A Survey of Exponential Dynamics

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Abstract In this paper we describe some of the interesting dynamics, topology, and geometry that arises in the iteration of the complex exponential $E_\lambda(z) = \lambda e^z$ where $\lambda > 0$. There are two quite distinct cases. When $\lambda \leq 1/e$, the Julia set for E_λ is a Cantor bouquet. When $\lambda > 1/e$, the Julia set suddenly explodes and fills the entire plane. We show that it is the appearance of indecomposable continua in the Julia set that accounts for this explosion.

Keywords Complex dynamics, exponential function, indecomposable continuum

AMS Subject Classification 37F10

1 Introduction

Our goal in this paper is to describe some of the interesting dynamics, topology, and geometry that arises in the iteration of entire functions such as the complex exponential $E_\lambda(z) = \lambda e^z$. We will see that the important invariant sets for this family possesses a extremely rich topological structure, including such objects as Cantor bouquets, Knaster continua, and explosion points.

For a complex analytic function E, the interesting orbits lie in the *Julia set*, which we denote by $J(E)$. This is the set on which the map is chaotic. For the exponential family, the Julia set of E_λ has three equivalent characterizations:

1. $J(E_\lambda)$ is the set of points at which the family of iterates of E_λ, $\{E_\lambda^n\}$, is not a normal family in the sense of Montel. This is the characterization that is most useful to prove theorems.

2. $J(E_\lambda)$ is the closure of the set of repelling periodic points of E_λ. This is the dynamical definition of the Julia set.

3. $J(E_\lambda)$ is the closure of the set of points whose orbits tend to ∞. This is the characterization that is most useful to compute the Julia set.

We remark that characterization 3 differs from the case of polynomial iterations, where the Julia set is the *boundary* of the set of escaping orbits. The reason for the difference is that E_λ has an essential singularity at ∞, while polynomials have superattracting fixed points at ∞. The equivalence of 1 and 2 was shown by Baker, see [Ba1]. The equivalence of 1 and 3 is shown in [DT].

In this paper we will concentrate on the dynamics of E_λ where λ is real. For λ positive, the Julia set for E_λ undergoes a remarkable transformation as λ passes through $1/e$. We will show below that E_λ possesses an attracting fixed point when $0 < \lambda < 1/e$. All points in the left half plane have orbits that tend to this fixed point. Indeed, the full basin of attraction of this fixed point is open and dense in the plane.

We will show that the complement of the basin, $J(E_\lambda)$, is a *Cantor bouquet* for $0 < \lambda \leq 1/e$. Roughly speaking, a Cantor bouquet has the property that all points in the set lie on a curve (or "hair") homeomorphic to a closed half line. Each of these curves in $J(E_\lambda)$ extend to ∞ in the right half-plane.

In Figures 1 and 2 we display a computer graphics rendering of the Julia set of E_λ for a particularλ with $0 < \lambda < 1/e$. This image was computed using characterization 3 of the Julia set: Points are shaded in white and grey if their orbits ever enter the region $\mathrm{Re}\, z > 50$. The complement of the Julia set is displayed in black. It appears that this Julia set contains large open sets, but this in fact is not the case. The Julia set actually consists of uncountably many curves lying in the Cantor bouquet and extending to ∞ in the right half plane. These curves are packed together so tightly that the resulting set has Hausdorff dimension 2, thus giving the appearance of an open set.

At $\lambda = 1/e$, E_λ undergoes a simple saddle-node bifurcation. The attracting fixed point merges with a repelling fixed point at this λ-value, producing a neutral fixed point. When $\lambda > 1/e$, this neutral fixed point gives way to a pair of repelling fixed points.

This apparently simple bifurcation has profound global ramifications. When $\lambda \leq 1/e$, the Julia set is a nowhere dense subset of the

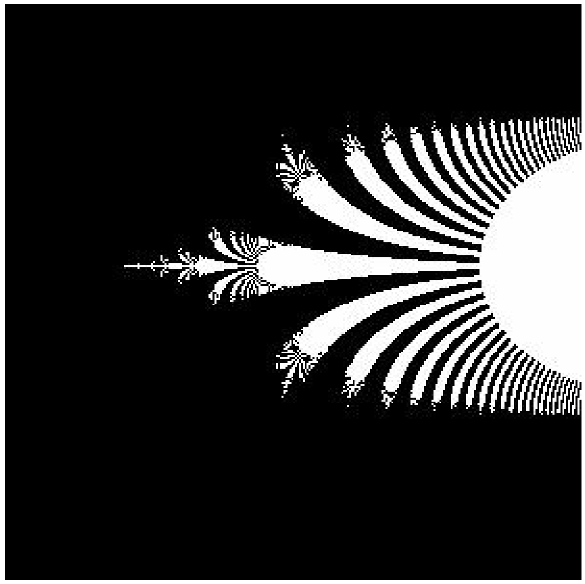

Figure 1: A Julia set for $\lambda < 1/e$.

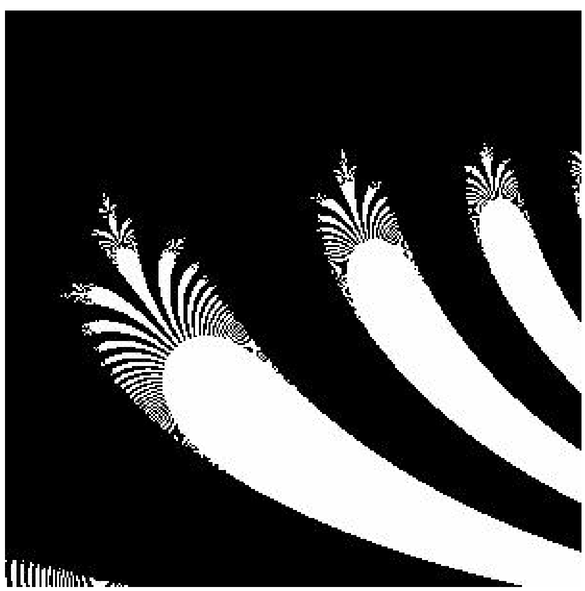

Figure 2: Magnification of a Julia set for $\lambda < 1/e$.

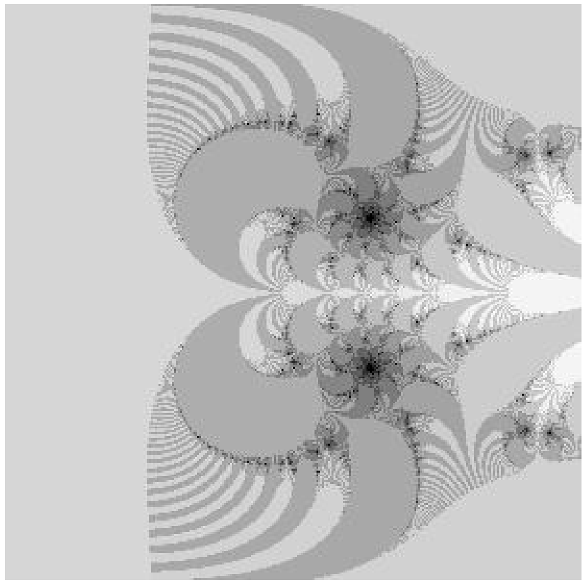

Figure 3: The Julia set for $\lambda > 1/e$.

right half plane. However, when $\lambda > 1/e$, $J(E_\lambda)$ suddenly becomes the whole plane. No new repelling periodic points (except the two fixed points involved in the saddle-node) are born in this bifurcation; all others simply move smoothly as λ crosses through $1/e$. Yet somehow, as soon as λ exceeds $1/e$, the repelling periodic points become dense in $\mathbb{C}$.

In Figure 3 we display the Julia set for E_λ for a particular $\lambda > 1/e$. Note the striking difference between this image and that in Figure 1.

At this bifurcation, the attracting fixed point and its entire basin of attraction disappear. Most of the curves in the Cantor bouquet remain as curves in the Julia set. However, some evolve into a new and interesting set called an indecomposable continuum.

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2 Exponential Dynamics

As in the often-studied quadratic family $Q_c(z) = z^2 + c$, it is the orbit of 0 that plays a crucial role in determining the dynamics of E_λ. For the exponential family, 0 is an asymptotic value (an omitted value) rather than a critical point. Nevertheless, the orbit of 0 plays a decisive role in the determination of the structure of $J(E_\lambda)$:

Theorem 2.1 *Suppose E_λ has an attracting or rationally neutral (parabolic) periodic point. Then $E_\lambda^n(0)$ must tend to the attracting or neutral cycle. If, on the other hand, $E_\lambda^n(0) \to \infty$, then $J(E_\lambda) = \mathbb{C}$.*

The proof of the first statement in this theorem is a classical fact that goes back to Fatou. The second follows from the Sullivan No Wandering Domains Theorem [Su], as extended to the case of the exponential by Goldberg and Keen [GK] and Eremenko and Lyubich [EL].

Consider for the moment the restriction of E_λ to the real line. The exponential family undergoes a saddle node bifurcation at $\lambda = 1/e$ since, when $\lambda = 1/e$, the graph of $E_{1/e}$ is tangent to the diagonal at 1. See Figure 4. We have $E_{1/e}(1) = 1$ and $E'_{1/e}(1) = 1$. When $\lambda > 1/e$, the graph of E_λ lies above the diagonal and all orbits (including 0) tend to ∞. When $\lambda < 1/e$, the graph of E_λ crosses the diagonal twice, at an attracting fixed point a_λ and a repeling fixed point r_λ. For later use note that $0 < a_\lambda < 1 < r_\lambda$. Note also that the orbit of 0 tends to a_λ, as it must by Fatou's theorem.

3 Cantor Bouquets

In this section, we begin the study of the dynamics of E_λ by considering the case where $\lambda \leq 1/e$. In this case $J(E_\lambda)$ is a *Cantor bouquet.* We will give a sketch of the construction of this object. For more details, see [D2]

Let $E(z) = (1/e)e^z$. We have $E(1) = 1$ and $E'(1) = 1$. If $x_0 \in \mathbb{R}$ and $x_0 < 1$, then $E^n(x_0)$ tends to the fixed point at 1. If $x_0 > 1$, then $E^n(x_0) \to \infty$ as $n \to \infty$. This can be shown using the web diagram as shown in Figure 5.

The vertical line $\operatorname{Re} z = 1$ is mapped to the circle of radius 1 centered at the origin. In fact, E is a contraction in the half plane H to the left of this line, since

$$|E'(z)| = \frac{1}{e}\exp(\operatorname{Re} z) < 1$$

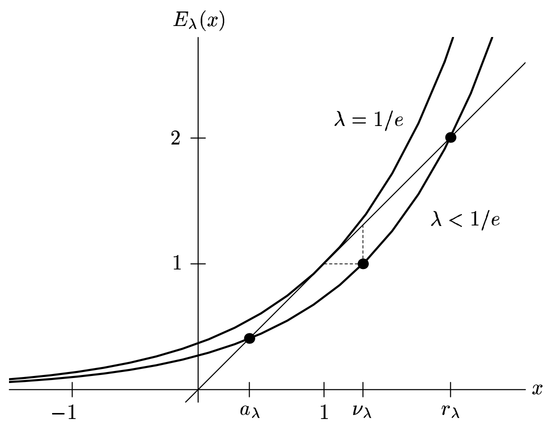

Figure 4: The graphs of E_λ for $\lambda = 1/e$ and $\lambda < 1/e$.

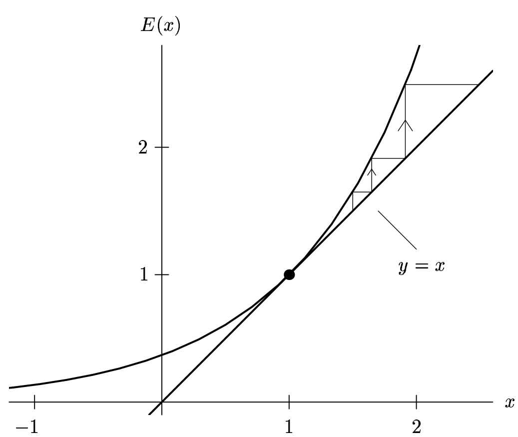

Figure 5: The graph of $E(x) = (1/e)e^x$.

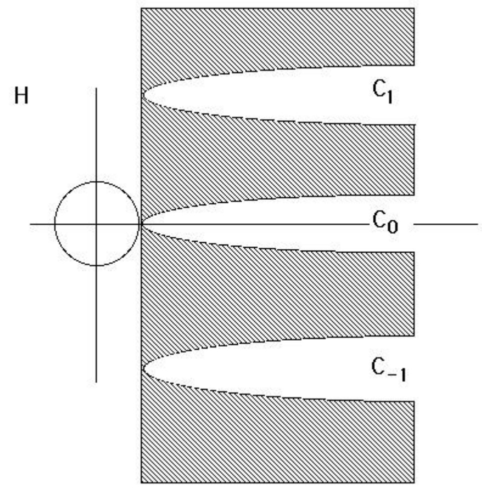

Figure 6: The preimage of H consists of H and the shaded region.

if $z \in H$. Consequently, all points in H have orbits that tend to 1. Hence this half plane lies in the stable set, i.e., in the complement of the Julia set. We will try to paint the picture of the Julia set of E by painting instead its complement.

Since the half plane H is forward invariant under E, we can obtain the entire stable set by considering all preimages of this half plane. Now the first preimage of H certainly contains the horizontal lines $\operatorname{Im} z = (2k+1)\pi$, $\operatorname{Re} z \geq 1$, for each integer k, since E maps these lines to the negative real axis which lies in H. Hence there are open neighborhoods of each of these lines that lie in the stable set. The first preimage of H is shown in Figure 6. The complement of $E^{-1}(H)$ consists of infinitely many "fingers." The fingers are $2k\pi i$ translates of each other, and each is mapped onto the complementary half plane $\operatorname{Re} z \geq 1$.

We denote the fingers in the complement of $E^{-1}(H)$ by C_j with $j \in \mathbb{Z}$, where C_j contains the half line $\operatorname{Im} z = 2j\pi$, $\operatorname{Re} z \geq 1$, which is mapped into the positive real axis. That is, the C_j are indexed by the integers in order of increasing imaginary part. Note that C_j is contained within the strip $-\frac{\pi}{2} + 2j\pi \leq \operatorname{Im} z \leq \frac{\pi}{2} + 2j\pi$.

Now each C_j is mapped in one-to-one fashion onto the entire half plane $\operatorname{Re} z \geq 1$. Consequently each C_j contains a preimage of each

Figure 7: The second preimage of H in one of the fingers C_j.

other C_k. Each of these preimages forms a subfinger which extends to the right in the half plane H. See Figure 7. The complement of these subfingers necessarily lies in the stable set.

Now we continue inductively. Each subfinger is mapped onto one of the original fingers by E. Consequently, there are infinitely many sub-subfingers which are mapped to the C_j's by E^2. So at each stage we remove the complement of infinitely many subfingers from each remaining finger.

This process is reminiscent of the construction of the Cantor set in the dynamics of polynomials when all critical points tend to ∞. In that construction, the complements of disks are removed at each stage; here we remove the complement of infinitely many fingers. As a result, after performing this operation infinitely many times, we do not end up with points. Rather, the intersection of all of these fingers, if nonempty, is a simple curve extending to ∞. See [DK].

This collection of curves forms the Julia set. E permutes these curves and each curve consists of a well-defined endpoint together with a "stem" which extends to ∞. It is tempting to think of this structure as a "Cantor set of curves," i.e., a product of the set of endpoints and the half-line. However, this is not the case as the set of endpoints is not closed.

Note that we can assign symbolic sequences to each point on these curves. To do this, we attach an infinite sequence $s_0 s_1 s_2 \dots$ to each curve in the Julia set via the rule: $s_j \in \mathbb{Z}$ and $s_j = k$ if the j^{th} iterate of the curve lies in C_k. The sequence $s_0 s_1 s_2 \dots$ is called the *itinerary* of the curve.

For example, the portion of the real line $\{x \mid x \geq 1\}$ lies in the Julia set since all points (except 1) tend to ∞ under iteration, not to the fixed point. These points all have itinerary $000\dots$.

One temptation is to say that there is a curve corresponding to every possible sequence $s_0 s_1 s_2 \dots$ This, unfortunately, is not true, as certain sequences simply grow too quickly to correspond to orbits of E. See [DeV].

So this is $J(E)$: a "hairy" object extending toward ∞ in the right-half plane. We call this object a *Cantor bouquet.* We will see that this bouquet has some rather interesting topological properties.

We remark that the same construction works if $0 < \lambda < 1/e$. We still define the half plane H as the set $\operatorname{Re} z < 1$. As we saw earlier, the point 1 on the real axis sits between the attracting fixed point a_λ and the repelling fixed point r_λ, and so $E_\lambda(1) < 1$ and as a consequence $E_\lambda(H)$ is strictly contained in H. The construction of the fingers now proceeds exactly as above.

The Cantor bouquet is a remarkable object from the topological and geometric point of view. Here are just a few of its properties:

Properties.

1. There are two types of points in the Cantor bouquet: the end-points and the points on the stem. It is known that all points on the stem have orbits that tend to ∞. Hence the set of bounded orbits is contained in the set of endpoints. In particular, the set of repelling periodic points lies in the set of endpoints. But these points are dense in $J(E_\lambda)$, so the set of endpoints accumulates on all points in J.

2. A result of Mayer [Ma] shows that the set of endpoints has the following intriguing structure: In the Riemann sphere the set of endpoints together with ∞ forms a connected set. However, the set of endpoints alone is totally disconnected! That is, removing just one point from this connected set not only disconnects the set, but also **totally** disconnects it!

3. McMullen [McM] has shown that the Hausdorff dimension of the Cantor bouquet constructed above is 2 but its Lebesgue measure is zero. This accounts for why figures 1 and 2 seem to have open regions in the Julia set.

4. Babinska has shown that the Hausdorff dimension of the set of stems is 1, but the Hausdorff dimension of the set of endpoints is 2!

4 Indecomposable Continua

We now consider the case $\lambda > 1/e$. Since the orbit of 0 tends to ∞, the Julia set is now the entire plane. For these λ values, the attracting basin for the attracting fixed point a_λ disappears. What replaces it is a collection of complicated sets known as indecomposable continua. We describe the construction of one such set in this section.

Consider the horizontal strip

$$S = \{z|\ 0 \leq \text{Im } z \leq \pi\}$$

(or its symmetric image under $z \to \overline{z}$). The exponential map E_λ takes the boundary of S to the real axis and the interior of S to the upper half plane. Thus, E_λ maps certain points outside of S while other points remain in S after one application of E_λ. Our goal is to investigate the set of points whose entire orbit lie in S. Call this set Λ. The set Λ is clearly invariant under E_λ. There is a natural way to compactify this set in the plane to obtain a new set Γ. Moreover, the exponential map extends to Γ in a natural way. Our main results in this section include:

Theorem 4.1 *Γ is an indecomposable continuum.*

Moreover, we will see that Λ is constructed in similar fashion to the well known *Knaster continua* described below. Thus the topology of Λ is quite intricate. Despite this, we will show that the dynamics of E_λ on Λ is quite tame. Specifically, we will prove:

Theorem 4.2 *The restriction of E_λ to $\Lambda - \{\text{orbit of } 0\}$ is a homeomorphism. This map has a unique repelling fixed point $w_\lambda \in \Lambda$, and the α-limit set of all points in Λ is w_λ. On the other hand, if $z \in \Lambda$, $z \neq w_\lambda$, then the ω-limit set of z is either*

1. *The point at ∞, or*

2. *The orbit of 0 under E_λ together with the point at ∞.*

Thus we see that E_λ possesses an interesting mixture of topology and dynamics in the case where the Julia set is the whole plane. In the plane the dynamics of E_λ are quite chaotic, but the overall topology is tame. On our invariant set Λ, however, it is the topology that is rich, but the dynamics are tame. For more details we refer to [D1].

4.1 Topological Preliminaries

In this section we review some of the basic topological ideas associated with indecomposable continua. See [Ku] for a more extensive introduction to these concepts.

Recall that a continuum is a compact, connected space. A continuum is decomposable if it is the (not necessarily disjoint) union of two proper subcontinua. Otherwise, it is indecomposable. A well known example of an indecomposable continuum is the Knaster continuum, K. One way to construct this set is to begin with the Cantor middle-thirds set. Then draw the semi-circles lying in the upper half plane with center at $(1/2, 0)$ that connect each pair of points in the Cantor set that are equidistant from $1/2$. Next draw all semicircles in the lower half plane which have for each $n \geq 1$ centers at $(5/(2 \cdot 3^n),\, 0)$ and pass through each point in the Cantor set lying in the interval

$$2/3^n \leq x \leq 1/3^{n-1}.$$

The resulting set is partially depicted in Figure 8.

For a proof that this set is indecomposable, we refer to [Ku]. Dynamically, this set appears as the closure of the unstable manifold of Smale's horseshoe map (see [Ba], [Sm]).

Note that the curve passing through the origin in this set is dense, since it passes through each of the endpoints of the Cantor set. It also accumulates everywhere upon itself. Such a phenomenon gives a criterion for a continuum to be indecomposable, as was shown by S. Curry.

Theorem 4.3 *Suppose X is a one-dimensional nonseparating plane continuum which is the closure of a ray that limits upon itself. Then X is indecomposable.*

We refer to [Cu] for a proof.

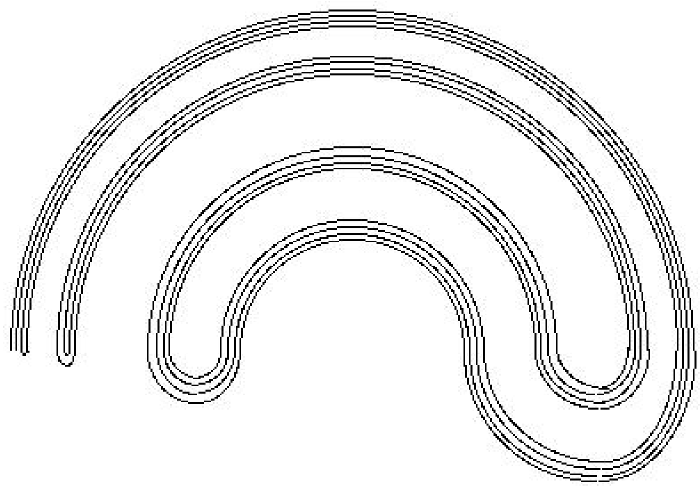

Figure 8: The Knaster Continuum.

4.2 Construction of Λ

Recall that the strip S is given by $\{z \mid 0 \leq \text{Im } (z) \leq \pi\}$. Note that E_λ maps S in one-to-one fashion onto $\{z \mid \text{Im } z \geq 0\} - \{0\}$. Hence E_λ^{-1} is defined on $S - \{0\}$ and, in fact, E_λ^{-n} is defined for all n on $S - \{\text{orbit of } 0\}$. We will always assume that E_λ^{-n} means E_λ^{-n} restricted to this subset of S.

Define

$$\Lambda = \{z \mid E_\lambda^n(z) \in S \text{ for all } n \geq 0\}.$$

If $z \in \Lambda$ it follows immediately that $E_\lambda^n(z) \in S$ for *all* $n \in \mathbf{Z}$ provided z does not lie on the orbit of 0. Our goal is to understand the structure of Λ.

Toward that end we define L_n to be the set of points in S that leave S at precisely the n^{th} iteration of E_λ. That is,

$$L_n = \{z \in S \mid E_\lambda^i(z) \in S \text{ for } 0 \leq i < n$$

$$\text{but } E_\lambda^n(z) \notin S\}.$$

Let B_n be the boundary of L_n.

Recall that E_λ maps a vertical segment in S to a semi-circle in the upper half plane centered at 0 with endpoints in $\mathbf{R}$. Either this semi-circle is completely contained in S or else an open arc lies outside S. As

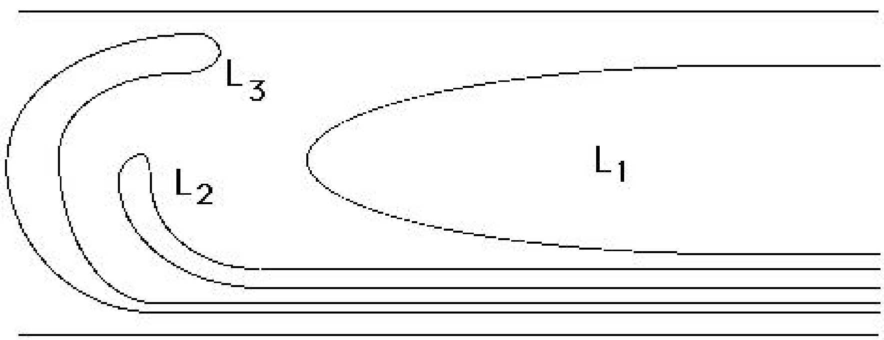

Figure 9: Construction of the L_n.

a consequence, L_1 is an open simply connected region which extends to ∞ toward the right in S as shown in Figure 9. There is a natural parameterization $\gamma_1: \mathbf{R} \to B_1$ defined by

$$E_\lambda(\gamma_1(t)) = t + i\pi.$$

As a consequence,

$$\lim_{t \to \pm\infty} \text{Re}\ \gamma_1(t) = \infty.$$

If $c > 0$ is large, the segment $\text{Re}\, z = c$ in S meets $S - L_1$ in two vertical segments v_+ and v_- with $\text{Im}\, v_- > \text{Im}\, v_+$. E_λ maps v_- to an arc of a circle in $S \cap \{z \,|\, \text{Re}\ z < 0\}$ while E_λ maps v_+ to an arc of a circle in $S \cap \{z \,|\, \text{Re} z > 0\}$. As a consequence, if c is large, v_+ meets L_2 in an open interval. Since $L_2 = E_\lambda^{-1}(L_1)$, it follows that L_2 is an open simply connected subset of S that extends to ∞ in the right half plane *below* L_1.

Continuing inductively, we see that L_n is an open, simply connected subset of S that extends to ∞ toward the right in S. We may also parameterize the boundary B_n of L_n by $\gamma_n: \mathbf{R} \to B_n$ where

$$E_\lambda^n(\gamma_n(t)) = t + i\pi$$

as before. Again

$$\lim_{t \to \pm\infty} \text{Re}\ \gamma_n(t) = \infty.$$

Since each L_n is open, it follows that Λ is a closed subset of S.

Proposition 4.4 *Let $J_n = \bigcup_{i=n}^{\infty} B_i$. Then J_n is dense in Λ for each $n > 0$.*

Proof. Let $z \in \Lambda$ and suppose $z \notin B_i$ for any i. Let U be an open connected neighborhood of z. Fix $n > 0$. Since $E_\lambda^i(z) \in S$ for all i, we may choose a connected neighborhood $V \subset U$ of z such that $E_\lambda^i(V) \subset S$ for $i = 0, \ldots, n$.

Now the family of functions $\{E_\lambda^i\}$ is not normal on V, since z belongs to the Julia set of E_λ. Consequently, $\bigcup_{i=0}^{\infty} E_\lambda^i(V)$ covers $\mathbf{C} - \{0\}$. In particular, there is $m > n$ such that $E_\lambda^m(V)$ meets the exterior of S. Since $E_\lambda^m(z) \in S$, it follows that $E_\lambda^m(V)$ meets the boundary of S. Applying E_λ^{-m}, we see that B_m meets V. $\square$

In fact, it follows that for any $z \in \Lambda$ and any neighborhood U of z, all but finitely many of the B_m meet V. This follows from the fact that E_λ has fixed points outside of S (in fact one such point in each horizontal strip of width 2π—see [DK]), so we may assume that $E_\lambda^m(V)$ contains this fixed point for all sufficiently large m. In particular, we have shown:

Proposition 4.5 *Let $z \in \Lambda$ and suppose that V is any connected neighborhood of z. Then $E_\lambda^m(V)$ meets the boundary of S for all sufficiently large m.*

Proposition 4.6 *Λ is a connected subset of S.*

Proof. Let G be the union of the boundaries of the L_i for all i. Since Λ is the closure of G, it suffices to show that G is connected. Suppose that this is not true. Then we can write G as the union of two disjoint sets A and B. One of A or B must contain infinitely many of the boundaries of the L_i. Say A does. But then, if $b \in B$, the previous proposition guarantees that infinitely many of these boundaries meet any neighborhood of b. Hence b belongs to the closure of A. This contradiction establishes the result.

We can now prove:

Theorem 4.7 *There is a natural compactification Γ of Λ that makes Γ into an indecomposable continuum.*

Proof. We first compactify Λ by adjoining the backward orbit of 0. To do this we identify the "points" $(-\infty, 0)$ and $(-\infty, \pi)$ in S: this

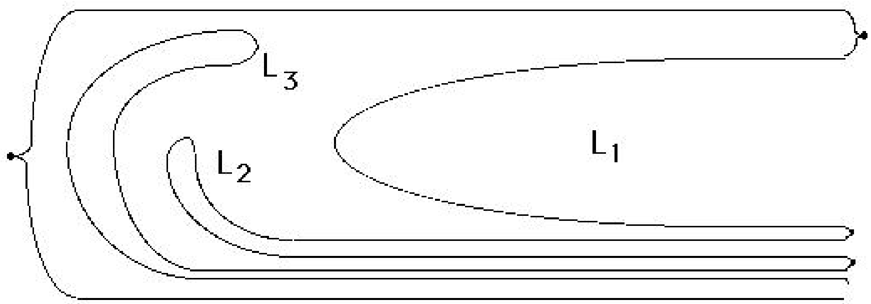

Figure 10: Embedding Γ in the plane.

gives $E_\lambda^{-1}(0)$. We then identify the points (∞, π) and $\lim_{t\to-\infty}\gamma_1(t)$. This gives $E_\lambda^{-2}(0)$. For each $n > 1$ we identify

$$\lim_{t\to\infty}\gamma_n(t)$$

and

$$\lim_{t\to-\infty}\gamma_{n+1}(t)$$

to yield $E_\lambda^{-n-1}(0)$. This augmented space Γ may easily be embedded in the plane. See Figure 10. Moreover, if we extend the B_i and the lines $y = 0$ and $y = \pi$ in the natural way to include these new points, then this yields a curve which accumulates everywhere on itself but does not separate the plane. See the proposition above. By a theorem of S. Curry [Cu], it follows that Γ is indecomposable.

$\square$

As a consequence of this theorem, Λ must contain uncountably many composants (see [Ku], p. 213). In fact, in [DK] it is shown that Λ contains uncountably many curves.

4.3 Dynamics on Λ

In this section we describe the dynamics of E_λ on Λ.

Proposition 4.8 *There exists a unique fixed point w_λ in S if $\lambda > 1/e$. Moreover, w_λ is repelling and, if $z \in S -$ orbit of 0, $E_\lambda^{-n}(z) \to w_\lambda$ as $n \to \infty$.*

Proof. First consider the equation

$$\lambda e^{y \cot y} \sin y = y.$$

Since $y \cot y \to 1$ as $y \to 0$ and $\lambda e > 1$, we have $\lambda e^{y \cot y} \sin y > y$ for y small and positive. Since the left-hand side of this equation vanishes when $y = \pi$, it follows that this equation has at least one solution y_λ in the interval $0 < y < \pi$.

Let $x_\lambda = y_\lambda \cot y_\lambda$. Then one may easily check that $w_\lambda = x_\lambda + i y_\lambda$ is a fixed point for E_λ in the interior of S. Since the interior of S is conformally equivalent to a disk and E_λ^{-1} is holomorphic, it follows from the Schwarz Lemma that w_λ is an attracting fixed point for the restriction of E_λ^{-1} to S and that $E_\lambda^{-n}(z) \to w_\lambda$ for all $z \in S$.

Remarks.
1. Thus the α-limit set of any point in Λ is w_λ.
2. The bound $\lambda > 1/e$ is necessary for this result, since we know that E_λ has two fixed points on the real axis for any positive $\lambda < 1/e$. These fixed points coalesce at 1 as $\lambda \to 1/e$ and then separate into a pair of conjugate fixed points, one of which lies in S.

We now describe the ω-limit set of any point in Λ. Clearly, if $z \in B_n$ then $E_\lambda^{n+1}(z) \in \mathbf{R}$ and so the ω-limit set of z is infinity. Thus we need only consider points in Λ that do not lie in B_n. We will show:

Theorem 4.9 *Suppose $z \in \Lambda$ and $z \neq w_\lambda$, $z \notin B_n$ for any n. Then the ω-limit set of z is the orbit of 0 under E_λ together with the point at infinity.*

To prove this we first need a lemma.

Lemma 4.10 *Suppose $z \in \Lambda$, $z \neq w_\lambda$. Then $E_\lambda^n(z)$ approaches the boundary of S as $n \to \infty$.*

Proof. Let h be the uniformization of the interior of S taking S to the open unit disk and w_λ to 0. Recall that E_λ^{-1} is well defined on S and takes S inside itself. Then $g = h \circ E_\lambda^{-1} \circ h^{-1}$ is an analytic map of the open disk strictly inside itself with a fixed point at 0. This fixed point is therefore attracting by the Schwarz Lemma. Moreover, if $|z| > 0$ we have $|g(z)| < |z|$. As a consequence, if $\{z_n\}$ is an orbit in Λ, we have $|h(z_{n+1})| > |h(z_n)|$, and so $|h(z_n)| \to 1$ as $n \to \infty$.

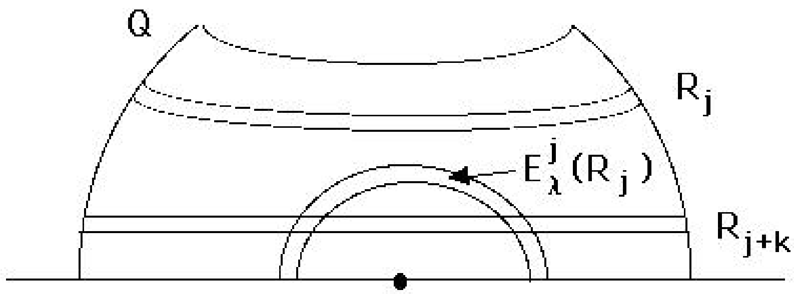

Figure 11: The return map on Q.

□

The remainder of the proof is essentially contained in [DK] (see pp. 45-49). In that paper it is shown that there is a "quadrilateral" Q containing a neighborhood of 0 in $\mathbf{R}$ as depicted in Figure 11. The set Q has the following properties:

1. If $z \in \Lambda - \bigcup_n B_n$ and $z \neq w_\lambda$, then the forward orbit of z meets Q infinitely often.
2. Q contains infinitely many closed "rectangles" $R_k, R_{k+1}, R_{k+2}, \ldots$ for some $k > 1$ having the property that if $z \in R_j$, then $E^j_\lambda(z) \in Q$ but $E^i_\lambda(z) \notin Q$ for $0 < i < j$.
3. If $z \in Q$ but $z \notin \bigcup_{j=k}^\infty R_j$, then $z \in L_n$ for some n.
4. $E^j_\lambda(R_j)$ is a "horseshoe" shaped region lying below R_j in Q as depicted in Figure 11.
5. $\lim_{j\to\infty} E^j_\lambda(R_j) = \{0\}$.

As a consequence of these facts, any point in Λ has orbit that meets the $\cup R_j$ infinitely often. We may thus define a return map

$$\Phi\colon \Lambda \cap (\cup_j R_j) \to \Lambda \cap \cup_j R_j$$

by

$$\Phi(z) = E^j_\lambda(z)$$

if $z \in R_j$. By item 4, $\Phi(z)$ lies in some R_k with $k > j$. By item 5, it follows that

$$\Phi^n(z) \to 0$$

for any $z \in \Lambda \cap Q$. Consequently, the ω-limit set of z contains the orbit of 0 and infinity.

For the opposite containment, suppose that the forward orbit of z accumulates on a point q. By the Lemma, q lies in the boundary of S. Now the orbit of z must also accumulate on the preimages of q. If q does not lie on the orbit of 0, then these preimages form an infinite set, and some points in this set lie on the boundaries of the L_n. But these points lie in the interior of S, and this contradicts the Lemma. Thus the orbit of z can only accumulate in the finite plane on points on the orbit of 0. Since the "preimage" of 0 is infinity, the orbit also accumulates at infinity.

□

5 Final Remarks

It is known [BD] that there are uncountably many curves in the λ-plane having the property that, if λ lies on one of these curves, then $E_\lambda^n(0) \to \infty$. Consequently, for such a λ-value, the Julia set of E_λ is again the complex plane. For these λ-values, a variant of the above construction also yields invariant indecomposable continua in the Julia set [MR]. Whether these continua are homeomorphic to any of those constructed above is an open question.

Douady and Goldberg [DoG] have shown that if $\lambda, \mu > 1/e$, then E_λ and E_μ are not topologically conjugate. Each such map possesses invariant indecomposable continua Λ_λ and Λ_μ in S, and the dynamics on each are similar, as shown above. In fact, we conjecture that each pair of these invariant sets is non-homeomorphic.

A simpler semilinear model mapping that mimics the behavior of E_λ has been constructed in [DMR]. The indecomposable continua constructed in this paper should be easier to deal with than those of E_λ, though we conjecture that they are homeomorphic to specific indecomposable continua in the exponential family.

It is also known that the set of points whose itineraries feature blocks of 0's whose length goes to ∞ quickly is an indecomposable continuum [DJ]. The exact structure of these sets, however, is far from understood.

M. Lyubich has shown that each Λ_λ is a set of measure 0 in S. Indeed, it follows from his work [Ly] that the set of points in $\mathbf{C}$ whose orbits have arguments that are equidistributed on the unit circle have full measure. In Λ_λ, the arguments of all orbits tend to 0 and/or π, and so Λ_λ has measure 0 in S.

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