Blowup Points and Baby Mandelbrot Sets for Singularly Perturbed Rational Maps

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Abstract. In this paper we describe the behavior of the family of rational maps of the form

$$f_\lambda(z) = \lambda\left(z + \frac{1}{z}\right).$$

We show that, in every neighborhood of the origin in parameter space, there are infinitely many copies of the Mandelbrot set as well as infinitely many "blowup" points, i.e., parameters for which the critical orbits map to ∞ so the Julia set is the entire plane.

1. Introduction

In recent years, much attention has been paid to families of rational maps that arise as singular perturbations of polynomials. These are families of rational maps that depend on a parameter λ and have the property that, when $\lambda = 0$, the map involved is a polynomial of degree n, but for all other parameters, the maps are rational with higher degree. As λ becomes nonzero, the Julia sets of these maps usually undergo a significant transformation.

Most of the work on these singularly perturbed rational maps has centered on families of the form

$$G_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

where $\lambda \in \mathbb{C}$ and n and d are positive integers. When $\lambda = 0$, these maps reduce to the special case $z \mapsto z^n$. So G_0 is a polynomial of degree n, there is a superattrating fixed point at the origin (when $n \geq 2$), and the Julia set is the unit circle. When λ becomes nonzero, the degree of G_λ increases from n to $n+d$, the origin becomes a pole, and the Julia sets change dramatically.

For example, if $n, d \geq 2$ (but n and d are not both equal to 2), results of McMullen [**4**] imply that there is an open neighborhood $\mathcal{M}$ of the origin in parameter space having the property that if $\lambda \in \mathcal{M}$ but $\lambda \neq 0$, then the Julia set of G_λ is a Cantor set of simple closed curves rather than a single circle.

The cases of low values of n and d are quite different. For example, if $n = d = 2$ or if $d = 1$ but $n > 1$, then it is known [**1**] that there are infinitely many parameters in any neighborhood of $\lambda = 0$ for which the Julia set of G_λ is a Sierpinski curve. A Sierpinski curve is a set that is homeomorphic to the well known Sierpinski carpet

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fractal. So all of these Julia sets are homeomorphic. However, it is known that the dynamics of G_λ on each of these sets is quite different in the sense that no two of the maps are conjugate on their Julia sets.

Our goal in this paper is to investigate another special case of these maps, namely when $n = d = 1$. For technical reasons discussed below, we change the form of our family slightly to consider the family

$$f_\lambda(z) = \lambda\left(z + \frac{1}{z}\right)$$

instead of G_λ. For this family, the singular perturbation away from $\lambda = 0$ results in very different behavior for the maps. One difference is that, when $n \geq 2$, the point at ∞ is always a superatttracting fixed point for G_λ. So there is always a basin of ∞ for these maps and hence the Julia sets in these cases can never be the whole plane. This is not the case for f_λ, as it has been shown (see [**3**], [**8**]) that there are infinitely many parameters in this family for which the Julia set is the entire plane.

It is known that, if $|\lambda| > 1$, then the Julia set of f_λ is a Cantor set on which f_λ is conjugate to the shift map on two symbols [**9**]. Also, as we show below, for this family, the open disks of radius $1/2$ centered at $\pm 1/2$ in the parameter plane each contain parameter values for which the dynamics of f_λ is stable and the corresponding Julia sets are all simple closed curves passing through both ∞ and 0. Since $\lambda = 0$ lies on the boundaries of these two disks, it follows that most singular perturbations away from $\lambda = 0$ result in relatively understandable changes in the Julia sets. However, when λ moves away from the origin in the positive or negative imaginary directions, the situation is quite different. Our goal is to investigate the structure of both the dynamical and parameter planes for singular perturbations in these directions. Our main result is:

Theorem. *In any neighborhood of the origin in parameter plane:*

- *There are infinitely many parameter values for which the Julia set of f_λ is the entire plane;*
- *There are infinitely many copies of Mandelbrot sets in any neighborhood of $\lambda = 0$. If λ lies in one of these sets, then there are subsets of the Julia set of f_λ that are homeomorphic to the Julia set of the quadratic polynomial that corresponds to the given parameter in the Mandelbrot set.*

The parameter plane for f_λ and a magnification are displayed in Figure 1. In the magnification note that there are several Mandelbrot sets in the regions between the two circles of radius $1/2$ centered at $\pm 1/2$.

2. Preliminaries

Throughout this paper we restrict attention to the family

$$f_\lambda(z) = \lambda\left(z + \frac{1}{z}\right)$$

where $\lambda \neq 0$ is complex. The *Julia set* of f_λ is defined to be the set of points at which the family of iterates of f_λ is not a normal family in the sense of Montel. Equivalently, it is known that the Julia set is the closure of the set of repelling periodic points of f_λ. We denote the Julia set by $J = J(f_\lambda)$.

For each λ, the map f_λ has two critical points given by ± 1. The critical values are $\pm 2\lambda$. Since $f_\lambda(-z) = -f_\lambda(z)$, it follows that the orbits of these critical points

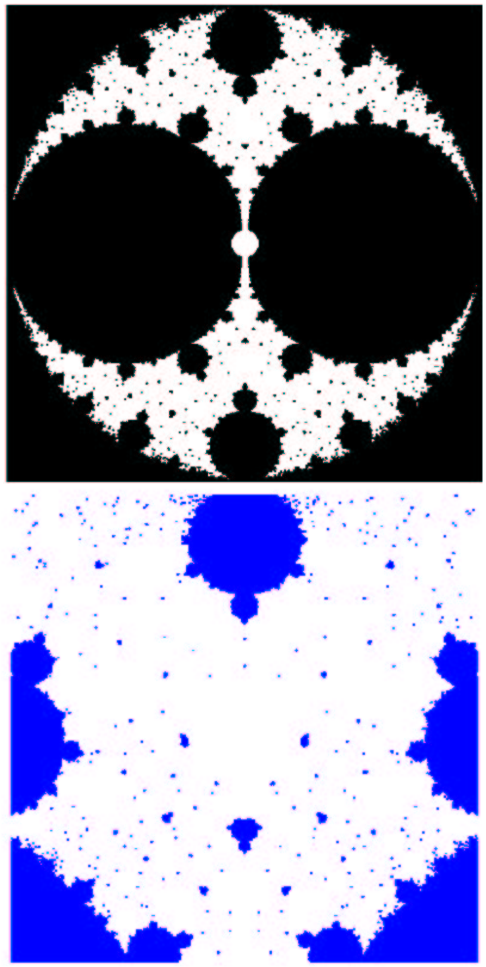

FIGURE 1. The parameter plane and a magnification for the family f_λ.

are symmetric with respect to $z \mapsto -z$. The orbits of the critical points are called the *critical orbits*. As is well known, the behavior of the critical orbits of a complex map determine to a large extent the dynamics of the map on the whole Riemann sphere. For this reason, we define the function $g_n : \mathbb{C} \to \hat{\mathbb{C}}$ via

$$g_n(\lambda) = f_\lambda^n(1) \tag{2.1}$$

Each function g_n is defined on parameter space and gives the location of the n^{th} point on the critical orbit in the dynamical plane.

The map f_λ has several useful mapping properties that we will exploit later. First, a straightforward computation shows that the unit circle is mapped in two-to-one fashion to the straight line segment connecting the critical values $\pm 2\lambda$ and that any other circle centered at the origin is mapped one-to-one onto an ellipse that surrounds this line segment. We call the unit circle the *critical circle* and denote it by C_λ. The image of C_λ is the *critical segment* and we denote it by S_λ. Both the interior and the exterior of the critical circle are mapped one-to-one over the entire

Riemann sphere minus the critical segment. Thus, every open set not intersecting the critical segment has a two preimages under f_λ: one inside the critical circle and one outside.

Note that, if $|\lambda| > 1$, the point at ∞ is an attracting fixed point of f_λ, whereas if $|\lambda| < 1$, ∞ is a repelling fixed point. In constrast, the family $G_\lambda(z) = z + \lambda/z$ is actually conjugate to G_1 for all λ and therefore, ∞ is always a neutral fixed point for G_λ. This motivates our selction of the modified family f_λ for study as opposed to the original family G_λ.

The following result appears to be well known (see [**7**]), but we include a partial proof here for completeness.

Theorem. *For the family f_λ:*

(1) *If $|\lambda| > 1$, the Julia set of f_λ is a Cantor set and f_λ is conjugate to the shift map on two symbols on $J(f_\lambda)$;*
(2) *If λ lies the open disk of radius $1/2$ centered at $1/2$ (resp., $-1/2$) in the parameter plane, then f_λ admits a pair of attracting fixed points (resp., an attracting two cycle), and $J(f_\lambda)$ is a simple closed curve passing through both ∞ and the origin that forms the boundaries of the two basins of these points.*

Proof: The proof of part 1 may be found in [**9**]. For part 2, we note that f_λ has fixed points at

$$z_\pm = \pm\sqrt{\frac{-\lambda}{\lambda - 1}}$$

and $f'_\lambda(z_\pm) = 2\lambda - 1$. So f_λ has an attracting fixed point when λ lies in the open disk of radius $1/2$ centered at $1/2$. If λ is real, then

$$f_\lambda(iy) = i\lambda\left(y - \frac{1}{y}\right).$$

Hence f_λ is two-to-one on the imaginary axis. Therefore the imaginary axis is completely invariant under f_λ and so this axis serves as the Julia set when $0 < \lambda < 1$. If λ lies in the open disk of radius $1/2$ about $1/2$, then standard arguments show that any two such maps are quasiconformally conjugate, and so the Julia sets are all quasicircles that necessarily contain the repelling fixed point at ∞ as well as its preimage at 0.

□

Therefore the dynamical behavior of f_λ is completely understood in the regions depicted in Figure 2. In the remainder of this paper, we will concentrate on the behavior of f_λ when λ is drawn from the two complementary regions in parameter plane.

3. Blowup Points and Superstable Parameters

In this section we prove the existence of a pair of special sequences of parameter values that lie along the imaginary axis in parameter space and converge to the origin. These sequences consist of blowup points and superstable parameters. A parameter value λ is a *blowup point of order n* if one and hence both of the critical orbits lands on zero after n iterations, i.e., if $f_\lambda^n(\pm 1) = 0$ or equivalently if $g_n(\lambda) = 0$. It is well known that such a parameter corresponds to a map whose Julia set is the entire plane [**6**]. A parameter value λ is *superstable* if one and hence both critical

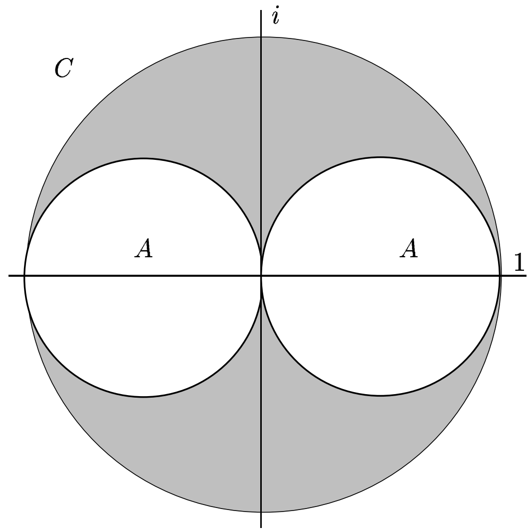

FIGURE 2. In the region marked C, $J(f_\lambda)$ is a Cantor set; In the regions marked A, $J(f_\lambda)$ is a simple closed curve passing through ∞ and 0.

points are periodic. In the following sections we will show that there exists an infinite sequence of baby Mandelbrot sets converging to each of the blowup points. Some of the superstable parameters will be the centers of the main cardioids of these Mandelbrot sets.

We select the imaginary axis for study since parameters on this axis have the desirable property that the real axis is invariant under the second iterate of f_λ. For simplicity, when λ is real, we henceforth denote the second iterate of $f_{i\lambda}$ by F_λ. We then have

$$F_\lambda(z) = f_{i\lambda}^2(z) = -\lambda^2\left(z + \frac{1}{z}\right) + \frac{1}{z + \frac{1}{z}} \tag{3.1}$$

where $\lambda \in \mathbb{R}$.

The graphs of this function for several λ values are shown in Figure 3. Note that, when $\lambda = 0$, the function f_0 vanishes identically, but the second iterate map does not vanish but rather is given by

$$F_0(z) = \frac{1}{z + \frac{1}{z}}.$$

In direct analogy with equation (2.1), we define a family of functions $G_n : \mathbb{R} \to \mathbb{R}$ by

$$G_n(\lambda) = F_\lambda^n(1).$$

Note that λ values for which $G_n(\lambda) = 0$ are blowup points of order $2n$ for the original map. As in the case of g_n, G_n is a rational map. However, unlike g_n, which is defined and continuous everywhere in the Riemann sphere, G_n is only defined on $\mathbb{R}$ and so is discontinuous at blowup points of order less than $2n$. We will use this fact to prove the following lemmas.

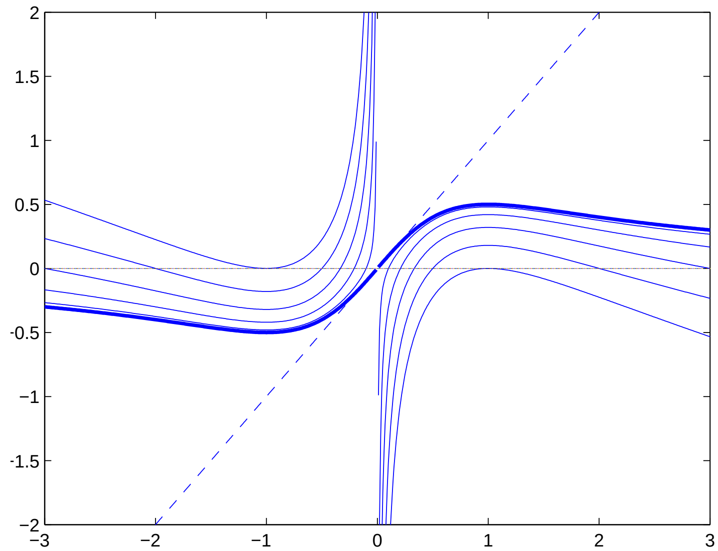

FIGURE 3. The graphs of the function $F_\lambda(x)$ for $\lambda = 0, 0.1, 0.2, 0.3, 0.4, 0.5$. For $x > 0$, $F_{0.1}(x) > F_{0.2}(x) > \cdots > F_{0.5}(x)$. Note that, for $x \neq 0$, as $\lambda \to 0$, these functions converge to F_0, which is well-defined at 0.

LEMMA 3.1. There exists an infinite decreasing sequence of purely imaginary parameter values $i\lambda_n$ such that $\lambda_n \to 0$ as $n \to \infty$ and:

(1) $G_n(\lambda_n) = 0$, i.e., λ_n is a blowup point of order $2n$;
(2) $G_n(\lambda) \in (0, 1/2)$ for $\lambda \in (0, \lambda_n)$;
(3) $G_n'(\lambda) \in (0, \infty)$ for $\lambda \in (0, \lambda_n]$ where the derivative here is with respect to λ.

PROOF. We prove this via induction. The $n = 1$ case is clear, for here we may compute G_1 explicitly as $G_1(\lambda) = -2\lambda^2 + 1/2$. Therefore $\lambda_1 = 1/2$. Moreover, $G_1(\lambda) \in (0, 1/2)$ if $\lambda \in (0, \lambda_1)$ and $G_1'(\lambda) = -4\lambda \neq 0$ for $0 < \lambda \leq \lambda_1 = 1/2$.

For the general case, we assume that there exists $\lambda_{n-1} \in (0, 1/2)$ with

- $G_{n-1}(\lambda_{n-1}) = 0$;
- $G_{n-1}(\lambda) \in (0, 1/2)$ for $\lambda \in (0, \lambda_{n-1})$;
- $G_{n-1}'(\lambda) \in (-\infty, 0)$ for $\lambda \in (0, \lambda_{n-1}]$.

Let $\lambda \in (0, \lambda_{n-1}]$. We compute

$$G_n(\lambda) = -\lambda^2(H_{n-1}(\lambda)) + \frac{1}{H_{n-1}(\lambda)}$$

where

$$H_j(\lambda) = G_j(\lambda) + \frac{1}{G_j(\lambda)}.$$

Since $G_{n-1}(\lambda) \in (0, 1/2)$, it follows that $H_{n-1}(\lambda) \in (2, \infty)$ for all $\lambda \in (0, \lambda_{n-1})$. Consequently, we have

$$H'_{n-1}(\lambda) = G'_{n-1}(\lambda)\left(1 - \frac{1}{G^2_{n-1}(\lambda)}\right).$$

Therefore $H'_{n-1}(\lambda) > 0$. We can therefore compute that

$$G'_n(\lambda) = -2\lambda H_{n-1} - \lambda^2 H'_{n-1} - \frac{1}{H^2_{n-1}} H'_{n-1} = -2\lambda H_{n-1} - H'_{n-1}\left(\lambda^2 + \frac{1}{H^2_{n-1}}\right).$$

However, using easy estimates on H and H', we conclude that $G'_n(\lambda) < 0$ for all $\lambda \in (0, \lambda_{n-1})$, which establishes assertion three in the lemma.

Now note that $G_n(\lambda_{n-1}) = F_{\lambda_{n-1}}(G_{n-1}(\lambda_{n-1}) = F_{\lambda_{n-1}}(0) = -\infty$. Also, $G_n(0) > 0$. So G_n maps the interval $[0, \lambda_{n-1})$ monotonically over $(-\infty, G_n(0)]$ and thus there exists a unique $\lambda_n \in (0, \lambda_{n-1})$ such that $G_n(\lambda_n) = 0$. The second assertion now follows directly. Furthermore, this sequence converges to zero. If this were not the case, then there would exist a nonzero λ-value whose critical orbit was always positive and decreasing which is clearly impossible given the nature of the equation (3.1). This completes the proof.

□

We have thus established that there are a countable number of parameters on the imaginary axis for which the Julia set is the entire Riemann sphere. We now proceed to show that, between any two of these parameter values, there are superstable parameters as well as other blowup points. We prove these assertions for arbitrary blowup points on the imaginary axis whose dynamics may be different from those described in Lemma 3.1.

LEMMA 3.2. Between any two blowup points of differing orders there exists a superstable λ-value.

PROOF. Let λ_n and λ_m be blowup points of order $2n$ and $2m$. Assume $n < m$. For fixed m there are a finite number of discontinuities of G_m in the interval (λ_m, λ_n). Furthermore, these discontinuities represent blowup points of order less than $2m$. Therefore, we restrict ourselves to a subinterval on which G_m is continuous and note that the result holding here is sufficient to establish the result in the general setting. Thus, without loss of generality, we assume that G_m is continuous on (λ_m, λ_n). Therefore, $G_m(\lambda_n) = \infty$ and $G_m(\lambda_m) = 0$. Consequently, by continuity of G_m, there exists a $\lambda_p \in (\lambda_m, \lambda_n)$ such that $G_m(\lambda_p) = 1$. This yields the superstable parameter value.

□

LEMMA 3.3. Between any blowup point and any superstable parameter there exists another blowup point.

PROOF. Let λ_p be the value of λ for which the critical orbit is periodic. Let λ_n be the value of λ for which the critical orbit lands on zero after n iterations of F_{λ_n} (or $2n$ iterations of f_{λ_n}). We will break this argument up into three cases. Without loss of generality, assume $\lambda_p < \lambda_n$.

Case 1: $n < p$. We prove this by contradiction. We assume that there are no such points between λ_p and λ_n. Recall that $G_m(\lambda)$ is continuous everywhere except at blowup points of order less than $2m$. Thus, we may assume that G_m is continuous on (λ_p, λ_n) for all m. So $G_p(\lambda_n) = \infty$ and $G_p(\lambda_p) = 1$. Therefore, by continuity of G_m there exists a $\lambda \in (\lambda_p, \lambda_n)$ for which the critical point is a preimage of zero. This contradicts our assumption that there are no blowup points between λ_p and λ_n.

Case 2: $p < n$. Choose $k \in \mathbb{Z}^+$ so that $kp > n$. Then a similar argument to Case 1 holds with the contradiction occurring after kp iterations.

Case 3: $p = n$. Again assume continuity of G_m for all m and repeat the argument detailed in Case 1, with $G_p(\lambda_n) = 0$.

□

We may now state the main theorem of the section.

Theorem. *On the imaginary axis, there exists a countably infinite set of parameter values that are blowup points as well as a countably infinite set of superstable parameter values. Both sets accumulate on $\lambda = 0$.*

PROOF. Lemma 3.1 establishes a countably infinite set of blowup points accumulating on zero. Repeated applications of Lemma 3.2 yields the set of superstable parameters.

□

Remark. Note that Lemma 3.3 was not necessary for establishing the theorem, but this result can be used to construct even more blowup points and hence, more superstable parameters.

4. Polynomial-Like Maps about $\lambda = i/2$

In this section, we restrict attention to the parameter value $\lambda = i/2$. For this parameter value, both critical orbits land on 0 after two iterations, so $i/2$ is a blowup point. Our goal is prove the existence of an infinite collection of small Mandelbrot sets inside any neighborhood of the parameter value $\lambda = i/2$. To do this, we invoke the theory of polynomial-like maps as derived in Douady and Hubbard [**2**].

Definition. *A map $h : \mathbb{C} \to \mathbb{C}$ is said to be polynomial-like of degree d if there exist open, simply connected subsets $U, U' \subset \mathbb{C}$ with U' relatively compact in U such that $h : U' \to U$ is an analytic map which is proper of degree d.*

With this definition in hand, we invoke a theorem due to Douady and Hubbard [**2**] to establish the existence of Mandelbrot sets.

THEOREM 4.1. [**2**] Let $W \subset \mathbb{C}$ be an open, simply connected set in parameter space such that the one-parameter family of maps $\{h_\mu : U'_\mu \to U_\mu \,|\, \mu \in W\}$ are each polynomial-like maps of degree two. Let c_μ denote the (unique) critical point of h_μ that lies in U'. Suppose that:

(1) The sets U'_μ and U_μ depend continuously on μ;
(2) On the boundary of W, $h_\mu(c_\mu) \cap U'_\mu = \emptyset$;
(3) The winding number of $h_\mu(c_\mu) - c_\mu$ as μ wraps around the boundary of W is one.

Then there exists a homeomorphic copy of the Mandelbrot set lying in W.

Given any $\epsilon > 0$, we will produce an open, simply connected set W_ϵ inside the ϵ ball about $i/2$ having the property that, for each n sufficiently large, the function f_λ^n

satisfies the hypotheses of the Theorem. Thus, for each such n, the set W_ϵ contains a distinct copy of the Mandelbrot set, and we conclude that any neighborhood of $i/2$ contains infinitely many copies of Mandelbrot sets that necessarily converge to $\lambda = i/2$.

Before turning our attention to the proof of this result, we prove a preliminary fact. Let $W_\epsilon = W_\epsilon(i/2)$ denote the open ball of radius ϵ about the parameter value $i/2$. We have that W_ϵ is a subset of parameter space whereas we recall that $g_j(W_\epsilon)$ lies in the dynamical plane.

LEMMA 4.2. There exists $\epsilon > 0$ such that for all $\lambda \in W_\epsilon$:

(1) $T_\epsilon = g_2(W_\epsilon)$ does not contain either of the critical values of f_λ;
(2) $f_\lambda^{-2}(T_\epsilon) \cap S_\lambda = \emptyset$ where $f_\lambda^{-2}(T_\epsilon)$ denotes the preimage of T_ϵ that surrounds the critical point 1;
(3) g_2 and g_3 are both one-to-one on W_ϵ.

PROOF. Recall that $g_1(i/2) = i$ and $g_2(i/2) = 0$ so, by continuity of g_1 and g_2, we can find an ϵ_1 such that $g_1(W_{\epsilon_1}) \cap g_2(W_{\epsilon_1}) = \emptyset$. Hence $T_{\epsilon_1} = g_2(W_{\epsilon_1})$ does not contain any critical values of f_λ when $\lambda \in W_{\epsilon_1}$.

For part 2 we first note that the critical segment $S_{i/2}$ lies along the imaginary axis whereas $f_{i/2}^{-2}(T_{\epsilon_1})$ is a neighborhood of 1. We may choose ϵ_1 small enough so that these two sets are disjoint. Since $f_\lambda(z)$ depends continuously on both λ and z, we may then choose $\epsilon_2 \leq \epsilon_1$ so that $f_\lambda^{-2}(T_{\epsilon_2}) \cap S_\lambda = \emptyset$ for all $\lambda \in W_{\epsilon_2}$.

For part 3, by Lemma 3.1,

$$\frac{\partial g_2}{\partial \lambda}(i/2) \neq 0.$$

Hence we can choose $\epsilon_3 \leq \epsilon_2$ small enough so that g_2 is one-to-one on the closure of W_{ϵ_3}. To show that g_3 is also one-to-one, we first compute

$$g_3'(\lambda) = \frac{\partial F}{\partial \lambda}(\lambda, g_2(\lambda)) = \left(g_2(\lambda) + \frac{1}{g_2(\lambda)}\right) + \left(\lambda - \frac{\lambda}{(g_2(\lambda))^2}\right) g_2'(\lambda) \tag{4.1}$$

where the prime as usual denotes differentiation with respect to λ. For λ close enough to $i/2$, g_3' is then arbitrarily close to ∞. Thus we may choose $\epsilon_4 \leq \epsilon_3$ so $g_3'(i/2) \neq 0$ on W_{ϵ_4} and g_3 maps the closure of W_{ϵ_4} one-to-one onto its image. Therefore, if $\epsilon \leq \epsilon_4$, the set W_ϵ has all of the required properties.

□

We henceforth call the set $T_\epsilon = g_2(W_\epsilon)$ the *target*.

Using this lemma we now construct a family of polynomial-like maps for parameter values in a neighborhood of $\lambda = i/2$.

THEOREM 4.3. There exists an ϵ-ball W surrounding $\lambda = i/2$ in parameter space such that for all $\lambda \in W$, f_λ^{n+1} is polynomial-like of degree 2 for all $n \geq N$, where N depends upon ϵ.

PROOF. Let W be the ϵ-ball in parameter space found via Lemma 4.2. Let $T = g_2(W)$ be the target. Notice that T is mapped in one-to-one fashion by f_λ onto a neighborhood of ∞ for all $\lambda \in W$. Call this neighborhood U_λ. Define $V = \cap_{\lambda \in W} U_\lambda$. By continuity V contains a nonempty neighborhood of ∞.

By Lemma 4.2, there are no critical values of f_λ in the interior of T. Therefore, the target has two preimages under f_λ: one surrounding i and one surrounding $-i$.

Both are mapped by f_λ in one-to-one fashion onto T. By convention, we will select T_λ^{-1} to be the connected component of the preimage surrounding i. In turn, the preimage of T_λ^{-1} has only one connected component surrounding 1. Call this set T_λ^{-2}. By Lemma 4.2, this set is disjoint from the critical segment for all λ. As a result, the preimage of this set has two connected components: one strictly inside the unit circle and one outside. Select T_λ^{-3} to be the preimage lying outside the unit circle. We may now proceed inductively to produce a sequence of sets, T_λ^{-j}. For each λ, these sets converge to ∞ since ∞ is attracting under f_λ^{-1}. Hence, for each λ, there exists a minimal $N_\lambda > 0$ such that for all $n \geq N_\lambda$, $\bar{T}_\lambda^{-n} \subset V$. Define $N = \max_\lambda N_\lambda$. We then set $U'_\lambda = T_\lambda^{-k}$ for some $k \geq N$.

We now observe that, for all $\lambda \in W$,

$$f_\lambda^{k+1}(U'_\lambda) = f_\lambda^k(T_\lambda^{-k+1}) = \cdots = f_\lambda^2(T_\lambda^{-1}) = f_\lambda(T) = U_\lambda. \tag{4.2}$$

Note that each T_λ^{-j} is mapped one-to-one onto T_λ^{-j+1} for $j = 1, 3, 4, \ldots, k$. However, when $j = 2$ the critical point 1 lies in T_λ^{-2} and T_λ^{-2} is then mapped two-to-one onto T_λ^{-1}. See Figure 4. Also, T is mapped one-to-one onto U_λ and therefore the polynomial-like map f_λ^{k+1} has degree 2.

Since (4.2) holds for all $k \geq N$ this concludes the proof.

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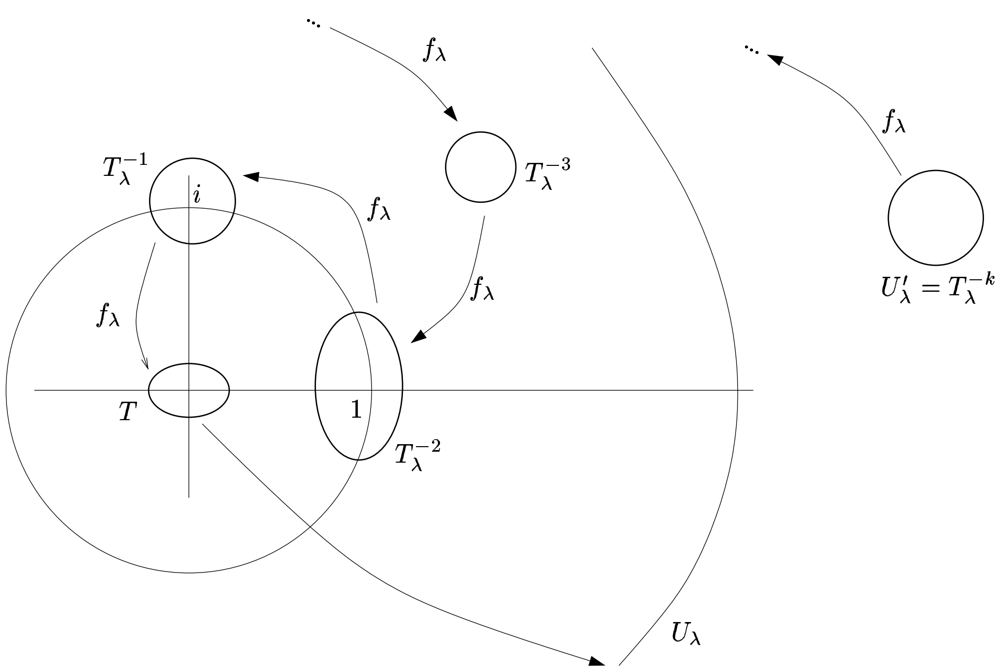

FIGURE 4. The map f_λ^{k+1} takes $U'_\lambda = T_\lambda^{-k}$ two-to-one onto $U_\lambda \supset U'_\lambda$.

COROLLARY 4.4. The family of functions, f_λ^{k+1} for all $\lambda \in W$ defined in Theorem 4.3 satisfy the hypothesis of Theorem 4.1 and hence there exists a Mandelbrot set in W. Furthermore, there exists an infinite sequence of such Mandelbrot sets corresponding to each value of $k \geq N$ in Theorem 4.3.

PROOF. First note that the sets U_λ and U'_λ vary continuously with λ since $f^k_\lambda(z)$ is continuous on the Riemann sphere with respect to both λ and z.

Let $c_\lambda \in U'_\lambda$ be the critical point of the polynomial-like map f^{k+1}_λ. In other words, $c_\lambda = f_\lambda^{-k+2}(1)$. For $\lambda \in \partial W$, $g_2(\lambda) = f^k_\lambda(c_\lambda) \in \partial T$. But $f_\lambda(\partial T) \cap \text{int}\,(V) = \emptyset$ by the definition of V.

Now consider $f^{k+1}_\lambda(c_\lambda) - c_\lambda$ for $\lambda \in \partial W$. Since c_λ always lies inside V and by Lemma 4.2 $g_3(\lambda) = f_\lambda(g_2(\lambda)) = f^{k+1}_\lambda(c_\lambda)$ is one-to-one which implies that the winding number is 1. Therefore we have a family of polynomial-like maps $\{f^{k+1}_\lambda(z) \mid \lambda \in W\}$ which satisfies the hypothesis of Theorem 4.1, so we conclude that we have a 1-1 branched cover of the Mandelbrot set lying inside W. Moreover, we can find a distinct family of such polynomial-like maps for each $k \geq N$, so therefore we have found an infinite family of Mandelbrot sets lying inside W. Now if we let $\epsilon \to 0$, we see that the Mandelbrot sets converge to $\lambda = i/2$.

$\square$.

REMARK 4.5. Not all of these Mandelbrot sets are neccesarily disjoint (i.e., distinct).

5. Baby Mandelbrot Sets for General Blow-up Points

In this section we extend the construction in section 5 to general blowup points. This extension will be valid given several conditions on f_λ at the blowup points. We will show that for the blowup points λ_n on the imaginary axis, these conditions are satisfied.

THEOREM 5.1. Let λ be a blowup point of order n. Then, provided that there exists an open ϵ-ball $W \subset \mathbb{C}$ and target $T = g_n(W)$ for which

(1) $f_\lambda^{-j}(T) \cap v_\lambda = \emptyset$ for all $\lambda \in W$ and $j = 0, 1, \ldots, n-2, n$;
(2) both g_n and g_{n+1} take W one-to-one onto their images;
(3) and there exists an $N > n$ such that, for $k \geq N$, the triple $(f^{k+1}_\lambda, U'_\lambda, U_\lambda)$ form a family of polynomial-like maps of degree two;

then there exists an infinite sequence of Mandelbrot sets converging to the blowup point λ.

To prove this theorem, we use exactly the same construction used in section 4 for the blowup point $\lambda = i/2$. This construction is valid for general blowup points provided that conditions one and two in Theorem 5.1 are satisfied. We note that the first condition can be satisfied for any arbitrary blowup point simply by continuity in a manner analogous to Lemma 4.2. The third condition follows similarly.

We remark that, at this point, we are unable to determine the validity of the second condition for general blowup points. However, for the set of blowup points of the form λ_n generated in Lemma 3.1 both derivative conditions can be easily established in a manner similar to that of Lemma 4.2. Since this sequence converges to the origin in parameter space, we have the following Corollary:

COROLLARY 5.2. There exists a sequence of Mandelbrot Sets in the parameter space of the function $f_\lambda(z) = \lambda(z + \frac{1}{z})$ converging to the origin along the imaginary axis.

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