Singular Perturbations of $z^n$

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February 22, 2006

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1 Introduction

Our goal in this paper is to give an overview of some of the many recent results dealing with the topology of and dynamics on certain Julia sets of functions drawn from the family of rational maps of the complex plane given by

\[ F_\lambda(z) = z^n + \frac{\lambda}{z^d} \]

where \( n, d \in \mathbb{Z}^+ \). While many of these results have appeared elsewhere, some of the results described below are new.

When \( \lambda = 0 \), these maps reduce to \( z \mapsto z^n \) and the dynamical behavior in this case is well understood: the Julia set of \( F_\lambda \) is just the unit circle and all other orbits tend either to \( \infty \) or to the superattracting fixed point at 0.

When \( \lambda \neq 0 \), several things happen. First of all, the map \( F_\lambda \) now has degree \( n + d \) rather than \( n \). Secondly, the origin is a pole rather than a fixed point. And, finally, there are \( n + d \) new critical points in addition to the original critical points at 0 and \( \infty \). As we discuss below, the orbits of all of these new critical points behave symmetrically, so we essentially have only one additional “free” critical orbit for each of these maps. As is well known in complex dynamics, the behavior of this critical orbit determines much of the structure of the Julia sets of these maps.

One of our main goals in this paper is to describe what happens to the Julia set when the parameter \( \lambda \) is nonzero but small. In this case, the map \( F_\lambda \) is called a singular perturbation of \( z^n \). The reason for the interest in such a perturbation arises from Newton’s method. Suppose we are applying Newton’s method to find the roots of a family of polynomials \( P_\lambda \) which has a multiple root at, say, the parameter \( \lambda = 0 \). For example, consider the especially simple case of \( P_\lambda(z) = z^2 + \lambda \). When \( \lambda = 0 \) this polynomial has a multiple root at 0 and the Newton iteration function is simply \( N_\lambda(z) = z/2 \). However, when \( \lambda \neq 0 \), the Newton iteration function becomes

\[ N_\lambda(z) = \frac{z^2 - \lambda}{2z} \]

and we see that, as in the family \( F_\lambda \), the degree jumps as we move away from \( \lambda = 0 \). In addition, instead of a fixed point at the origin, after the perturbation, there is a pole at the origin.

For the families \( F_\lambda \), there are a number of different cases to consider depending on the values of \( n \) and \( d \). When \( n \geq 2 \), the point at \( \infty \) is a
superattracting fixed point whereas when $n = 1$ this point is a parabolic fixed point. Since much of the interesting dynamical behavior occurs when the free critical points tend to or land at $\infty$, the singular perturbations therefore behave very differently in these two cases.

One of the main results that we describe below is the following. When $n \geq 2$, we have an immediate basin of attraction $B_\lambda$ of the superattracting fixed point at $\infty$. Note that $F_\lambda$ is $n$ to 1 on a neighborhood of $\infty$ in $B_\lambda$. Since 0 is a pole of order $d$, the only other preimages of points in $B_\lambda$ lie in a neighborhood of the origin. We let $T_\lambda$ be the preimage of $B_\lambda$ surrounding the origin. (The sets $B_\lambda$ and $T_\lambda$ may or may not be disjoint.) As we shall show, for $\lambda$ small, it is possible that the critical orbits eventually land in $B_\lambda$ and hence tend to $\infty$. In this case, we have the following result described in Section 3.

**Theorem (The Escape Trichotomy).** Suppose $n \geq 2$ and that the orbits of the free critical points of $F_\lambda$ tend to $\infty$. Then

1. If one of the critical values lies in $B_\lambda$, then $J(F_\lambda)$ is a Cantor set and $F_\lambda | J(F_\lambda)$ is a one-sided shift on $n + d$ symbols. Otherwise, the preimage $T_\lambda$ is disjoint from $B_\lambda$.

2. If one of the critical values lies in $T_\lambda \neq B_\lambda$, then $J(F_\lambda)$ is a Cantor set of simple closed curves (quasicircles).

3. If one of the critical values lies in a preimage of $B_\lambda$ different from $T_\lambda$, then $J(F_\lambda)$ is a Sierpinski curve.

Several Julia sets illustrating this trichotomy and drawn from the family where $n = d = 3$ are included in Figure 1.

A **Sierpinski curve** is a very interesting topological space. By definition, a Sierpinski curve is a planar set that is homeomorphic to the well-known Sierpinski carpet fractal. But a Sierpinski curve has an alternative topological characterization: any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by disjoint simple closed curves is known to be homeomorphic to the Sierpinski carpet [24]. Moreover, such a set is a universal planar set in the sense that it contains a homeomorphic copy of any compact, connected, one-dimensional subset of the plane.
Figure 1: Some Julia sets for $z^3 + \lambda/z^3$: if $\lambda = 0.23$, $J(F_\lambda)$ is a Cantor set; if $\lambda = 0.0006$, $J(F_\lambda)$ is a Cantor set of circles; and if $\lambda = 0.125i$, $J(F_\lambda)$ is a Sierpinski curve.
When \( n \geq 2 \), there are certain cases of this Theorem that may or may not hold, depending on the value of \( d \). For example, if \( n \) and \( d \) satisfy
\[
\frac{1}{n} + \frac{1}{d} < 1,
\]
then there is a neighborhood of \( \lambda = 0 \) for which the critical values all lie in \( T_\lambda \neq B_\lambda \) and so the Julia set is a Cantor set of simple closed curves. This phenomenon was first observed by McMullen for small \( \lambda \) (see [14]) and so we call the regime in the \( \lambda \)-plane where this occurs the *McMullen domain*. There is no McMullen domain if this inequality does not hold, i.e., if \( n \) or \( d \) is equal to 1 or if \( n = d = 2 \). Instead, in the special cases where \( n = d = 2 \) or \( n > 1, d = 1 \), we have the following result which is described in Section 4:

**Theorem.** Suppose \( n = d = 2 \) or \( n > 1, d = 1 \). Then, in every neighborhood of the origin in the parameter plane, there are infinitely many disjoint open sets \( \mathcal{O}_j \) with \( j = 1, 2, 3, \ldots \) of parameters having the following properties:

1. If \( \lambda \in \mathcal{O}_j \), then the Julia set of \( F_\lambda \) is a Sierpinski curve, so that if \( \lambda \in \mathcal{O}_j \) and \( \mu \in \mathcal{O}_k \), the Julia sets of \( F_\lambda \) and \( F_\mu \) are homeomorphic;

2. But if \( k \neq j \), the maps \( F_\lambda \) and \( F_\mu \) are not topologically conjugate on their respective Julia sets.

The case where \( n = 1 \) is fundamentally different from the other cases since the function
\[
F_\lambda(z) = z + \frac{\lambda}{z^d}
\]
has a parabolic fixed point at \( \infty \). Furthermore, any map of this form is linearly conjugate to the case where \( \lambda = 1 \). So, instead of considering this case, we adjust the family slightly to deal instead with the family
\[
F_\lambda(z) = \lambda \left( z + \frac{1}{z^d} \right)
\]
when \( n = 1 \). For this family \( \infty \) is an attracting fixed point when \( |\lambda| > 1 \) and is repelling when \( |\lambda| < 1 \). So when \( |\lambda| < 1 \), \( \infty \) is in the Julia set, and we may have that the critical orbits map onto \( \infty \). In this case, the Julia set is the entire Riemann sphere. Much else occurs near \( \lambda = 0 \), for, as we show in Section 5, we have:

**Theorem.** Let \( F_\lambda(z) = \lambda(z + 1/z) \). Then, in any neighborhood of \( \lambda = 0 \) in the parameter plane:
1. There are infinitely many parameter values $\lambda$ for which the Julia set of $F_\lambda$ is the entire Riemann sphere;

2. There are also infinitely many parameter values for which the critical orbit is superattracting.

Unlike the situation that is described in the previous two theorems for $\lambda$ near 0, in the case where we have a McMullen domain, the dynamical behavior of $F_\lambda$ is the same for any $\lambda$ sufficiently close to 0. However, away from this region, $F_\lambda$ exhibits a rich array of different dynamical behavior. For example, in Section 6 we show that there are many different ways that the Julia sets may be Sierpinski curves. In the previous Sierpinski curve examples, the complement of $J(F_\lambda)$ was simply $B_\lambda$ together with all of its preimages. However, there are parameter values in these families for which the Julia set is a Sierpinski curve whose complementary domains consist of a variety of different attracting basins (not just $B_\lambda$) together with their preimages. Again, while these Julia sets are all homeomorphic to one another, the dynamics on different pairs of these sets is often quite different.

There is another famous Sierpinski “object” in fractal geometry, namely, the Sierpinski gasket or triangle. Objects similar in construction to this shape also occur in these families. In Section 7 we construct infinitely many “Sierpinski gasket-like” Julia sets for $F_\lambda$. Unlike the Sierpinski curves, each pair of these Julia sets are topologically as well as dynamically distinct.

This paper is respectfully dedicated to the memory of Professor Noel Baker. Professor Baker’s numerous contributions to the field of complex dynamics have been an inspiration to all of us.

2 Preliminaries

We consider the maps

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

where $n, d \in \mathbb{Z}^+$. The Julia set of $F_\lambda$, $J(F_\lambda)$, is defined to be the set of points at which the family of iterates of $F_\lambda$ fails to be a normal family in the sense of Montel. Equivalently, the Julia set is the closure of the set of repelling periodic points for $F_\lambda$ or, alternatively, the set of points on which $F_\lambda$ behaves chaotically. The complement of the Julia set is called the Fatou set.
There are $n + d$ finite and nonzero critical points for $F_{\lambda}$ and all are of the form $\omega^k c_{\lambda}$ where $c_{\lambda}$ is one of the critical points and $\omega^{n+d} = 1$. Similarly, the critical values are arranged symmetrically with respect to $z \mapsto \omega z$, though there need not be $n + d$ of them. For example, if $n = d$, the $n + d$ critical points are given by $\lambda^{1/2n}$, while there are only two critical values given by $\pm 2\sqrt{\lambda}$. There are $n + d$ prepoles at the points $(-\lambda)^{1/(n+d)}$.

Note that $F_{\lambda}(\omega z) = \omega^n F_{\lambda}(z)$. Hence the orbits of points of the form $\omega^j z$ all behave “symmetrically” under iteration of $F_{\lambda}$. For example, if $F_{\lambda}(z) \to \infty$, then $F_{\lambda}^k(\omega^k z)$ also tends to $\infty$ for each $k$. If $F_{\lambda}(z)$ tends to an attracting cycle, then so does $F_{\lambda}^k(\omega^k z)$. Note, however, that the cycles involved may be different depending on $k$ and, indeed, they may even have different periods. Nonetheless, all points lying on this set of attracting cycles are of the form $\omega^j z_0$ for some $z_0 \in \mathbb{C}$. In particular, all $n + d$ critical points have orbits that behave symmetrically, so this is why there is only one free critical orbit for $F_{\lambda}$.

We now restrict attention to the case $n \geq 2$; the case $n = 1$ will be dealt with in Section 5. The point at $\infty$ is a superattracting fixed point for $F_{\lambda}$ and it is well known that $F_{\lambda}$ is conjugate to $z \mapsto z^n$ in a neighborhood of $\infty$, so we have an immediate basin of attraction $B_{\lambda}$ at $\infty$. Since $F_{\lambda}$ has a pole of order $d$ at $0$, there is an open neighborhood of $0$ that is mapped $d$ to $1$ onto a neighborhood of $\infty$ in $B_{\lambda}$. If the entire basin of $\infty$ is disjoint from this neighborhood around the origin, then there is a open set about $0$ that is mapped $d$ to $1$ onto $B_{\lambda}$, and this entire set is disjoint form $B_{\lambda}$. This set is called the \textit{trap door} and we denote it by $T_{\lambda}$. Since the degree of $F_{\lambda}$ is $n + d$, all points in the preimage of $B_{\lambda}$ lie either in $B_{\lambda}$ or in $T_{\lambda}$.

Using the symmetry $F_{\lambda}(\omega z) = \omega^n F_{\lambda}(z)$, it is straightforward to check that all of $B_{\lambda}$, $T_{\lambda}$, and $J(F_{\lambda})$ are symmetric under $z \mapsto \omega z$. We say that these sets possess $n + d$-fold symmetry. In particular, since the critical points are arranged symmetrically about the origin, it follows that if one of the critical points lies in $B_{\lambda}$ (resp., $T_{\lambda}$), then all of the critical points lie in $B_{\lambda}$ (resp., $T_{\lambda}$).

For other components of the Fatou set, the symmetry situation is somewhat different: either a component contains $\omega^j z_0$ for a given $z_0$ in the Fatou set and all $j \in \mathbb{Z}$, or else such a component contains none of the $\omega^j z_0$ with $j \neq 0$ mod $n + d$.

\textbf{Symmetry Lemma.} \textit{Suppose $U$ is a connected component of the Fatou set of $F_{\lambda}$. Suppose also that both $z_0$ and $\omega^j z_0$ belong to $U$, where $\omega^j \neq 1$. Then}
in fact, \( \omega^i z_0 \) belongs to \( U \) for all \( i \) and, as a consequence, \( U \) has \( n + d \)-fold symmetry and surrounds the origin.

See [8] for a proof of this fact.

3 The Escape Trichotomy

For the well-studied family of quadratic maps \( Q_c(z) = z^2 + c \) with \( c \) a complex parameter there is the well known Fundamental Dichotomy:

1. If the orbit of the one free critical point at 0 tends to \( \infty \), then the Julia set of \( Q_c \) is a Cantor set;

2. If the orbit of 0 does not tend to \( \infty \), then the Julia set is a connected set.

In this section we discuss a similar result for \( F_\lambda \) that we call the Escape Trichotomy. Unlike the family of quadratic maps \( Q_c \), there exist three different "ways" that the critical orbit for \( F_\lambda \) can tend to infinity. If the critical orbit tends to infinity, then all of the critical values must lie in \( B_\lambda \) or one of its preimages. These three different scenarios lead to three distinct classes of Julia sets for \( F_\lambda \) that comprise the Escape Trichotomy.

3.1 Critical Values in \( B_\lambda \)

We first assume that one of the critical values of \( F_\lambda \) lies in \( B_\lambda \). In this case, \( J(F_\lambda) \) is a Cantor set. We sketch a proof of this fact here (for more details, see [8]).

By symmetry, if one of the critical values lies in \( B_\lambda \), then all of the critical values do so as well. Let \( v \) be a critical value of \( F_\lambda \) and let \( c \) be a critical point such that \( F_\lambda(c) = v \). Let \( U \) be an open disk in \( B_\lambda \) containing both \( v \) and \( \infty \) with \( F_\lambda(U) \subset U \). We may assume that \( U \) has \( (n + d) \)-fold symmetry. Let \( V \) be the preimage of \( F_\lambda(U) \) containing the origin. We may also assume that \( U \) and \( V \) are disjoint.

Let \( \gamma \) be an arc in \( U \) connecting \( v \) to \( \infty \). The preimage of \( \gamma \) is an arc \( \gamma' \) that contains \( c \) and is mapped two-to-one onto \( \gamma \). One portion of \( \gamma' \) connects \( c \) to \( \infty \). The curve \( \gamma' \) must therefore also lie in \( B_\lambda \), and so we see that \( c \) and hence all of the critical points must lie in \( B_\lambda \).
Since $c$ is a critical point, it follows that $\gamma'$ contains a second preimage of $\infty$. One checks easily that this second preimage of $\infty$ is 0, not $\infty$, and so $\gamma'$ extends all the way from 0 to $\infty$. In particular, $\gamma'$ meets both $U$ and $V$, and so both of these sets lie in $B_\lambda$. Therefore $B_\lambda$ and $T_\lambda$ are not disjoint sets. Let $W$ be the preimage of $U$. It follows that $W$ contains $U$, $V$, and a neighborhood of $\gamma'$.

Since $v$ was an arbitrary critical value of $F_\lambda$ we can repeat this process and obtain $n + d$ arcs connecting 0 and $\infty$ such that each arc contains a distinct critical point. Furthermore, these arcs may be chosen so that they do not intersect and are symmetric under $z \to \omega z$ where $\omega^{n+d} = 1$. Each of these arcs also lies in $W$ and so $W$ consists of the Riemann sphere with $n + d$ disjoint and symmetric disks $A_j$ for $j = 1, \ldots, n + d$ removed. Finally, it is easy to check that each $A_j$ in the complement of $W$ is mapped univalently over the complement of $U$ and hence over all of the other $A_i$. Therefore, each of the $n + d$ sets $A_j$ contain preimages of all of the other $A_i$, and the Julia set is contained in the union of these $(n + d)^2$ sets. See Figure 2. Standard arguments then show that the Julia set is a Cantor set and $F_\lambda$ is a one-sided shift on $n + d$ symbols on this set. Figure 1a displays an example of a Julia set for which the critical values lie in $B_\lambda$.

![Diagram of sets A1, A2, A3 and their preimages](image)

Figure 2: The sets $A_j$ and their preimages.
3.2 Critical Values in $T_\lambda$

Assume now that $B_\lambda$ and $T_\lambda$ are disjoint and that one, and hence all, of the critical values of $F_\lambda$ now lie in $T_\lambda$. In this case, $J(F_\lambda)$ is a Cantor set of simple closed curves. To see this, note first that, since $B_\lambda$ and $T_\lambda$ are both open disks, the Riemann-Hurwitz formula shows that preimage of $T_\lambda$ is an open annulus surrounding the origin and located between $T_\lambda$ and $B_\lambda$. We denote this preimage of $T_\lambda$ by $T_\lambda^{-1}$ and the $n^{th}$ preimage of $T_\lambda$ by $T_\lambda^{-n}$. The annulus $T_\lambda^{-1}$ contains all of the critical points of $F_\lambda$ and its closure divides the region between $T_\lambda$ and $B_\lambda$ into two open subannuli that are mapped onto $\mathbb{C} - (\overline{B_\lambda} \cup \overline{T_\lambda})$. We call these subannuli $A_{in}$ and $A_{out}$, with $A_{in}$ the subannulus bordering $T_\lambda$ and $A_{out}$ the subannulus bordering $B_\lambda$. Note that since the boundary of $T_\lambda^{-1}$, $\partial T_\lambda^{-1}$, is mapped onto $\partial T_\lambda$, whereas both $\partial T_\lambda$ and $\partial B_\lambda$ are both mapped onto $\partial B_\lambda$, it must be the case that $\partial T_\lambda^{-1}$ is disjoint from $\partial T_\lambda$ and $\partial B_\lambda$. See Figure 3. Let $A$ denote the union of the three annuli $A_{in}$, $A_{out}$, and $T_\lambda^{-1}$.

![Diagram](image)

Figure 3: The sets $A_{in}$, $A_{out}$, $T_\lambda$, $B_\lambda$ and $T_\lambda^{-1}$.

Since all of the critical points lie in $T_\lambda^{-1}$, the annuli $A_{in}$ and $A_{out}$ are mapped as coverings onto $\mathbb{C} - (\overline{B_\lambda} \cup \overline{T_\lambda})$. Hence there exist preimages of $T_\lambda^{-1}$ in each of these subannuli. Note that there will be two annular components of $T_\lambda^{-2}$, one in $A_{in}$ and one in $A_{out}$. See Figure 4. Continuing in this fashion, we see that $T_\lambda^{-n}$ consists of $2^n$ subannuli. In [8], quasiconformal surgery was
used to show that the boundaries of $B_\lambda$, $T_\lambda$, and all of the preimages of $T_\lambda$ are simple closed curves surrounding the origin. Hence the Julia set is given by a nested intersection of closed annuli and the result follows exactly as in the case described by McMullen in [14].

![Diagram showing inverse images of $T_\lambda$.](image)

Figure 4: Inverse images of $T_\lambda$.

We remark that, by the covering properties of $F_\lambda$ on $A_{in}$ and $A_{out}$, we must have

$$\text{mod } A > \text{mod } A_{in} + \text{mod } A_{out} = \left(\frac{1}{d} + \frac{1}{n}\right) \text{mod } A$$

where mod $A$ denotes the modulus of $A$. Hence, as in the McMullen result, we must have $1/d + 1/n < 1$ in order for $v$ to lie in the trap door. Therefore, if $1/d + 1/n > 1$, then $v$ cannot lie in the trap door, so part 2 of the Escape Trichotomy Theorem cannot occur if $d = n = 2$ or if either $n$ or $d$ is equal to 1. In Figure 1b we display a Julia set for which the critical values all lie in $T_\lambda$.

### 3.3 Critical Values in a Preimage of $T_\lambda$

We now describe the final case where the critical values have orbits that eventually escape through the trap door, but the critical values do not them-
selves lie in the trap door. In this case the Julia set is a Sierpinski curve. We first observe that the Julia set of \( F_\lambda \) is compact, connected, locally connected, and nowhere dense. Indeed, since we are assuming that the critical orbit eventually enters the basin of \( \infty \), we have that the Julia set is given by \( \mathbb{C} - \bigcup F_\lambda^{-J}(B_\lambda) \). That is, \( J(F_\lambda) \) is \( \mathbb{C} \) with countably many disjoint, simply connected, open sets removed. Hence \( J(F_\lambda) \) is compact and connected. Since \( J(F_\lambda) \neq \mathbb{C} \), \( J(F_\lambda) \) cannot contain any open sets, so \( J(F_\lambda) \) is also nowhere dense.

Finally, since the critical orbits all tend to \( \infty \) and hence do not lie in or accumulate on \( J(F_\lambda) \), it follows that \( F_\lambda \) is hyperbolic on \( J(F_\lambda) \) and standard arguments show that \( J(F_\lambda) \) is locally connected (see [16]). In particular, since \( B_\lambda \) is a simply connected component of the Fatou set, it follows that the boundary of \( B_\lambda \) is locally connected. Hence \( J(F_\lambda) \) fulfills the first four of the conditions to be a Sierpinski curve.

To finish showing that \( J(F_\lambda) \) is a Sierpinski curve we need to show that the boundaries of \( B_\lambda \) as well as all of the preimages of \( B_\lambda \) are simple closed curves and that these boundary curves are pairwise disjoint. To see this, we first claim that \( \mathbb{C} - \overline{B_\lambda} \) is a connected open set. This should be contrasted with the situation for quadratic polynomial Julia sets where \( \mathbb{C} - \overline{B_\lambda} \) often consists of infinitely many disjoint open sets (consider the Julia sets known as the basilica or Douady’s rabbit, for example). Assume that \( \mathbb{C} - \overline{B_\lambda} \) has more than one component. Let \( W_0 \) be the component of \( \mathbb{C} - \overline{B_\lambda} \) that contains the origin. Note that \( T_\lambda \subset W_0 \). Since \( F_\lambda(\partial T_\lambda) = \partial B_\lambda \supset \partial W_0 \), it follows that there are points in \( W_0 \) whose images also lie in \( W_0 \) and consequently \( F_\lambda(W_0) \supset W_0 \). Now if one of the prepoles lies in a component of \( \mathbb{C} - \overline{B_\lambda} \) that is disjoint from \( W_0 \), then by symmetry all of the prepoles have this property. But this then gives us too many preimages of points in \( W_0 \), and so all of the prepoles must in fact lie in \( W_0 \). It then follows that all of the preimages of any point in \( W_0 \) lie in \( W_0 \).

If there were another component of \( \mathbb{C} - \overline{B_\lambda} \), then the boundary of this set must eventually be mapped over the boundary of \( W_0 \) since \( \partial W_0 \subset J(F_\lambda) \), and so there must be additional preimages of points in \( W_0 \). But again, this is impossible. Therefore \( W_0 \) is the only component of \( \mathbb{C} - \overline{B_\lambda} \). Standard arguments [10] using external rays then show that the boundary of \( W_0 \) must in fact be a simple closed curve. So too are the boundaries of all of the preimages of \( B_\lambda \). One then checks that all of these curves are disjoint, for a point that lies in the intersection of one of these curves must either be a critical point or one of its preimages, but we know that all critical points have orbits that tend to \( \infty \). This completes the proof that the Julia set is a
Sierpinski curve.

In Figure 5 we show $B_\lambda$, $T_\lambda$ and the first two preimages of $T_\lambda$ in the special case where $n = d = 2$ and under the assumption that there are no critical points in $T_\lambda^{-1}$ or $T_\lambda^{-2}$. An actual Julia set for which the critical points lie in $T_\lambda^{-2}$ is depicted in Figure 1c.

![Figure 5: $B_\lambda$, $T_\lambda$, $T_\lambda^{-1}$ and $T_\lambda^{-2}$.](image)

In Figure 6, we show the $\lambda$ plane in the case $n = d = 4$. The outside grey region in this image consists of $\lambda$-values for which $J(F_\lambda)$ is a Cantor set. The central grey region is the McMullen domain in which $J(F_\lambda)$ is a Cantor set of simple closed curves. The region between these two sets is called the connectedness locus as the Julia sets are always connected when $\lambda$ lies in this region. The other grey regions in this figure correspond to Sierpinski holes in which the corresponding Julia sets are Sierpinski curves.

## 4 The Case $n = d = 2$

As mentioned earlier, the cases where $n = d = 2$ or $n > 1, d = 1$ are significantly different from the other cases where $n \geq 2$ since there is no McMullen domain in parameter space. In these cases, we instead have infinitely many open sets of parameters in any neighborhood $\lambda = 0$ in parameter space in which the critical orbits eventually enter $B_\lambda$ and hence the Julia set is a Sierpinski curve. In each of these open sets the number of iterations that it takes
for the critical orbit to enter $B_\lambda$ is different, and so two maps drawn from
different open sets are dynamically distinct in the sense that these maps are
not topologically conjugate.

We sketch the proof of this when $n = d = 2$. We show that there
are infinitely many open intervals in $\mathbb{R}$ in any neighborhood of the origin
in parameter space in which the critical orbit eventually escapes. Similar
results hold when $n > 1$, $d = 1$, though the real axis need not be the home
of these open sets.

When $n = d = 2$, the four critical points and four prepoles of $F_\lambda$ all lie
on the circle of radius $|\lambda|^{1/4}$ centered at the origin. We call this circle the
*critical circle*. The case $n = d = 2$ is especially simple since the second image
of the critical points is given by

$$F_\lambda^2(c_\lambda) = 4\lambda + \frac{1}{4}$$

and so $\lambda \mapsto F_\lambda^2(c_\lambda)$ is an analytic function of $\lambda$ that is a homeomorphism.
If $-1/16 < \lambda < 0$, then one checks easily that the critical circle is mapped
strictly inside itself. Therefore, as in the previous section, $J(F_\lambda)$ is a con-}

ected set and $B_\lambda$ and $T_\lambda$ are disjoint. In particular, the second image of the
critical point lands on the real axis and lies in the complement of $B_\lambda$ in $\mathbb{R}$.  

Figure 6: The parameter plane when $n = d = 4$. 

13
Figure 7: Sierpinski curve Julia sets for various negative values of $\lambda$ when $n = d = 2$. All of these sets are homeomorphic, but the dynamics on each is different.
Proposition. There is an increasing sequence $\lambda_2, \lambda_3, \ldots$ in $\mathbb{R}$ with $\lambda_j \to 0$ and $F_{\lambda_j}^2(c_{\lambda_j}) = 0$.

Proof: Since $F_{\lambda}^2(c_{\lambda}) = 4\lambda + 1/4$, this quantity increases monotonically toward $1/4$ as $\lambda \to 0$. Now the orbit of $1/4$ remains in $\mathbb{R}^+$ for all iterations of $F_0$ and decreases monotonically to 0. Hence, given $N$, for $\lambda$ sufficiently small, $F_{\lambda}^j(c_{\lambda})$ also lies in $\mathbb{R}^+$ for $2 \leq j \leq N$ and moreover this finite sequence is decreasing.

Now suppose $\beta < \alpha < 0$. We have $F_{\beta}(x) < F_{\alpha}(x)$ for all $x \in \mathbb{R}^+$. Also, $F_{\beta}^2(c_{\beta}) < F_{\alpha}^2(c_{\alpha}) < 1/4$. Hence $F_{\beta}^j(c_{\beta}) < F_{\alpha}^j(c_{\alpha})$ for all $j$ for which $F_{\beta}^j(c_{\beta}) \in \mathbb{R}^+$. The result then follows by continuity of $F_{\lambda}$ with respect to $\lambda$. \hfill \Box

Note that $\lambda_2 = -1/16$. Using the previous Proposition, we may find open intervals $I_j$ about $\lambda_j$ for $j = 2, 3, \ldots$ having the property that, if $\lambda \in I_j$, then $F_{\lambda}^j(c_{\lambda}) \in T_{\lambda}$, and so $F_{\lambda}^{j+1}(c_{\lambda}) \in B_{\lambda}$. Therefore, $F_{\lambda}^n(c_{\lambda}) \to \infty$ as $n \to \infty$, and the Escape Trichotomy then shows that $J(F_{\lambda})$ is a Sierpinski curve.

Now let $C(c_{\lambda})$ denote the component of the Fatou set of $F_{\lambda}$ containing $c_{\lambda}$. The map $F_{\lambda}$ is two-to-one on each of the four sets $C(c_{\lambda})$ containing these critical points, and we have $F_{\lambda}^2(C(c_{\lambda})) = T_{\lambda}$ for some $j$. Now suppose that $F_{\lambda}|J(F_{\lambda})$ is conjugate to $F_{\alpha}|J(F_{\alpha})$ for some $\alpha \in \cup I_k$ for some $k > 1$. This conjugacy must take the boundaries of $B_{\lambda}$ and $T_{\lambda}$ to the corresponding boundaries of $B_{\alpha}$ and $T_{\alpha}$. Similarly the boundaries of the four regions $C(c_{\lambda})$ must be mapped to one of the corresponding regions by the conjugacy, since these are the only complementary domains (besides $B_{\lambda}$ and $T_{\lambda}$) on which $F_{\lambda}$ is two-to-one. If, however, $\lambda \in I_j$ and $\alpha \in I_k$ with $j \neq k$, then these maps cannot be conjugate, since a conjugacy maps each of the $j^{th}$ preimages of $T_{\lambda}$ to one of the $j^{th}$ preimages of $T_{\alpha}$. Such a conjugacy would also have to map boundaries of domains on which $F_{\lambda}$ and $F_{\alpha}$ were two-to-one to each other. Since $j \neq k$, this is impossible. We therefore have:

**Theorem.** Let $\lambda \in I_j$ and $\alpha \in I_k$ with $j \neq k$. Then $F_{\lambda}$ is not conjugate to $F_{\alpha}$ on their corresponding Julia sets.

In Figure 7 we display several dynamically distinct Sierpinski curve Julia sets for $\lambda$ close to 0.

In Figure 8 we display the parameter plane for the case $n = d = 2$ as well as a magnification around $\lambda = 0$. In contrast to the image in Figure 6, all of the internal grey regions in this image are Sierpinski holes. There is no McMullen domain when $n = d = 2$. 

15
5 The case \( n = 1 \)

In this section we restrict attention to the family of functions

\[
F_\lambda(z) = z + \frac{\lambda}{z}
\]

so that \( n = 1 \). The dynamics of these maps are quite different from those for which \( n > 1 \). First, one checks easily that, for each \( \lambda \), the map \( F_\lambda \) is conjugate to the function

\[
F_1(z) = z + \frac{1}{z}.
\]

Hence this family does not really depend on a parameter. Therefore we change the family slightly so that we consider instead

\[
F_\lambda(z) = \lambda \left( z + \frac{1}{z} \right).
\]

This family is conjugate to the family

\[
G_\lambda(z) = \lambda z + \frac{1}{z},
\]

and so can be regarded as a linear perturbation of the involution \( z \mapsto 1/z \).
The main difference between this family and our original family is that these functions have a repelling fixed point at infinity whenever $|\lambda| < 1$. Consequently, 0 lies in the Julia set and thus there is no trap door as in the case where $n > 1$.

![Image](image_url)

**Figure 9:** The $\lambda$ plane for the function $F_\lambda(z) = \lambda(z + \frac{1}{z})$.

As in the previous cases, we are mainly concerned with the case of $\lambda$ small, so that we are perturbing away from the identically zero function. It has been shown by Yongcheng [25] that for $0 < |\lambda| \leq 1$ the Julia set is connected while if $|\lambda| > 1$, it is a Cantor set. The parameter space is plotted in Figure 9. Similar figures have been produced by Hawkins [12] and Milnor [17]. Note that most of the interesting behavior seems to occur as we approach the parameter 0 along the imaginary axis. In fact, it is easy to check that, for $0 < \lambda \leq 1$, $J(F_\lambda)$ is the imaginary axis, and all other points have orbits that are attracted to one of two attracting fixed points. For $-1 \leq \lambda < 0$, $J(F_\lambda)$ is the real axis, and this set separates the basins of an attracting two-cycle. In both of the large black circular regions in parameter space flanking the origin, the Julia sets are similar curves passing through the origin and $\infty$. In contrast, the dynamical behavior along the imaginary axis is much more complicated.
Given nonzero \( \lambda \), the function \( F_\lambda \) is a degree-two rational map with two critical points at \( \pm 1 \). The orbits of these critical points behave symmetrically under \( F_\lambda \). For purely imaginary parameter values, this function has the desirable property that, in the dynamical plane, the real axis is mapped to the imaginary axis and vice versa. Therefore, for such parameter values we will consider the second iterate map restricted to the real axis, that is, we restrict attention to the behavior of \( F^2_{i\lambda} \) on \( \mathbb{R} \), where \( \lambda \) is now a real parameter.

We compute

\[
F^2_{i\lambda}(x) = -\lambda^2 \left( x + \frac{1}{x} \right) + \frac{1}{x + \frac{1}{x}}.
\]

Note that, for small \( \lambda \), this second iterate map can be viewed as a perturbation of the \( \lambda \)-independent function

\[
x \mapsto \frac{1}{x + \frac{1}{x}}.
\]

When one and hence both of the critical points land on the repelling fixed point at \( \infty \), the Julia set is known to be the entire Riemann sphere [16]. We will refer to such parameter values as blowup points, with the convention that a blowup point of order \( n \) is one such that \( F^n_{i\lambda}(1) = 0 \). Parameter values for which this occurs are also known as \( m \)-ergodic rational maps (although \( m \)-ergodicity describes a larger set of maps than just those for which a critical point lands on a repelling cycle). Rees [21] has proved that \( m \)-ergodic maps comprise a set of positive Lebesgue measure in the parameter space of most rational maps. Hawkins [12] developed a computer algorithm for finding and plotting these parameter values. In that paper, it was shown numerically that the \( m \)-ergodic maps accumulate on the origin along the imaginary axis in parameter space. We formalize this observation via the following theorem.

**Theorem.** For the family of functions \( F_{i\lambda}(z) = i\lambda(z + 1/z) \), in any neighborhood of \( \lambda = 0 \), there exists:

1. A countably infinite set of \( \lambda \)-values lying in \((-1,1)\) for which the Julia set is the entire Riemann sphere;

2. A countably infinite set of \( \lambda \)-values lying in \((-1,1)\) for which the critical point is part of a superattracting cycle.

**Proof:** To prove the first assertion, we will define a function \( G_n : \mathbb{R} \to \mathbb{R} \) via \( G_n(\lambda) = F^2_{i\lambda_n}(1) \) where \( \lambda \in \mathbb{R} \). For \( \lambda_1 = .5 \), \( G_1(\lambda_1) = 0 \). Also, \( G_1(0) = 1/2 \).
Further, note that $G_m(\lambda)$ is continuous except at blowup points of order less than $m$. We now see that $G_2$ maps $(0, \lambda_1)$ to $(-\infty, 1/2)$. Thus, by continuity of $G_2$ in this interval, there exists a $\lambda_2 \in (0, \lambda_1)$ such that $G_2(\lambda_2) = 0$. If more than one $\lambda$ value exists, we will chose the smallest to be $\lambda_2$. This ensures that $G_3$ will be continuous on $(0, \lambda_2)$. Iterating this process we obtain the desired sequence.

Now suppose that this sequence does not accumulate on the origin. In other words, there exists some interval $(0, \hat{\lambda})$ such that $G_n(\lambda) > 0$ for all $n$ and $\lambda \in (0, \hat{\lambda})$. Since the graph of $F^2_{\lambda}$ lies strictly below the diagonal on $(0, 1)$ and $F^2_{\lambda}$ is monotonically increasing there, the interval $(0, 1)$ is mapped inside itself. Thus, by the contraction mapping principle there exists a fixed point in $(0, 1)$, which is a contradiction.

To prove the second part of the assertion, let $\lambda_n$ and $\lambda_m$ be blowup points of order $n$ and $m$. Assume $n < m$. For fixed $m$ there are a finite number of discontinuities of $G_m$ in the interval $(\lambda_m, \lambda_n)$. Furthermore, these discontinuities represent blowup points of order less than $m$. Therefore, we will restrict ourselves to a subinterval on which $G_m$ is continuous and note that the result holding here is sufficient to establish the result in the general setting. Thus, without loss of generality, assume that $G_m$ is continuous on $(\lambda_m, \lambda_n)$. Therefore, $G_m(\lambda_n) = \infty$ and $G_m(\lambda_m) = 0$. By continuity of $G_m$ there exists $\lambda_p \in (\lambda_m, \lambda_n)$ such that $G_m(\lambda_p) = 1$.

We will now briefly turn our attention to the case where $d > 1$. In this case the critical points are $c = d^{-1/2}$. As in the $d = 1$ case the critical points do not depend on the parameter value. Also there exist lines, analogous to the imaginary axis for the case $d = 1$ and passing through the origin in parameter space, for which $F^{d+1}_\lambda$ is invariant over $\mathbb{R}$. The parameter planes for several of these functions are plotted in Figure 10. For this class of rational functions, the results of Rees [21] guarantee a set of positive Lebesgue measure in parameter space for which the Julia set is the whole Riemann sphere. However, it is unknown whether this behavior accumulates on the origin and hence whether a corollary to the Theorem is true for $d > 1$.

6 Buried Sierpinski Curves

In this section, we discuss an infinite collection of dynamically distinct Sierpinski curve Julia sets for the family $F_\lambda$ where the Fatou components are
Figure 10: The $\lambda$ plane for the functions $F_\lambda(z) = \lambda(z + 1/z^2)$ and $F_\lambda(z) = \lambda(z + 1/z^3)$.

quite different than those described in previous sections. Instead of being preimages of a single superattracting basin at $\infty$, we give examples where the complementary domains consist of a collection of different attracting basins together with the basin at $\infty$ and all of the preimages of these basins. As before, we sketch a proof that the dynamics on these Julia sets are all distinct from one another as well as from those mentioned above, but again, all of these Julia sets are homeomorphic.

For simplicity, we restrict attention in this section to the special family $F_\lambda(z) = z^2 + \lambda/z$ with $\lambda \in \mathbb{R}^-$. In Figure 11, we display the Julia set of $F_\lambda$ when $\lambda = -0.327$. For this map, there are attracting basins of period 3 and period 6 together with the basin at $\infty$. We also display the case where $\lambda = -0.5066$ for which there are three different attracting basins of period 4 together with the basin at $\infty$. The basins of the finite cycles are displayed in black.

There is a positive real fixed point for $F_\lambda$ which we denote by $p(\lambda)$. Also, $c(\lambda) = (\lambda/2)^{1/3}$ is a critical point and

$$v(\lambda) = \frac{3}{2^{2/3}}\lambda^{2/3}$$

is a critical value. Note that, for $\lambda \in \mathbb{R}^-$, both $c(\lambda)$ and $v(\lambda)$ are real.
Figure 11: The Julia sets for $F_{\lambda}(z) = z^2 + \lambda/z$ where $\lambda = -0.327$ and $\lambda = -0.5066$.

Let $\lambda^* = -16/27$. Straightforward calculations show that $p(\lambda^*) = 4/3$ and $p(\lambda^*)$ is repelling. Further, the real critical point $c(\lambda^*) = -2/3$ is prefixed, i.e., $F_{\lambda^*}(c(\lambda^*)) = 4/3 = p(\lambda^*)$. For $\lambda$-values slightly larger than $\lambda^*$, the real critical value lies to the left of $p(\lambda)$ and hence subsequent points on the orbit of the critical value begin to decrease. Graphical iteration shows that there is a sequence of $\lambda$-values tending to $\lambda^*$ for which the critical orbit decreases along the positive axis and then, at the next iteration, lands back at $c(\lambda)$. See Figure 12. Thus, for these $\lambda$-values, we have a superattracting cycle. More precisely, we have:

**Theorem.** Let $F_{\lambda}(z) = z^2 + \lambda/z$ with $\lambda \in \mathbb{R}^-$. There is a decreasing sequence $\lambda_n \in \mathbb{R}^-$ for $n \geq 3$ with $\lambda_n \to \lambda^* = -16/27$ and having the property that $F_{\lambda_n}$ has a superattracting cycle of period $n$ given by $x_j(\lambda_n) = F_{\lambda_n}(x_{j-1}(\lambda_n))$, where

1. $x_0(\lambda_n) = x_n(\lambda_n) = c(\lambda_n)$, and
2. $x_0 < 0 < x_{n-1} < x_{n-2} < \cdots < x_1 = v(\lambda_n) < p(\lambda_n)$.

For a proof see [6]. Now fix a particular parameter value $\lambda = \lambda_n$ for which $F_{\lambda}$ has a superattracting periodic point $x_0$ lying in $\mathbb{R}^-$ as described in the previous Theorem. We say that a basin of attraction of $F_{\lambda}$ is *buried* if the
boundary of this basin is disjoint from the boundaries of all other basins of attraction (including $B_{\lambda}$). We remark that buried basins are quite different from buried components of Julia sets. Note that, if the basin of one point on an attracting cycle is buried, then so too are all forward and backward images of this basin, so the entire basin of the cycle is buried. In [6] the following was shown:

**Theorem.** All of the basins of $F_{\lambda}$ are buried and $J(F_{\lambda})$ is a Sierpinski curve.

As discussed earlier, any two Sierpinski curves are homeomorphic. Hence $J(F_{\lambda_n})$ is topologically equivalent to $J(F_{\lambda_m})$ for any $n$ and $m$. However, each of these Julia sets is dynamically distinct from the others since the periods of the superattracting cycles are different.

In Figure 13 we display the parameter plane for the degree three family

$$F_{\lambda}(z) = z^2 + \frac{\lambda}{z}$$

together with a magnification of a certain region along the negative real axis.

The grey holes in this parameter plane correspond to parameter values for which the critical orbit eventually escapes to $\infty$ through the trap door, so the Julia set is a Sierpinski curve as discussed in Section 3. These are the Sierpinski holes. Note the existence of a small copy of a Mandelbrot set along the negative real axis in this image. In fact, there are infinitely many such
Mandelbrot sets converging to the left tip of the parameter space, which is the parameter $\lambda^*$. See [4] for a proof of this in a more general setting. The parameters for which we have the superattracting cycles constructed above form the centers of the main cardioids of certain of these Mandelbrot sets.

We remark that there appear to be two very different types of baby Mandelbrot sets in this picture, some of which touch the outer boundary of the connectedness locus, and some that do not. It is known [3] that those Mandelbrot sets that touch the outer boundary actually touch infinitely many of the Sierpinski holes as well. We conjecture that the Mandelbrot sets corresponding to the $\lambda_n$ in this section are also “buried,” this time in the sense that these sets do not touch any of the Sierpinski holes, nor the outer boundary.

7 Sierpinski Gasket-like Julia Sets

One of the outstanding theorems in the study of the families of polynomials $z \mapsto z^d + c$ is the Landing Theorem, due to A. Douady and J. H. Hubbard [11], which states that every external ray in the parameter plane whose external angle is rational lands at a unique point in the boundary of the connectedness locus. Recently, C. Petersen and G. Ryd [20] have shown that
this result may be extended to many other one-parameter families of maps with a single free critical orbit, including the family $F_\lambda$ when $n \geq 2$. In this section we will concentrate on $\lambda$-values that correspond to external rays whose external angles are of the special form $p/n^j$ with $p, j \in \mathbb{Z}$. The Landing Theorem implies that such a $\lambda$-value is a parameter for which the critical orbits eventually land on a fixed point in the boundary of $B_\lambda$. We call the corresponding maps Misiurewicz-Sierpinski maps, or MS maps, for short.

In Figures 14 and 15 we display several examples of Julia sets corresponding to Misiurewicz parameters for $z \mapsto z^2 + \lambda/z$ and $z \mapsto z^2 + \lambda/z^2$ respectively. Clearly, these sets are no longer homeomorphic to the Sierpinski curve, as infinitely many boundaries of the complementary domains intersect other complementary boundaries at one or more points. In particular, the Julia set in the left-hand side of Figure 14 is homeomorphic to the well-known Sierpinski gasket (or triangle). Although the second Julia set in Figure 14 looks similar to the Sierpinski gasket, these two Julia sets are not homeomorphic, as we explain below.

![Figure 14: Julia sets from the family $z \mapsto z^2 + \lambda/z$ with $\lambda \approx -0.59257$ and $-0.03804 + i0.42622$.](image)

The Julia sets in Figure 15 can be thought of as generalizations of a Sierpinski gasket set with four distinguished vertices. We will see that these Julia sets are again not homeomorphic to each other. A generalized Sierpinski gasket set with four distinguished vertices is constructed as follows.
Figure 15: Julia sets from the family $z \mapsto z^2 + \lambda/z^2$ with $\lambda \approx -0.36428$ and $\lambda \approx -0.01965 + i0.2754$.

Consider the closed unit disk in the plane from which we remove an open rectangular region whose vertices lie in the boundary of the disk. We assume that the removed rectangle is symmetric under rotation of the disk by angle $\pi/2$. We are left with four symmetric closed sets which we denote by $I_0, I_1, I_2$ and $I_3$. From each of the $I_j$ we next remove an open “generalized” rectangle whose vertices lie on the boundary of $I_j$. We stipulate that exactly two of these vertices lie on the boundary of the previously removed rectangle and that the newly removed sets are all symmetrically arranged. This leaves sixteen sets whose only intersection points are vertices of the removed rectangles. We continue in this fashion by removing at each stage open generalized rectangles with exactly two vertices lying in the boundary of the previously removed rectangle. In the limit this produces a set which we call a generalized Sierpinski gasket or a Sierpinski gasket-like set.

For simplicity, in this section we consider only the special case where $n = d = 2$, although all of the results go over with minor modifications to the more general family of maps with $n \geq 2, d \geq 1$. See [9].

Theorem. Let $F_\lambda(z) = z^2 + \lambda/z^2$ be an MS map. Then the Julia set $J(F_\lambda)$ is a generalized Sierpinski gasket with four distinguished vertices. Moreover, if we assume that $\lambda$ and $\mu$ are chosen so that $F_\lambda$ and $F_\mu$ are MS maps from
the same family, then their Julia sets are homeomorphic if and only if \( \lambda = \overline{\mu} \).

7.1 Topology of Julia Sets

Suppose \( F_\lambda \) is an MS map with \( n = d = 2 \). Since the post-critical orbit is finite, the map is sub-hyperbolic and thus the boundary of each Fatou component is locally connected (see [16], page 191). Moreover, as shown in [8], there is only one component to the set \( \mathbb{C} - B_\lambda \) and the boundary of this set is a simple closed curve which is also the boundary of \( B_\lambda \). Denote the boundary of \( B_\lambda \) by \( \beta_\lambda \) and the boundary of the trap door by \( \tau_\lambda \). Our assumption implies that the four finite and non-zero critical points \( c_\lambda = \lambda^{1/4} \) lie in both \( \beta_\lambda \) and \( \tau_\lambda \). A straightforward argument given in [7] shows that if the set \( \beta_\lambda \cap \tau_\lambda \) is non-empty, then the critical points are the only points in this intersection. We call these points the corners of the trap door. The four corners separate \( \tau_\lambda \) into four edges.

Using the fact that \( F_\lambda \) is conjugate to \( z \mapsto z^2 \) in \( B_\lambda \), and that this conjugacy extends to \( \beta_\lambda \), there exist four disjoint smooth curves, \( \gamma_j \) for \( j = 0, 1, 2, 3 \), connecting each of the critical points \( c_j \) to \( \infty \) in \( B_\lambda \). The \( \gamma_j \) are the external rays landing at \( c_j \). Let \( H_\lambda(z) = \sqrt{\lambda} z \). One checks easily that the two involutions \( H_\lambda \) interchange \( B_\lambda \) and \( T_\lambda \) and satisfy \( F_\lambda((H_\lambda(z))) = F_\lambda(z) \).

Let \( \nu_j \) denote the image of \( \gamma_j \) under the involution \( H_\lambda \) that fixes \( c_j \). Then the curve \( \eta_j = \gamma_j \cup \nu_j \) connects 0 to \( \infty \) and meets \( J(F_\lambda) \) only at \( c_j \). Moreover, the \( \eta_j \) are pairwise disjoint (except at 0 and \( \infty \)). Hence these four curves divide the Julia set into four symmetric pieces \( I_0, \ldots, I_3 \) where we assume that \( c_j \in I_j \) but \( c_j \) does not lie in the other three regions. Let \( I_0 \) be the component that contains the repelling fixed point \( p(\lambda) \) that lies in \( \beta_\lambda \). Note that the \( I_j \) are neither open nor closed subsets of \( J(F_\lambda) \).

Since there are no critical points in any of the preimages of the trap door, it follows that each of its preimages is mapped in one-to-one fashion onto the trap door by \( F_\lambda \). Hence each component of \( F_\lambda^{-k}(\tau_\lambda) \) also has four corners and edges, and each of these corners is mapped by \( F_\lambda^k \) onto a distinct critical point in \( \tau_\lambda \).

To see that \( J(F_\lambda) \) is a Sierpinski gasket-like set, we require the following lemma.

**Lemma.** For \( k \geq 1 \), let \( \tau_\lambda^k \) be the union of all of the components of \( F_\lambda^{-k}(\tau_\lambda) \) and let \( A \) be a particular component in \( \tau_\lambda^k \). Then exactly two of the corner points of \( A \) lie in a particular edge of a single component of \( \tau_\lambda^{k-1} \).
Proof: The case \( k = 1 \) is seen as follows. We have that \( F_\lambda \) maps each \( I_j \) for \( j = 0, \ldots, 3 \) in one-to-one fashion onto all of \( J(F_\lambda) \), with \( F_\lambda(I_j \cap \beta_\lambda) \) mapped onto one of the two halves of \( \beta_\lambda \) lying between two critical values (which, by assumption, are not equal to any of the critical points). Hence \( F_\lambda(I_j \cap \beta_\lambda) \) contains exactly two critical points. Similarly, \( F_\lambda(I_j \cap \tau_\lambda) \) maps onto the other half of \( \beta_\lambda \) and so also meets two critical points. The preimages of these latter two critical points in \( \tau_\lambda \) are precisely the corners of the component of \( \tau_\lambda \) that lies in \( I_j \). Thus we see that each component in \( \tau_\lambda \) meets the boundary of one of the \( I_j \)'s in two points lying in \( \beta_\lambda \) and two points lying in \( \tau_\lambda \). In particular, two of the corners lie in the edge of \( \tau_\lambda \) that meets \( I_j \).

Now consider a component in \( \tau_\lambda^k \) with \( k > 1 \). \( F_\lambda^k \) maps each component in \( \tau_\lambda^k \) onto \( \tau_\lambda \) and therefore \( F_\lambda^{k-1} \) maps the components in \( \tau_\lambda^k \) onto one of the four components of \( \tau_\lambda^1 \). Since each of these four components meets a particular edge of \( \tau_\lambda \) in exactly two corner points, it follows that each component of \( \tau_\lambda^k \) meets an edge of one of the components of \( \tau_\lambda^{k-1} \) in exactly two corner points as claimed.

\( \square \)

We may now show the Julia set of an MS map is a gasket-like set as follows. Let \( K_0 = \overline{\mathbb{C}} - B_\lambda \) and \( K_1 = K_0 - T_\lambda \). Then \( K_1 \) consists of the union of the four sectors \( I_j \) which are mapped in a one-to-one fashion onto \( K_0 \). Define recursively the sets \( K_{n+1} = K_n - F_\lambda^{-n}(T_\lambda) \). Each \( K_n \) is a nested collection of closed and connected subsets of the Riemann sphere with exactly \( 4^n \) generalized rectangles removed at each \( n \)th step. Moreover, the above lemma shows that for each \( n \), the removed rectangles satisfy the two corner restriction given in the definition of gasket-like sets. Is not hard to see that \( \cap_{n=0}^\infty K_n \) coincides with \( J(F_\lambda) \) and hence is a Sierpinski gasket-like set.

### 7.2 Homeomorphisms Between Julia Sets

Before proceeding with the discussion of homeomorphisms between Julia sets of MS maps, we provide a topological characterization of the critical points and the corners of every \( \tau_\lambda^k \). Proofs of the following propositions may be found in [9].

**Proposition.** (The Disconnection Property.) The four corners of the trap door are the only set of four points in the Julia set whose removal disconnects \( J(F_\lambda) \) into exactly four components. Any other set of four points removed from \( J(F_\lambda) \) will yield at most three components.
Clearly the corners of each component $A$ in $\tau^k_\lambda$ inherit the disconnection property when restricted to the largest connected component of $\tau^{k-1}_\lambda$ that contains $A$. Any homeomorphism between Julia sets of MS maps must then preserve this topological invariant as described in the following result.

**Proposition.** Suppose $F_\lambda$ and $F_\mu$ are MS maps. If there exists a homeomorphism $h : J(F_\lambda) \to J(F_\mu)$, then

1. The map $h$ takes the corners of $F^{-k}_\lambda(\tau_\lambda)$ to the corners of $F^{-k}_\mu(\tau_\mu)$ when $k \geq 0$.

2. For $k \geq 1$, each component of $F^{-k}_\lambda(\tau_\lambda)$ is mapped to a unique component of $F^{-k}_\mu(\tau_\mu)$.

For a proof of this result, see [9].

Suppose $\lambda$ and $\mu$ are given parameters that correspond to MS maps of the degree four family. Unless these parameters are complex conjugate, the Theorem in the previous section states their Julia sets are not homeomorphic. To prove this assertion, we have developed a recursive algorithm based solely on the configuration of the corners of a finite number of preimages of $\tau_\lambda$ along $\beta_\lambda$. The configuration is completely determined by the itinerary associated to the finite critical orbit. If the itineraries for $\lambda$ and $\mu$ disagree at the $(n+1)^{\text{st}}$ entry, then the algorithm shows that the corner configurations of $\tau^n_\lambda$ and $\tau^n_\mu$ differ along the respective boundaries of the basin at infinity. Hence there is no homeomorphism between these Julia sets. We illustrate this algorithm with the two examples given in Figure 15.

Using the partition given by the sectors $I_j$ we define the itinerary of a point $z \in J(F_\lambda)$ as the infinite sequence $S(z) = (s_0s_1s_2\ldots) \in \{0,1,2,3\}^\mathbb{N}$ defined in the natural way by its orbit in the regions $I_j$. Hence the itinerary of the accessible fixed point $p_\lambda$ is $\overline{0} = (000\ldots)$, the itinerary of $-p_\lambda$ is $2\overline{0} = (2000\ldots)$, and so forth.

By assumption the itinerary of any critical point of a MS map ends with an infinite string of 0's. Due to the four-fold symmetry and the existence of a unique free critical orbit, we will only concentrate on the itinerary of the critical point $c_\lambda$ that lies in the first quadrant.

The two examples displayed in Figure 15, with $\lambda \approx -0.36428$ and $\mu \approx -0.01965 + 0.2754i$, correspond to the landing points of external rays with arguments 1/2 and 1/4 respectively. The extension of the Landing Theorem to the case of our rational families implies that the external rays of the
same argument must land in the dynamical plane at the second iterate of the critical point. Thus, the itinerary of $F^2_\lambda(c_\lambda)$ is $(2\overline{0})$ and the itinerary of $F^2_\mu(c_\mu)$ is $(12\overline{0})$. It follows that the itinerary of $c_\lambda$ is $(112\overline{0})$ while the itinerary of $c_\mu$ is $(1112\overline{0})$.

Since these itineraries differ at the third entry, we only need to look at the configuration of the corners of the second preimage of the trap door.

We start with the case $\lambda \approx -0.36428$. The ray 1/8 lands at the critical point $c_1 = c_\lambda$. By symmetry, the ray 7/8 lands at $c_0$. Thus, the preimages of $c_1$ and $c_0$ in $I_0$ are landing points of the rays 1/16 and 15/16 respectively. Note that these points are two corners of the component of $\tau^1_\lambda$ that lies in $I_0$. The remaining two corners of this component lie in the arc of $\tau_\lambda$ contained in $I_0$ and are mapped onto the critical points $c_2$ and $c_3$.

By four-fold symmetry, we can compute the external rays landing on the corners of each component of $\tau^1_\lambda$ in each remaining sector $I_j$ by adding a proper multiple of $\pi/2$. In particular, two corners of the component of $\tau^1_\lambda$ in $I_1$ correspond to landing points of the rational rays 5/16 and 3/16.

Now we compute the configurations of the components of $\tau^2_\lambda$. For our purposes it suffices to find the configuration of the corners of $B \subset \tau^2_\lambda$ lying along the arc $\gamma \subset \beta_\lambda$ bounded by the rays 1/16 and 1/8. Under $F_\lambda$, $\gamma$ is mapped onto an arc bounded by rays 1/8 and 1/4. Since the ray 3/16 lands at a corner of the component of $\tau^1_\lambda$ in $I_1$, this implies that the ray 3/32 lands at a corner of the component $B$ in $\tau^2_\lambda$ along $\gamma$. A similar analysis can be done to compute the locations of the remaining three corners of $B$. See Figure 16.

For the case $\mu \approx -0.01965 + i0.2754$, let $c_1 = c_\mu$ be the critical point lying in the first quadrant which is the landing point of the ray 1/16. By symmetry, the ray 13/16 lands at $c_0$. Hence the first preimages of $c_1$ and $c_0$ in $I_0$ are landing points of the rays 1/32 and 29/32 respectively. We may compute the external rays of the remaining corners in $\tau^1_\mu$ by addition of a multiple of $\pi/2$ as before. In particular the external rays landing at corner points of $\tau^1_\mu$ in $I_1$ are 9/32 and 5/32.

Let $\gamma$ denote the arc of $\beta_\mu$ bounded by the landing points of the rays 1/16 and 1/32. Then $\gamma$ is mapped onto the arc bounded by the rays 1/8 and 1/16. In this case, the image of $\gamma$ fails to contain a corner point of $\tau^1_\mu$ in $I_1$ as $1/16 < 5/32$. This implies that there is no corners of the component $B$ in $\tau^2_\mu$ along $\gamma$. See Figure 16.

The previous proposition implies that a homeomorphism between $J(F_\lambda)$ and $J(F_\mu)$ must preserve the configurations shown in Figure 16, which is
impossible. Therefore these Julia sets cannot be homeomorphic.

Figure 16: A schematic representation of the Julia set $J(F_\lambda)$ and $J(F_\mu)$, respectively, up to second preimage of the trap door. For clarity, only the relevant rational rays and certain preimages of the trap door in sectors $I_0$ and $I_1$ have been displayed.

References


