 \ldots s_k)$, there exists a baby Mandelbrot set in the parameter plane for $F_\lambda$ for which the cusp of the main cardioid meets $\partial B_\lambda$ at the landing point of the corresponding external ray.

In Figure 6, we display some of the external rays that land at the cusps of accessible Mandelbrot sets for the family $z^2 + \lambda/z^2$.

We now consider the exceptional case where $s = (120 \ldots 0120 \ldots 0)$. (The case $(213 \ldots 3213 \ldots 3)$ may be handled in the same fashion.) We first assume that this sequence is not $(1212)$. In this case the connecting region $I_{120 \ldots 0}$ meets the internal ray of angle $1/2$ in an arc $\gamma$ that connects to $\partial T_\lambda$. The smaller connecting region $I_s$ meets this ray in a smaller arc $\gamma'$ that is contained in $\gamma$. So we may extend the connecting region $I_s$ as we did earlier by adding another “rectangular” region that now lies in $I_2$ and lies slightly below $\gamma'$. We may choose this region so that $F^{k+1}_\lambda$ maps it onto a larger rectangular region that is attached to $I_{120 \ldots 0}$ along $\gamma$ and strictly contains the original smaller rectangular region.

Now, using the $H_\lambda$ symmetry, we may attach another rectangular region to $I_s$, this time along the internal ray of angle $1/4$. These are the only two places where the boundaries of $I_s$ and its image under $F^{k+1}_\lambda$ meet. As before, we call this enlarged connecting region $I_s$. This region now sits prop-
Figure 6: The parameter plane for the family $z^2 + \lambda/z^2$ together with some external rays that land on accessible Mandelbrot sets.
erly inside its image under $F_{\lambda}^{k+1}$, so the polynomial-like map argument used previously now works in this case as well.

When $s = (1212)$, the boundaries of the regions $I_s$ and $I_{12}$ do meet, but not along the internal ray of angle $1/2$. However, proving the existence of a Mandelbrot set with base period 2 in this case is straightforward. Suppose that $I_{12}$ is a connecting region lying in $I_1$. So, by symmetry, the connecting region in $I_2$ is $I_{20}$. Thus there is a non-connecting region $I_{21}$ lying in $I_2$ that is mapped one-to-one onto $I_1$. Hence $F_{\lambda}^2$ maps $I_{21}$ two-to-one onto $I_2 \cup I_3 \supset I_{21}$. Also, the critical value of $F_{\lambda}^2$ is just $F_{\lambda}(c_1)$, so there is an open disk of parameters for which this critical value winds once around the boundary of $I_{21}$ as $\lambda$ moves around the boundary of this disk. This proves that there is a Mandelbrot set corresponding to the itinerary $(12)$. A similar argument produces the other base period 2 Mandelbrot set corresponding to the itinerary $(21)$.

4 Final Comments

In this paper we have concentrated for the most part on the family $z^n + \lambda/z^n$ where $n = 2$. When $n > 2$ the proof of the main result is essentially the same; only a few minor modifications are necessary. For example, in the construction of the internal rays, we used the Cantor necklace that lies in the sectors $S_0 \cup S_2$. When $n > 2$, as described in Section 1, this necklace now lies in $S_0 \cup S_n$. When $n = 2$ we then used two preimages of this necklace at the first stage to generate the internal rays of angles $1/4$ and $3/4$; now we need more preimages to generate the internal rays of angles $j/2n$, but the procedure to do this is exactly the same. Then we continue inductively as before.

Also, the uniformizing map on $B_\lambda$ now conjugates $F_\lambda$ on this basin to
Figure 7: The parameter plane for the family $z^3 + \lambda/z^3$ together with some external rays that land on accessible Mandelbrot sets.
the map \( z \mapsto z^n \), so the internal angles (and boundary curves) that are now produced by the earlier construction are of the form \( j/n^k \), while the periodic external rays that land at the cusps of accessible Mandelbrot sets are now of the form \( j/(n^k - 1) \). In Figure 7 we display the parameter plane for the case \( n = 3 \) as well as some of the external rays that land on accessible Mandelbrot sets. In [4], we proved the existence of the \( n - 1 \) principal Mandelbrot sets that do not appear when \( n = 2 \).

Finally, recall that, in the case where \( n \) is odd, it is not necessarily true that both of the critical values lie on the same critical orbit. However, since there are only two critical values, there can be at most two critical orbits. If there are two such orbits, then they must be symmetric under \( z \mapsto -z \), since one orbit contains \( v_\lambda \) and the other \(-v_\lambda \). Hence there is one critical point \( c_j \) that maps to \( v_\lambda \) in the first orbit (and \( n - 1 \) other critical points map onto \( v_\lambda \)). Then, by symmetry, the other orbit contains \(-c_j \) and \(-v_\lambda \) (and the other \( n - 1 \) critical points map onto \(-v_\lambda \)). Then, assuming that these orbits correspond to allowable repeating sequences, the polynomial-like map argument above applied to either of the corresponding connecting regions produces a single Mandelbrot set associated to such parameters. For each \( \lambda \) in this Mandelbrot set, there are now two distinct and symmetric orbits that behave similarly. For example, in the parameter plane for \( n = 3 \) (see Figure 7), for \( \lambda \)-values on the positive real line in the main cardioid of the right hand principal Mandelbrot set, there are two distinct attracting fixed points, one lying in \( \mathbb{R}^+ \), the other in \( \mathbb{R}^- \).

The other possibility is that both of the critical values lie on the same critical orbit. These now do not map to the same point as in the case where \( n \) is even. Rather, by the \( z \mapsto -z \) symmetry, each of these critical values is a distinct point on this orbit. Hence there must be two critical points on this orbit which, by symmetry, must be of the form \( \pm c_j \). Then we have
$F^k_{\lambda}(c_j) = -c_j$ for some $k$ and therefore $F^{2k}_{\lambda}(c_j) = c_j$. If we were to invoke the polynomial-like mapping argument in such a case, there would then be a problem. If the corresponding allowable sequence is $s = (s_0 \ldots s_{2k-1})$, then $F^{2k}_{\lambda}$ maps $I_{ss}$ four-to-one over itself since $F^k_{\lambda}$ maps this connecting region over the region $I_{s_k \ldots s_{2k-1} s}$ which contains the other critical point. This would not then yield an actual Mandelbrot set as in the previous cases.

However, we may remedy this as follows. Consider the connecting region $I_{s_0 \ldots s_{2k-1} s_0}$ that contains the critical point $c_{s_0}$. $F^k$ maps this region two-to-one over a different connecting region that contains $-c_{s_0}$, namely $-I_{s_0 \ldots s_{k-1} s_0}$. Now consider the map $-F^k_{\lambda}$ on this region. This now maps $I_{s_0 \ldots s_{k-1} s_0}$ two-to-one over itself. Then the polynomial-like mapping argument above produces a Mandelbrot set of base period $k$ for the family $-F^k_{\lambda}$. But $-F^k_{\lambda} \circ -F^k_{\lambda} = F^{2k}_{\lambda}$, so we would therefore have the same structure in the parameter plane for the map $F^{2k}_{\lambda}$ as we do for $-F^k_{\lambda}$. But this then gives a Mandelbrot set of base period $k$ for $F_{\lambda}$ (even though the corresponding attracting periodic orbits have twice this period).

For example, in the left hand principal Mandelbrot set in the case $n = 3$, parameters drawn from the main cardioid now have an attracting cycle of period 2. The corresponding quadratic-like Julia sets are not basilicas in this case; rather, there are two disjoint attracting basins for this cycle, one in the upper half-plane, and the other symmetrically located in the lower half-plane.

References


30