Generalized Baby Mandelbrot Sets Adorned with Halos in Families of Rational Maps

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Abstract

We consider the family of rational maps given by $F_\lambda(z) = z^n + \lambda/z^d$ where $n, d \in \mathbb{N}$ with $1/n + 1/d < 1$, the variable $z \in \widehat{\mathbb{C}}$ and the parameter $\lambda \in \mathbb{C}$. It is known [1] that when $n = d \geq 3$ there are $n-1$ small copies of the Mandelbrot set symmetrically located around the origin in the parameter $\lambda-$plane. These baby Mandelbrot sets have "antennas" attached to the boundaries of Sierpiński holes. Sierpiński holes are open simply connected subsets of the parameter space for which the Julia sets of F_λ are Sierpiński curves. In this paper we generalize the symmetry properties of F_λ and the existence of the $n-1$ baby Mandelbrot sets to the case when $1/n + 1/d < 1$ where n is not necessarily equal to d.

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1 Introduction

In this paper we consider the family of complex rational maps[1] $F_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

where $n, d \in \mathbb{N}$ with $1/n + 1/d < 1$ and the parameter $\lambda \in \mathbb{C}$. Let $m = n + d$ denote the degree of F_λ. When comparing different maps from the family F_λ it will be convenient to use $< n, d >$ to denote $z^n + \lambda/z^d$.

McMullen introduced F_λ in [2] where he shows that when $\lambda \neq 0$ is sufficiently small, then the Julia set of F_λ is a Cantor set of simple closed curves, see also [3]. The condition $1/n + 1/d < 1$ is equivalent to $nd > m$ and this defines the set of values $n, d \geq 2$ with $m \geq 5$. The dynamics of some of these maps, the topological structures of their Julia sets and the structure of the parameter $\lambda-$planes have been widely studied by several authors, see for example [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. A recent survey of results involving some of the maps in this family is given in [17]. For background results in complex dynamics see for example [18, 19, 20, 21, 22].

The function F_λ has $2(n+d)-2$ critical points counted with multiplicity, namely, $n-1$ at ∞, $d-1$ at 0 and $n+d$ additional critical points whose orbits depend on the value of the parameter λ. One of the reasons this family of maps has gained so much attention is the fact that these "free" critical points all behave symmetrically. This implies that there is essentially one critical orbit and then the $\lambda-$plane is a natural parameter plane for each of these families.

Since $n \geq 2$, the point at ∞ is a superattracting fixed point for any value of λ. Let B_λ denote the immediate basin of attraction of ∞ and notice that B_λ is mapped to itself at least in an $n-$to-1 fashion. When all the critical points are in B_λ the Julia set of the map is a Cantor set of points and B_λ is mapped to itself in an $m-$to-1 fashion. Let T_λ be the preimage of B_λ that contains the origin. When T_λ is disjoint from B_λ then T_λ is a simply connected set that is mapped $d-$to-1 onto B_λ. We call T_λ the *trap door*, since every point that escapes to infinity and it is not in B_λ must fall throuth T_λ along its orbit.

When the orbits of all the critical points of F_λ are attracted to ∞, the Julia set of F_λ can have exactly three different topological structures. The

[1]We use $\mathbb{C}$ for the complex plane and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for the Riemann sphere.

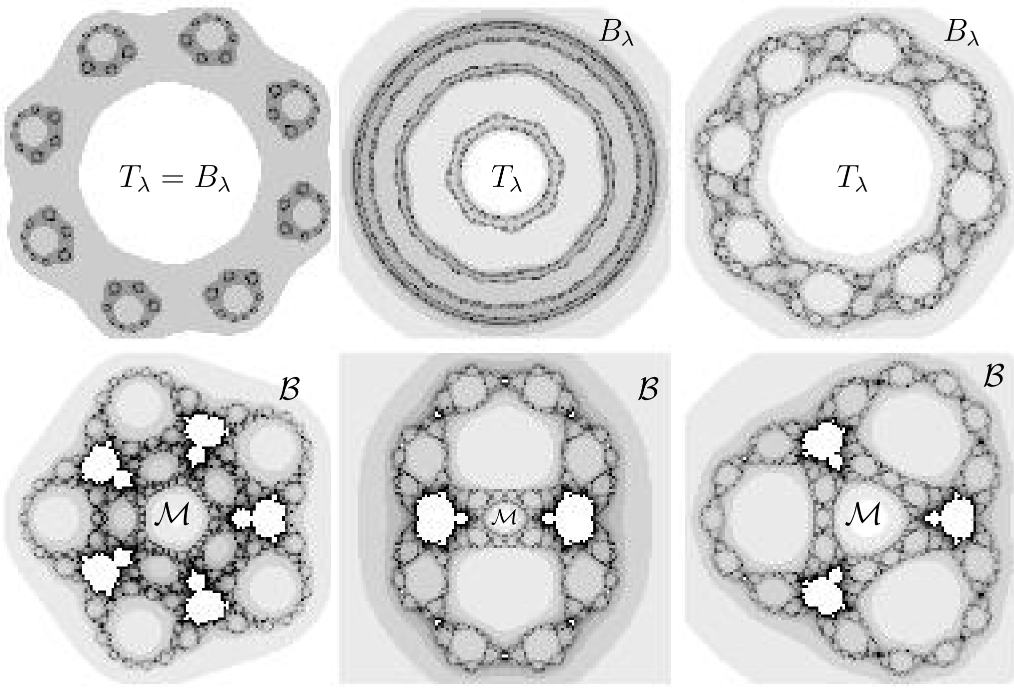

Figure 1: Several Julia sets from the family $F_\lambda(z)$ that illustrate the three topological structures presented in the Escape Trichotomy Theorem 1.1. The top line of pictures shows, from left to right: a Cantor set of points for the case $< n, d >=< 6, 2 >$ with $\lambda = 0.5(1 + i)$, a Cantor set of simple closed curves for the case $< 3, 5 >$ with $\lambda = 0.001(1 + 2i)$ and a Sierpiński curve for the case $< 4, 4 >$ with $\lambda = i/6$. The basin of attraction of ∞ is B_λ and its preimage containing the origin is the trap door T_λ. All these maps have the same degree $m = 8$ and therefore there are 8 sectors of the sphere that are equal to each other under rotation by $\pi/4$. The bottom line of pictures shows (from left to right) the parameter $\lambda-$planes for $< 6, 2 >$, $< 3, 5 >$ and $< 4, 4 >$. Notice the $n - 1$ white baby Mandelbrot sets symmetrically distributed around the origin $\lambda = 0$. The unbounded region is the Cantor set locus $\mathcal{B}$, the disk centered at the origin is the McMullen domain $\mathcal{M}$ and the other shaded disks in the connectedness locus correspond to Sierpiński holes.

following result is included in [3].

Theorem 1.1 *(The Escape Trichotomy). Fix $n, d \in \mathbb{N}$ with $1/n + 1/d < 1$ then,*

1. *If the critical values of F_λ lie in B_λ, then the Julia set is a Cantor set of points.*
2. *If the critical values of F_λ lie in $T_\lambda \neq B_\lambda$, then the Julia set is a Cantor set of simple closed curves.*
3. *If the critical values of F_λ lie in any other preimage of T_λ, then the Julia set is a Sierpiński curve.*

Case 1 corresponds to λ in the Cantor set locus, that is, an open connected set that surrounds infinity in the $\lambda-$plane where the Julia set of F_λ is a Cantor set of points. In this case the Fatou set consists of one infinitely connected region, i.e., $B_\lambda = T_\lambda$. Case 2 corresponds to the *McMullen domain*, that is, a punctured (at the origin) open disk that is bounded by a simple closed curve in the $\lambda-$plane where all the maps F_λ have Julia sets that are Cantor sets of simple closed curves. Then the Fatou set consists of 2 simply connected domains (B_λ and T_λ) and infinitely many concentric annuli that are preimages of T_λ. Case 3 is very different; the parameter plane of the family F_λ shows infinitely many *Sierpiński holes*, that is, disjoint simply connected domains with parameters for which the Julia set of F_λ is a Sierpiński curve. Unlike cases 1 and 2, Sierpiński curve Julia sets happen also when the free critical points do not escape to infinity, see for example [23, 24, 25]. Figure 1 illustrates the Escape Trichotomy Theorem 1.1.

A *Sierpiński curve* is a planar set that is characterized by the following five properties: it is a compact, connected, locally connected and nowhere dense set whose complementary domains are bounded by simple closed curves that are pairwise disjoint. It is known that every Sierpiński curve is homeomorphic to the well-known Sierpiński carpet fractal, see [26].

When $n = d \geq 3$ there are $n - 1$ small copies of the Mandelbrot set symmetrically distributed around the origin $\lambda = 0$, see [1]. These Mandelbrot sets have halos that consists of Sierpiński holes attached to antennas emanating from the baby Mandelbrot sets. In this paper we extend this result to the case when n and d satisfy $1/n + 1/d < 1$ and n is not necessarily equal to d. Our goal then is to prove the following theorem.

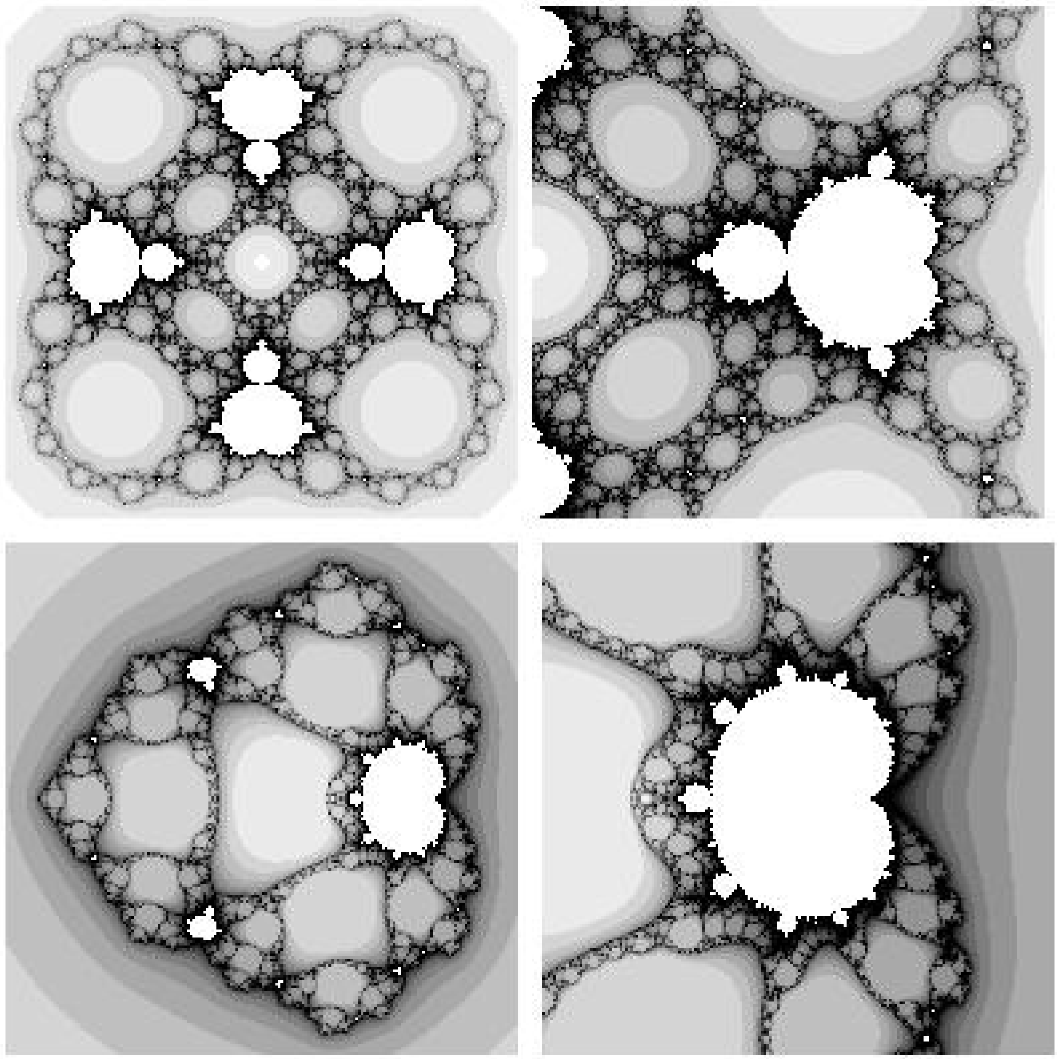

Figure 2: On the left we show the parameter planes for $< 5, 2 >$ (top) and $< 2, 7 >$ (bottom), and on the right we show magnifications around the baby Mandelbrot sets in the region W of Proposition 5.1. Notice the Sierpiński holes surrounding the Mandelbrot sets, these are the "halos" in Theorem 1.2. The maps with $n = 2$ show a smaller period-2 bulb for the baby Mandelbrot sets compared to the cases when $n \geq 3$. See also Figure 4.

Theorem 1.2 *(Generalized Principal baby Mandelbrot sets with halos) Fix $n, d \in \mathbb{N}$ with $1/n + 1/d < 1$, then there exists a small copy of the Mandelbrot set in parameter $\lambda-$plane for F_λ in each of the $n-1$ sectors of the form*

$$\frac{(2j-1)\pi}{n-1} < \text{Arg } \lambda < \frac{(2j+1)\pi}{n-1}, \qquad j = 0, 1, 2, ..., n-2.$$

Each of these baby Mandelbrot sets have infinitely many "halos" attached, i.e., infinitely many points on the boundary of Sierpinki holes.

The proof of Theorem 1.2 is similar to the one given in [1]. See Figures 1, 2 and 4.

2 Symmetries in dynamical plane

In this section we explain the symmetries that arise in the dynamical plane of the family F_λ. Let $\nu = e^{i\frac{2\pi}{m}}$ where $i = \sqrt{-1}$ be a primitive mth root of unity so that $\nu^m = \nu^{n+d} = 1$. It is easy to check that $F_\lambda(\nu z) = \nu^n F_\lambda(z)$ and then for all $k \in \mathbb{N}$,

$$F_\lambda^k(\nu z) = \nu^{n^k} F_\lambda^k(z). \tag{2.1}$$

Hence the orbits of points of the form $\nu^j z$ all behave "symmetrically" under iteration of F_λ. For example, if $F_\lambda^k(z) \to \infty$, then $F_\lambda^k(\nu^j z)$ also tends to ∞ for each j. If $F_\lambda^k(z)$ tends to an attracting cycle, then so does $F_\lambda^k(\nu^j z)$. Note, however, that the cycles involved may be different depending on j and, indeed, they may even have different periods. Nonetheless, all points lying on these attracting cycles are of the form $\nu^j z_0$ for some $z_0 \in \mathbb{C}$.

Let $\lambda = |\lambda| e^{i\psi}$, then the m "free" critical points c_j of F_λ are given by $c_j = c_\lambda \nu^j$ where $c_\lambda = \left(\frac{d}{n}|\lambda|\right)^{\frac{1}{m}} e^{i\frac{\psi}{m}}$ and $j = 0, 1, ..., m-1$. We see that the critical points all lie on a circle of radius $|c_\lambda|$, the *critical circle*, and are symmetrically distributed around the origin. Then Equation 2.1 implies that the critical orbits behave symmetrically as well. There are essentially three possibilities described in the next theorem.

Theorem 2.1 *(Generalized symmetries in dynamical plane) Let $n, d \in \mathbb{N}$ with $1/n + 1/d < 1$ and let $\nu = e^{i\frac{2\pi}{m}}$, then there exist $r, q \in \mathbb{N}$ such that for all $k \geq r$,*

$$F_\lambda^{q+k}(\nu z) = \nu^{n^k} F_\lambda^{q+k}(z). \tag{2.2}$$

Moreover,

(a) *If every prime factor of m is a prime factor of n, then there exists* $r \in \mathbb{N}$ *such that* $\forall k \geq r$ *the orbits of* νz *and* z *coincide, that is,* $F_\lambda^k(\nu z) = F_\lambda^k(z)$.

(b) *If m and n are relatively prime, then there exists* $q \in \mathbb{N}$ *such that* F_λ^q *is conjugate to itself under* $z \mapsto \nu z$*, that is,* $F_\lambda^q(\nu z) = \nu F_\lambda^q(z)$.

Let $gcd(m, n)$ be the greatest common divisor of m and n. Two integers m and n are said to be relatively prime if $gcd(m, n) = 1$. To prove Theorem 2.1 we need a classical property of the integers that we state here without proof as a proposition for the natural numbers, see Theorem 0.2 in page 5 of [30].

Proposition 2.2 *Let* $m, n \in \mathbb{N}$ *and* $gcd(m, n) = h$*. Then there exists* $a \in \mathbb{N}$ *and* $b \in \mathbb{Z}$ *with* $b \geq 0$ *such that* $an = bm + h$*. Moreover,* h *is the smallest integer of this form.*

Lemma 2.3 *Fix* $m, n \in \mathbb{N}$ *and for each* $k \in \mathbb{N}$ *let* $h_k = n^k (mod\ m)$ *so that* $h_k \in \{0, 1, 2, \ldots, m-1\}$*. Then there exist* $r, q \in \mathbb{N}$ *such that* $\forall k \geq r,\ h_{q+k} = h_k$*. Moreover,*

(a) $h_k = 0$ *for all* $k \geq r$ *if and only if every prime factor of m is a prime factor of n.*

(b) *If* $h_r \neq 0$*, then* $h_q = 1$ *if and only if m and n are relatively prime. In this case for all* $k \in \mathbb{N}, h_{q+k} = h_k$.

Proof of Lemma 2.3. Let $r, s \in \mathbb{N}$ with $s > r$ and such that $n^r = n^s \pmod{m}$ so that $h_r = h_s$. If we let $q = s - r$, then $h_{q+r} = h_s = h_r$. Also, $h_j h_k \pmod{m} = n^j n^k \pmod{m} = n^{j+k} \pmod{m} = h_{j+k}$. If $k \geq r$ then we can write $h_k = h_j h_r \pmod{m}$ for some $j \geq 0$, and then $h_{q+k} = h_q h_k \pmod{m} = h_q h_j h_r \pmod{m} = h_j h_{q+r} \pmod{m} = h_j h_r \pmod{m} = h_k$.

Since $h_{q+r} = h_q h_r \pmod{m} = h_r$ then $h_r = 0$, $h_q = 1$ or both. By definition $h_r = 0$ means $n^r = 0 \pmod{m}$ so that every prime factor of m is a prime factor of n and then part (a) follows. Instead, if $h_r \neq 0$ we see that, $h_q = 1$ means $n^q = 1 \pmod{m}$ and then it follows from Proposition 2.2 that m and n are relatively prime. Also, if $gcd(m, n) = 1$ then $gcd(m, n^q) = 1$ so that $h_q = 1$. Clearly, $\forall k \in \mathbb{N}$ we have $h_{q+k} = h_q h_k = h_k$ and the lemma follows. □

PROOF OF THEOREM 2.1. For each $k \in \mathbb{N}$ let $h_k = n^k \pmod{m}$. It follows from Lemma 2.3 that there exist $r, q \in \mathbb{N}$ such that $\forall k \geq r,\ h_{q+k} = h_k$, then $n^{q+k} = n^k \pmod{m}$ and $\nu^{n^{q+k}} = \nu^{n^k}$. Using (2.1) we see that $F_\lambda^{q+k}(\nu z) = \nu^{n^k} F_\lambda^{q+k}(z)$.

If every prime factor of m is a prime factor of n, it follows from Lemma 2.3 that there exists $r \in \mathbb{N}$ such that $\forall k \geq r,\ h_k = 0$ so that $n^k = 0 \pmod{m}$ and $\nu^{n^k} = 1$. Then $q = 1$, equation (2.2) reduces to $F_\lambda^k(\nu z) = F_\lambda^k(z)$ and part (a) follows.

For part (b) assume that m and n are relatively prime. Then from Lemma 2.3 there exists $q \in \mathbb{N}$ with $h_q = 1$ so that $n^q = 1 \pmod{m}$ and $\forall k \in \mathbb{N},\ h_{q+k} = h_k$. It follows that $\nu^{n^q} = \nu$ and (2.2) reduces to $F_\lambda^q(\nu z) = \nu F_\lambda^q(z)$, as we wanted to show. □

Using Proposition 2.2 it is easy to show that $gcd(m,n) = gcd(m,d) = gcd(n,d)$, and then a map $< n, d >$ is in one of the above cases in Theorem 2.1 if and only if the map $< d, n >$ belongs to that same case. Also, notice that there are $m - 3$ maps of each degree m. We now present several examples to illustrate Theorem 2.1.

When m and n are relatively prime, then one of the iterates of F_λ is conjugate to itself under $z \mapsto \nu z$. For example, consider the family $F_\lambda(z) = z^2 + \lambda/z^3$ where $n = 2, d = 3, m = 5$, and $gcd(5,2) = 1$. We get

$$F_\lambda(\nu z) = \nu^2 F_\lambda(z),$$

$$F_\lambda^2(\nu z) = \nu^4 F_\lambda^2(z),$$

$$F_\lambda^3(\nu z) = \nu^8 F_\lambda^3(z) = \nu^3 F_\lambda^3(z), \text{and}$$

$$F_\lambda^4(\nu z) = \nu^6 F_\lambda^4(z) = \nu F_\lambda^4(z).$$

Then $q = 4$ in part (b) of Theorem 2.1. The conjugacy of F_λ^4 with itself under $z \mapsto \nu z$ implies that the Julia set of F_λ^4, and therefore the Julia set of F_λ, is symmetric under rotation by $2\pi/5$. It is easy to check that $q = 4$ also for the maps $< 3, 7 >, < 5, 8 >$, and $< 5, 11 >$; $q = 3$ for $< 2, 5 >$ and $< 4, 5 >$; $q = 2$ for $< 3, 5 >$ and $< 4, 11 >$; $q = 5$ for $< 4, 7 >$ and $< 5, 6 >$; and $q = 16$ for $< 6, 11 >$.

Moreover, all maps with prime degree m correspond to part (b) as well as all maps of the form $< 2^k, 2j + 1 >, < 3^k, 3j + 1 >, < 3^k, 3j + 2 >$, in general $< a^k, aj + b >$ for all $a, b, k, j \in \mathbb{N}$, with $0 \leq b < a$, and many others. Actually, most maps correspond to part (b). See Figure 3.

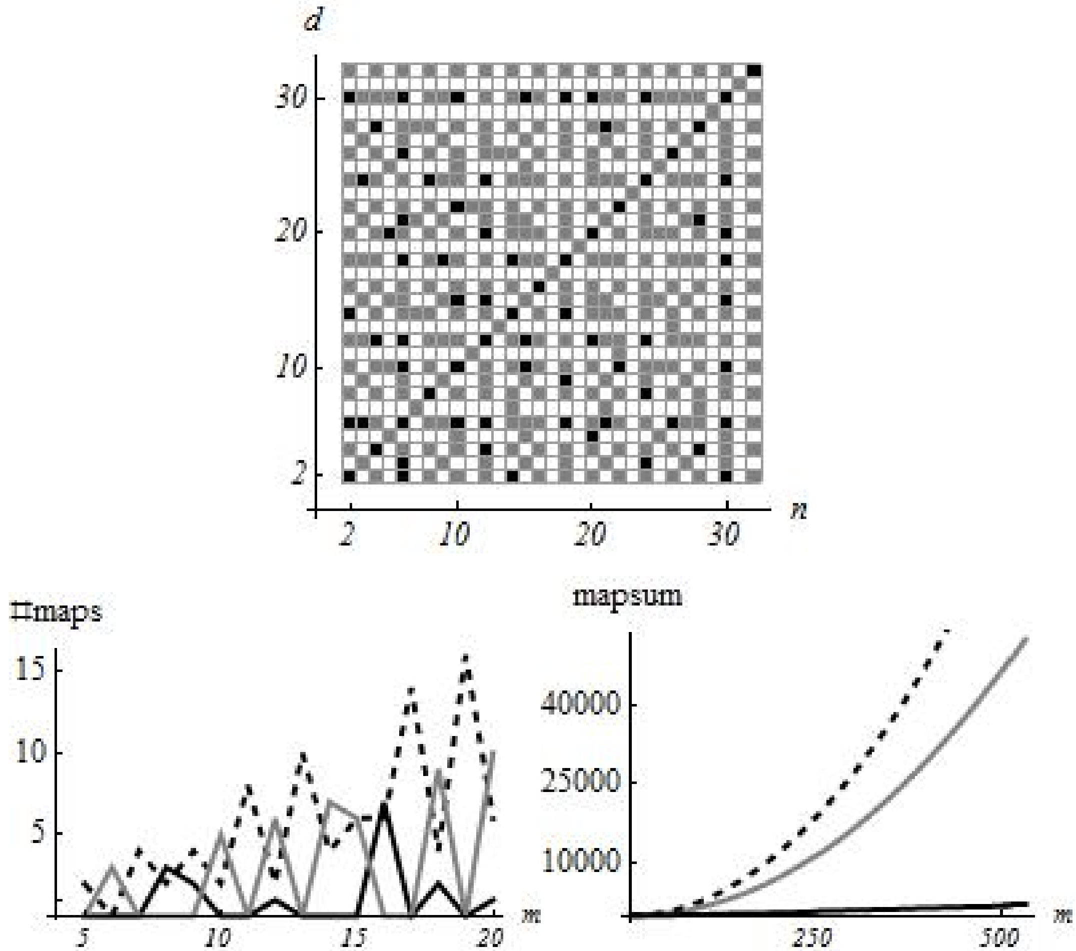

Figure 3: Top: The maps $< n, d >$ for $2 \leq n, d \leq 32$. The gray levels represent the three different cases in Theorem 2.1 namely, if m and n are relatively prime the square with coordinates $< n, d >$ is painted white; if every prime factor of m is a prime factor of n the square is painted black, and gray otherwise. In this area there are $31^2 = 961$ squares with 584 white, 72 black and 305 gray. Maps of the same degree lie along the diagonals with slope -1. For example, the diagonal from $< 30, 2 >$ to $< 2, 30 >$ contains 15 black maps and 14 white maps. There are no gray maps of degree 32. The same is true for all maps of degree $m = 2^k$, where $m/2 - 2$ maps are white and $m/2 - 1$ maps are black. Bottom: Graphs of the number of white (dashed), black and gray maps (left) and the cumulative number of white (dashed), black and gray maps (right) as a function of the degree m.

On the other hand, when every prime factor of m is a prime factor of n the orbits of the critical points collapse to one orbit. For example, consider the family $F_\lambda(z) = z^6 + \lambda/z^2$ where $n = 6, d = 2, m = 8$, and $gcd(8, 6) = 2$. We compute

$$F_\lambda(\nu z) = \nu^6 F_\lambda(z),$$
$$F_\lambda^2(\nu z) = \nu^{36} F_\lambda^2(z) = \nu^4 F_\lambda^2(z), \text{and}$$
$$F_\lambda^3(\nu z) = \nu^{24} F_\lambda^3(z) = F_\lambda^3(z).$$

Then $r = 3$ in part (a) of Theorem 2.1. Let $\lambda = |\lambda| e^{i\psi}$, then the critical points of F_λ are given by $c_j = c_\lambda \nu^j$ with $c_\lambda = (|\lambda|/3)^{\frac{1}{8}} e^{i\frac{\psi}{8}}$ and $j = 0, 1, ..., 7$. Then we see that there are 8 critical points mapped to 4 critical values $v_k = v_\lambda \nu^k$ where $v_\lambda = F_\lambda(c_\lambda) = 4(|\lambda|/3)^{\frac{3}{4}} e^{i\frac{3\psi}{4}}$ and $k = 6j \pmod 8$ so that $k = 0, 2, 4, 6$. These 4 critical values are mapped to 2 points $w_l = w_\lambda \nu^l$ with $w_\lambda = F_\lambda(v_\lambda)$ and $l = 6k \pmod 8$ so that $l = 0, 4$. Finally, since $6l = 0 \pmod 8$ we see that the two images of the critical values are mapped to a single point $x_\lambda = F_\lambda(\pm w_\lambda) = F_\lambda^2(v_\lambda)$.

For example, all maps where $n = d \geq 3$ studied in [1, 8, 10, 11] with n even correspond to part (a) of the theorem since $gcd(m, n) = gcd(2n, n) = n \geq 3$. A short computation shows that $r = 2$, so that $F_\lambda^2(\nu z) = F_\lambda^2(z)$. Therefore the points z and $\nu^j z$ for all $j \in \mathbb{N}$, and in particular the critical points, land on the same orbit after two iterations and so their orbits have the same eventual behavior.

The third case corresponds to $gcd(m, n) \neq 1$ and not every prime factor of m is a prime factor of n. In this case there is a partial collapse of orbits after a finite number of iterations and none of the iterates of the maps is conjugate to itself under $z \mapsto \nu z$, see (2.2). For example, all maps where $n = d \geq 3$ studied in [1, 8, 10, 11] with n odd correspond to this case since $gcd(m, n) = gcd(2n, n) = n \geq 3$. In this case, the orbits of $F_\lambda(z)$ and $F_\lambda(\nu^j z)$ are either the same or else they are the negatives of each other after the first iteration. In either case it follows that the orbits of $\nu^j z$ behave symmetrically and each of the free critical points eventually maps onto one of two symmetric orbits.

To further illustrate this case we focus on the maps with degree $m = 12$. This is the smallest value of m that shows maps of all three types in Theorem 2.1. The map $< 6, 6 >$ corresponds to part (a) with $r = 2$, the maps $< 5, 7 >$ and $< 7, 5 >$ correspond to part (b) with $q = 2$ and the other 6 maps of degree 12 correspond to the third case. For the maps $< 2, 10 >$

$, <4,8>, <8,4>$ and $<10,2>$, the critical points behave in 3 groups of 4 critical points each. For $<2,10>$ and $<8,4>$ if c_λ is eventually fixed then a total of 4 critical points will be fixed and the other 8 will be in a two cycle. Instead, for $<10,2>$ and $<4,8>$ if c_λ is eventually fixed then all the critical points will be fixed and distributed in groups of 4 critical points per fixed point. For the maps $<3,9>$ and $<9,3>$ the critical points divide in 4 groups of 3. For $<3,9>$ if c_λ is eventually fixed then a total of 6 critical points will be fixed and distributed among two fixed points, and the other 6 critical points land in a two cycle. Finally, for $<9,3>$ if c_λ is eventually fixed then all the critical points will be fixed and distributed among 4 fixed points. We see that in every case these cycles and fixed points are located at the vertices of regular polygons with a number of sides that is a factor of 8 and the Julia set of F_λ is symmetric under rotation by $\pi/4$.

In general, the orbits of the critical points are located at the vertices of regular polygons with a number of sides that is a factor of m and the Julia set of F_λ is symmetric under rotation by $2\pi/m$.

3 Symmetries in parameter plane

The parameter plane also possesses several symmetries, see Figure 4. First of all, we have $\overline{F_\lambda(z)} = F_{\overline{\lambda}}(\overline{z})$ so that F_λ and $F_{\overline{\lambda}}$ are conjugate via the map $z \mapsto \overline{z}$. Therefore the parameter plane is symmetric under the map $\lambda \mapsto \overline{\lambda}$.

We also have $(n-1)$-fold symmetry in the parameter plane for F_λ.

Theorem 3.1 *(Symmetries in parameter $\lambda-$plane) Let $n, d \in \mathbb{N}$ with $1/n + 1/d < 1$ and let $m = n + d$. Let $\omega = e^{i\frac{2\pi}{n-1}}$. Then there are $k, p \in \mathbb{N}$ such that F_λ^p is conjugate to $F_{\omega\lambda}^p$ under $z \mapsto \omega^{\frac{k}{p}} z$, that is,*

$$F_{\omega\lambda}^p(\omega^{\frac{k}{p}} z) = \omega^{\frac{k}{p}} F_\lambda^p(z).$$

It follows that the parameter λ-plane is symmetric under the map $\lambda \mapsto \omega\lambda$.

Proof. Let $m = ps$ with $gcd(s, n-1) = 1$. We will chose $p = gcd(m, n-1)$ unless the condition on s is not satisfied, in this case we will chose $p = m$ and then $s = 1$. In any case, it follows from Proposition 2.2 that there exist $k \in \mathbb{N}$ and $b \in \mathbb{Z}$ with $b \geq 0$ such that $ks = b(n-1) + 1$. Since $\omega^{n-1} = 1$, we have $\omega^{ks} = \omega^{\frac{k}{p}m} = \omega$ and $\omega / \omega^{\frac{k}{p}d} = \omega^{\frac{k}{p}m} / \omega^{\frac{k}{p}d} = \omega^{\frac{k}{p}n}$. We get $F_{\omega\lambda}(\omega^{\frac{k}{p}}z) = \omega^{\frac{k}{p}n}F_\lambda(z)$, and for all $j \in \mathbb{N}$, then $F^j_{\omega\lambda}(\omega^{\frac{k}{p}}z) = \omega^{\frac{k}{p}n^j}F^j_\lambda(z)$. Consider the expansion,

$$\frac{k}{p}n^p = \frac{k}{p}(n-1+1)^p = \frac{k}{p} + k\sum_{i=1}^{p}\frac{B^i_p}{p}(n-1)^i,$$

where

$$B^i_p = \frac{p!}{i!(p-i)!} \in \mathbb{N},$$

are the binomial coefficients with $1 \le i \le p$. We get $B^1_p = p$ and $(n-1)^i$ is divisible by p for all i. It follows that there exists $a \in \mathbb{N}$ such that

$$\frac{k}{p}n^p = \frac{k}{p} + (n-1)a.$$

Then $\omega^{\frac{k}{p}} = \omega^{\frac{k}{p}n^p}$ and $F^p_{\omega\lambda}(\omega^{\frac{k}{p}}z) = \omega^{\frac{k}{p}}F^p_\lambda(z)$, as we wanted to show. □

The cases $n = d \ge 3$ can be derived from the theorem. Since $m = 2n$, when n is even $gcd(n, n-1) = 1$ and $gcd(2, n-1) = 1$, so that $h = gcd(m, n-1) = 1$. We can use $k = 1, p = h = 1$ and $\omega = \omega^n$ to see that $F_{\omega\lambda}(\omega z) = \omega F_\lambda(z)$. Since $(n/2)n = n/2 + (n/2)(n-1)$ and n is even, we see that we can also choose $k = n/2$ to obtain the conjugacy under the map $z \mapsto \omega^{\frac{n}{2}}z$, see [10, 1, 8, 11].

When n is odd $h = gcd(m, n-1) = 2$ and with $k = n$ and $p = h = 2$ we can write

$$\frac{n}{2}n^2 = \frac{n}{2} + \frac{n}{2}(n+1)(n-1)$$

and since $n+1$ is even we get $\omega^{\frac{n}{2}} = \omega^{\frac{n}{2}n^2}$ and then $F^2_{\omega\lambda}(\omega^{\frac{n}{2}}z) = \omega^{\frac{n}{2}}F^2_\lambda(z)$, as in [10, 1, 8, 11].

For example, when $m = 10$ with $n = 6$ and $d = 4$. Then $h = gcd(10, 5) = 5$ then with $p = h = 5$, and $s = m/h = 2$ and we can write $2k = b(n-1)+1$. For $b = 1$ we get $k = 3$, it follows that $F^5_{\omega\lambda}(\omega^{\frac{3}{5}}z) = \omega^{\frac{3}{5}}F^5_\lambda(z)$.

Finally, consider the case $m = 9$ with $n = 7$ and $d = 2$ so that $h = gcd(9, 6) = 3$. Let $p = 9$ and $s = 1$ then since $gcd(1, 6) = 1$ we can write $k = 6b+1$. For $b = 0$ we get $k = 1$, it follows that $F^9_{\omega\lambda}(\omega^{\frac{1}{9}}z) = \omega^{\frac{1}{9}}F^9_\lambda(z)$.

In this paper we show the existence of one baby Mandelbrot set in the sector of the parameter plane given by

$$\frac{-\pi}{n-1} < \operatorname{Arg}\lambda < \frac{\pi}{n-1}.$$

and then invoke the $(n-1)$-fold symmetry of the parameter plane to prove the existence of the other $n-2$ babies.

Notice that the cases with $n=2$ and $d>2$ show only one principal baby Mandelbrot set straddling the real axis. These babies are slightly different from the ones in the rest of the family in the sense that the tail of these Mandelbrot sets and the period-2 bulbs are smaller. When $n=d=2$, a map that is not in our family, the tail seems to completely disappear. See Figures 2 and 4.

4 The Connectedness Locus

Let $\mathcal{C}$ denote the *connectedness locus*, that is, the set of λ-values for which the Julia set of F_λ is a connected set. The complement of this set is the *Cantor set locus* that we denote by $\mathcal{B}$. Let $\mathcal{M}$ denote the *McMullen domain*, that is, the set of $\lambda-$values for which the Julia set is a Cantor set of simple closed curves. The next results bound the regions $\mathcal{C}, \mathcal{B}$ and $\mathcal{M}$ in the $\lambda-$plane. Fix $n, d \in \mathbb{N}$ with $1/n + 1/d < 1$.

Proposition 4.1 *(Cantor set locus bound) Suppose that* $|\lambda| \geq \frac{n}{d}(\frac{2d}{m})^{\frac{m}{n}}$. *Then* $v_\lambda \in B_\lambda$ *so that* $\lambda \in \mathcal{B}$.

PROOF. Let $|\lambda| \geq \frac{n}{d}\left(\frac{2d}{m}\right)^{\frac{m}{n}}$. Suppose that $|z| \geq |v_\lambda| = \left(\frac{m}{d}\right)\left(\frac{d}{n}|\lambda|\right)^{\frac{n}{m}}$. Then $|z| \geq |v_\lambda| \geq 2$ and since $\frac{n}{d}\left(\frac{d}{m}\right)^{\frac{m}{n}} = \frac{n}{m}\left(\frac{d}{m}\right)^{\frac{d}{n}} < 1$ we have

$$\begin{aligned} |F_\lambda(z)| &\geq |z|^n - \frac{|\lambda|}{|z|^d} \\ &\geq |z|^n - \frac{\frac{n}{d}\left(\frac{d}{m}|z|\right)^{\frac{m}{n}}}{|z|^d} \\ &> |z|^n - \frac{1}{|z|^{d-\frac{m}{n}}} \\ &> |z|^{n-1}. \end{aligned}$$

Inductively, it follows that $|F_\lambda^k(z)| > |z|^{(n-1)^k}$ so that $z \in B_\lambda$. In particular, $v_\lambda \in B_\lambda$ and then $\lambda \in \mathcal{B}$ as we wanted to show. □

Proposition 4.2 *(General Escape Criterion) Suppose that* $|\lambda| \leq \frac{n}{d}\left(\frac{2d}{m}\right)^{\frac{m}{n}}$. *If* $|z| \geq 2$ *then* $z \in B_\lambda$. *If* $|z| \leq \frac{1}{2}\left(\frac{d}{n}|\lambda|\right)^{\frac{1}{d}}$ *then* $|F_\lambda(z)| \geq 2$, *so that* $F_\lambda(z) \in B_\lambda$.

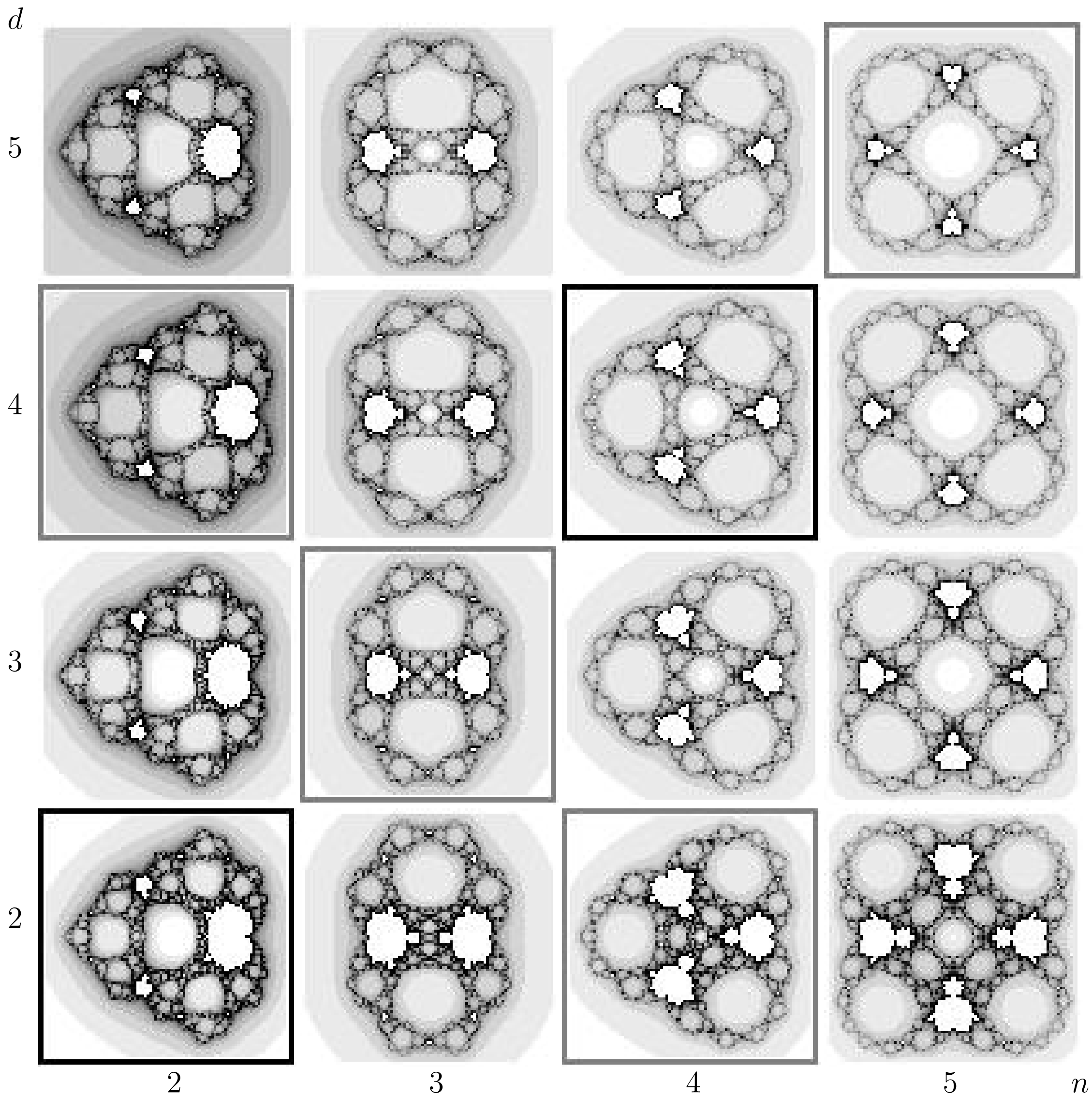

Figure 4: The parameter $\lambda-$planes for maps with $2 \leq n, d \leq 5$. The horizontal axis is n and the vertical axis d. Maps of the same degree are located along the diagonals with slope -1. Notice the $n-1$ white baby Mandelbrot sets symmetrically distributed around the origin $\lambda = 0$. The parameter planes with a black frame correspond to a map where every prime factor of m is a prime factor of n. The gray frame shows the cases where some but not all prime factors of m are prime factors of n. The rest of the pictures show cases in which m and n are relatively prime.

Proof. For the first part let $|z| \geq 2$. Notice that

$$|\lambda| \leq \frac{n}{d}\left(\frac{2d}{m}\right)^{\frac{m}{n}} = 2^{\frac{m}{n}}\left(\frac{n}{m}\right)\left(\frac{d}{m}\right)^{\frac{d}{n}} < 2^{\frac{m}{n}}.$$

We have

$$\begin{aligned} |F_\lambda(z)| &\geq |z|^n - \frac{|\lambda|}{|z|^d} \\ &> |z|^n - \frac{2^{\frac{m}{n}}}{|z|^d} \\ &> |z|^{n-1}. \end{aligned}$$

Then $|F_\lambda^k(z)| > |z|^{(n-1)^k}$ and $z \in B_\lambda$. The second part differs from the proof in [1] because there is no involution preserving the Julia set in the general case. Let $|\lambda| \leq \frac{n}{d}\left(\frac{2d}{m}\right)^{\frac{m}{n}}$ and suppose that $|z| \leq \frac{1}{2}\left(\frac{d}{n}|\lambda|\right)^{\frac{1}{d}}$. Then we have

$$\begin{aligned} |F_\lambda(z)| &\geq \frac{|\lambda|}{|z|^d} - |z|^n \\ &\geq \frac{n}{d}2^d - \frac{\left(\frac{d}{n}|\lambda|\right)^{\frac{n}{d}}}{2^n} \\ &\geq n2^{d-1} - \frac{\left(\frac{2d}{m}\right)^{\frac{m}{d}}}{2^n} \\ &> 2n - \frac{1}{2^{n-\frac{m}{d}}} > 2. \end{aligned}$$

It follows from the first part that $F_\lambda(z) \in B_\lambda$. ☐

Proposition 4.3 *(McMullen domain bound) If* $|\lambda| \leq \frac{n}{d}\left(\frac{d}{2m}\right)^{\frac{md}{nd-m}}$ *then* $v_\lambda \in T_\lambda$ *and then* $\lambda \in \mathcal{M}$. *Moreover, the disk of radius* $\frac{1}{2}\left(\frac{d}{n}|\lambda|\right)^{\frac{1}{d}}$ *is contained in* T_λ.

Proof. Since $md/(nd-m) > m/n$ and $d/m < 1$ we have that

$$|\lambda| \leq \frac{n}{d}\left(\frac{d}{2m}\right)^{\frac{md}{nd-m}} < \frac{n}{d}\left(\frac{d}{2m}\right)^{\frac{m}{n}} < \frac{n}{d}\left(\frac{2d}{m}\right)^{\frac{m}{n}}.$$

On the other hand, since $|\lambda| \leq \frac{n}{d}\left(\frac{d}{2m}\right)^{\frac{md}{nd-m}}$ we get

$$\frac{m}{d}\left(\frac{d|\lambda|}{n}\right)^{\frac{n}{m}} \cdot \left(\frac{d|\lambda|}{n}\right)^{-\frac{1}{d}} \leq \frac{1}{2}.$$

Therefore

$$|v_\lambda| = \frac{m}{d}\left(\frac{d|\lambda|}{n}\right)^{\frac{n}{m}} \leq \frac{1}{2}\left(\frac{d|\lambda|}{n}\right)^{\frac{1}{d}}. \qquad (4.1)$$

It follows from Proposition 4.2 that $F_\lambda(v_\lambda) \in B_\lambda$. It remains to show that T_λ is disjoint from B_λ and $v_\lambda \in T_\lambda$. Consider the ratio between $|v_\lambda|$ and $|c_\lambda| = \left(\frac{d}{n}|\lambda|\right)^{\frac{1}{m}}$. Using the inequality 4.1 and the bound for $|\lambda|$ we get

$$\frac{|v_\lambda|}{|c_\lambda|} \leq \frac{1}{2}\left(\frac{d|\lambda|}{n}\right)^{\frac{1}{d}-\frac{1}{m}} \leq \frac{1}{2}\left(\frac{d}{2m}\right)^{\frac{n}{nd-m}} < 1.$$

Then the image of the critical circle lies strictly inside itself. Hence we may choose δ slightly greater than $|c_\lambda|$ so that the circle of radius δ about the origin is also mapped strictly inside itself. Now consider the annular region A given by $\delta \leq |z| \leq 2$. The boundaries of A are mapped strictly outside of A and there are no critical points in A. Hence F_λ is a covering map of A onto its image. By the Riemann-Hurwitz Theorem, it follows that $F_\lambda^{-1}(A) \cap A$ is a subannulus of A that is mapped onto A. Then A contains a closed invariant set that surrounds the origin. Therefore B_λ cannot meet the inner boundary of A and in particular, B_λ cannot meet the disk of radius $\frac{1}{2}\left(\frac{d}{n}|\lambda|\right)^{\frac{1}{d}}$. Thus v_λ must lie in T_λ and $\lambda \in \mathcal{M}$, as we wanted to show. □

5 Baby Mandelbrot sets

In this section we prove the existence of $n-1$ baby mandelbrot sets in the connectedness locus $\mathcal{C}$ of the family $F_\lambda(z)$. We first recall the Douady-Hubbard theory of polynomial-like maps. See [27] for more details. Suppose $U' \subset U$ are a pair of bounded, open, simply connected subsets of $\mathbb{C}$ with U' relatively compact in U. A map $G : U' \to U$ is called a $polynomial-like$ map of degree two if G is analytic and proper of degree two. Hence such a map has a unique critical point $c \in U'$. The filled Julia set of G is defined in the natural manner as the set of points whose orbits never leave the subset U' under iteration of G. By the results in [27] it is known that G is topologically conjugate to a quadratic polynomial on a neighborhood of the polynomial's filled Julia set in $\mathbb{C}$, hence the name polynomial-like.

Now suppose that we have a family of polynomial-like maps $G : U'_\lambda \to U_\lambda$ depending on a parameter λ and satisfying:

(1) The parameter λ lies in an open set in $\mathbb{C}$ that contains a closed disk W, and the boundaries of U'_λ and U_λ vary analytically as λ varies;

(2) The map $(\lambda, z) \to G_\lambda(z)$ depends holomorphically on both λ and z;

(3) Each $G : U'_\lambda \to U_\lambda$ is polynomial-like of degree two.

Then we may consider the set of parameters in W for which the orbit of the critical point, c_λ, does not escape from U'_λ and so the corresponding filled Julia set is connected. Suppose that for each λ on the boundary of W we have that $G_\lambda(c_\lambda)$ lies in $U_\lambda - U'_\lambda$ and that, moreover, $G_\lambda(c_\lambda) - c_\lambda$ winds once around 0 as λ winds once around the boundary of W. Then, in this case, Douady and Hubbard also prove [27] that the set of λ-values for which the orbit of c_λ does not escape from U'_λ is holomorphic to the Mandelbrot set and that the polynomial to which G_λ corresponds under this homeomorphism is conjugate to G_λ on some neighborhood of its Julia set. This result thus gives a criterion for proving the existence of small copies of the Mandelbrot set.

We first define W to be the set of λ-values in the right half plane enclosed by arcs of the circles given by

$$|\lambda| = \frac{n}{d}\left(\frac{d}{2m}\right)^{\frac{md}{nd-m}} \quad \text{and} \quad |\lambda| = \frac{n}{d}\left(\frac{2d}{m}\right)^{\frac{m}{n}}$$

and by portions of the rays

$$\operatorname{Arg}\lambda = \pm\frac{\pi}{n-1}.$$

Later we use the symmetry in the system to consider parameter values drawn from rotationally symmetric sectors. Now let μ and Γ be the two circles in the $z-$plane given by

$$\mu: \ |z| = \frac{1}{2}\left(\frac{d|\lambda|}{n}\right)^{\frac{1}{d}} \quad \text{and} \quad \Gamma: \ |z| = 2.$$

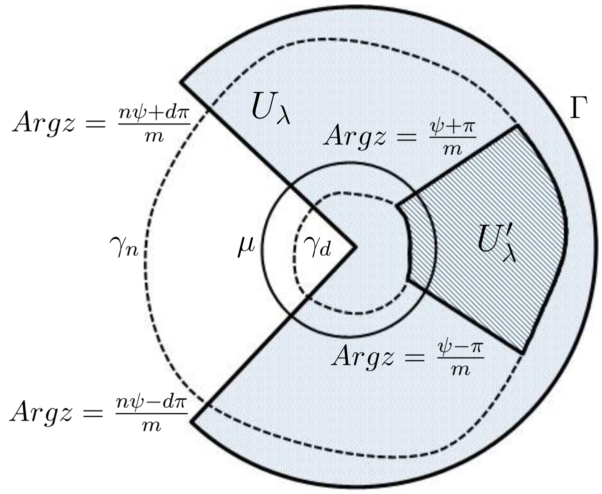

Figure 5: The sets U'_λ and U_λ where $U'_\lambda \subset F_\lambda(U'_\lambda) = U_\lambda$. The function F_λ is a polynomial like family on U'_λ. The circles $\Gamma \in B_\lambda$ and $\mu \in T_\lambda$. The curve $\gamma_d \in T_\lambda$ is mapped $d-$to-1 onto Γ and the curve $\gamma_n \in B_\lambda$ is mapped $n-$to-1 onto Γ.

The circle μ lies in T_λ and the circle Γ in B_λ. By Propositions 4.2 and 4.3 we know that when $\lambda \in W$, the disk bounded by μ and the region outside Γ are both mapped outside of Γ. Let $\gamma_d = \gamma_d(\lambda)$ be the preimage of Γ that lies inside μ in T_λ, and let $\gamma_n = \gamma_n(\lambda)$ be the preimage of Γ that lies inside Γ in B_λ. That is $F_\lambda(\gamma_d) = F_\lambda(\gamma_n) = \Gamma$. Notice that the disk bounded by γ_d is mapped d-to-1 onto the exterior of Γ and the region exterior of γ_n is mapped n-to-1 onto the exterior of Γ. See Figure 5.

For each $\lambda \in W$ let $\psi = \operatorname{Arg}\lambda$. We define the sector U'_λ to be points in the open region bounded by arcs of the two simple closed curves γ_d and γ_n and portions the rays

$$\operatorname{Arg} z = \frac{\psi \pm \pi}{m}.$$

A straightforward computation shows that there is a unique critical point lying in U'_λ given by $c_\lambda = \left(\frac{d}{n}|\lambda|\right)^{\frac{1}{m}} e^{i\frac{\psi}{m}}$ and the straight boundaries of U'_λ each contain a prepole p_λ, i.e., a preimage of 0, such that $p_\lambda = (-\lambda)^{\frac{1}{m}} = |\lambda|^{\frac{1}{m}} e^{i\frac{\psi \pm \pi}{m}}$.

Proposition 5.1 *The family of maps F_λ defined on U'_λ with $\lambda \in W$ is a polynomial like family of degree 2.*

Proof. Let $U_\lambda = F_\lambda(U'_\lambda)$. It follows that for each $\lambda \in W$, the two curve boundaries of U'_λ, that is, γ_d and γ_n are both mapped to the same circle Γ

in B_λ. Now consider the images of the straight line boundaries of U'_λ. Let $z = re^{i\frac{\psi \pm \pi}{m}}$, then

$$F_\lambda(z) = \frac{r^m e^{i(\psi\pm\pi)} + |\lambda| e^{i\psi}}{r^d e^{i\frac{d(\psi\pm\pi)}{m}}} = \left(\frac{|\lambda| - r^m}{r^d}\right) e^{i\frac{n\psi \mp d\pi}{m}}.$$

This means that the straight line boundaries of U'_λ are mapped onto the straight lines with argument $(n\psi \mp d\pi)/m$ that pass through the origin. Since

$$-\frac{\pi}{n-1} < \psi < \frac{\pi}{n-1}$$

and

$$(1-n)\psi < (n-1)\psi < (d-1)\pi,$$

we get

$$-\pi < \frac{n\psi - d\pi}{m} < \frac{\psi - \pi}{m} < \frac{\psi + \pi}{m} < \frac{n\psi + d\pi}{m} < \pi.$$

Therefore the image of the straight line boundaries of U'_λ also lie outside U'_λ and F_λ maps these boundaries to the images in a 2-to-1 fashion. It follows that $F_\lambda(U'_\lambda)$ contains U'_λ in its interior and $F_\lambda : U'_\lambda \to U_\lambda$ is a polynomial-like family of degree 2. □

We now prove the first part of Theorem 1.2 that we state here as a lemma for reference.

Lemma 5.2 *There exists a small copy of the Mandelbrot set in parameter $\lambda-$plane for F_λ in each of the $n-1$ sectors of the form*

$$\frac{(2j-1)\pi}{n-1} < \text{Arg } \lambda < \frac{(2j+1)\pi}{n-1}, \qquad j = 0, 1, 2, ..., n-2.$$

Proof. We first deal with λ-values in the region W defined above, so that $j = 0$ and

$$\frac{-\pi}{n-1} < \text{Arg}\,\lambda < \frac{\pi}{n-1}.$$

We consider the location of the critical value and the critical point for λ in each of the four different boundary curves of W. We must show that the critical value winds once around the exterior of U'_λ as λ winds once around the boundary of W. Suppose first that λ lies on the outer circular boundary of W, so that

$$|\lambda| = \frac{n}{d}\left(\frac{2d}{m}\right)^{\frac{m}{n}}$$

and then $|v_\lambda| = 2$, so v_λ lies outside U'_λ. If λ lies on the inner circular boundary of W we have

$$|\lambda| = \frac{n}{d}\left(\frac{d}{2m}\right)^{\frac{md}{nd-m}} \quad \text{and then} \quad |v_\lambda| = \frac{1}{2}\left(\frac{d|\lambda|}{n}\right)^{\frac{1}{d}}.$$

Then again v_λ lies outside the domain U'_λ. Now suppose that λ lies on the upper straight line boundary of W so that $\operatorname{Arg}\lambda = \psi = \pi/(n-1)$. Then for z in the upper straight boundary of U'_λ we have

$$\operatorname{Arg} z = \frac{\psi + \pi}{m} = \frac{n\pi}{m(n-1)} = \operatorname{Arg} v_\lambda.$$

Then again the result holds. The lower boundary of W is handled analogously. We see that $F_\lambda(c_\lambda) - c_\lambda$ winds once around the outside of U'_λ as λ winds once around the boundary of W. We conclude that there is a small copy of the Mandelbrot set inside the sector

$$\frac{-\pi}{n-1} < \operatorname{Arg}\lambda < \frac{\pi}{n-1}.$$

To find the Mandelbrot sets in the other $n-2$ symmetrically arranged sectors in the parameter plane we invoke the $(n-1)$-fold symmetry represented by the map $\lambda \to \omega\lambda$ in the parameter plane through a conjugacy between F_λ^p and $F_{\omega\lambda}^p$ under $z \to \omega^{\frac{k}{p}} z$, for some $k, p \in \mathbb{N}$, see Theorem 3.1. If a critical point of F_λ^p is eventually fixed or periodic, or tends to a periodic orbit, then there is a corresponding critical point of $F_{\omega\lambda}^p$ that has exactly the same eventual behavior after the same number of iterations and with the same periods. This concludes the proof of the lemma. □

The proof of the existence of the halos attached to the baby Mandelbrot sets is identical to the one in [1] and we omit it. See Figure 2. This finishes the proof of Theorem 1.2.

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