A Century of Complex Dynamics *

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1 Introduction

Like the MAA, the field of mathematics known as complex dynamics has been around for about one hundred years. Unlike the MAA, complex dynamics has had its ups and downs during this period. While the origins of complex dynamics stretch back into the late 1800s, the foundations of the contemporary study were established in the last years of World War I with the pioneering work of Gaston Julia and Pierre Fatou. Although one hundred years ago complex dynamics was a predominantly French field, there are some important American connections dating back to 1915, with some interesting historical connections to the MAA.

Fatou and Julia continued to explore and expand complex dynamics in the 1920s, but as open questions were successfully addressed, developments slowed. After World War II, aside from a growing body of work by Irvine Noel Baker beginning in the early 1950s concerning the iteration of entire maps, and a few isolated papers, such as those by Hans Brolin and Thomas Cherry in the mid-1960s, interest in the subject dwindled, and to an outside observer the field appeared dormant. This changed dramatically around 1980 with the discovery of the Mandelbrot set when the availability of computers and computer graphics suddenly revealed the beautiful objects that Julia and Fatou could only see in their minds. Throngs of mathematicians (including Fields medalists John Milnor, William Thurston, Jean-Pierre Yoccoz, and Curt McMullen, as well as numerous other eminent individuals) entered the field and complex dynamics was reborn.

In this paper we give a brief overview of the early and later history of the development of complex dynamics, including a discussion of the early American connections. For more historical details see [1], [2], and [3]. We also include a brief description of some of the major results that have come forward during the past century, and we describe briefly some of the dynamical behavior on what are now known as the Julia and Fatou sets, at least for the simplest types of complex functions, namely those with a single free critical orbit.

2 Preliminaries

In complex dynamics, the goal is to understand what happens when an analytic function in the complex plane $\mathbb{C}$ (or the Riemann sphere $\overline{\mathbb{C}}$) is iterated.

Recently, this goal has been expanded to include iteration in $\mathbb{C}^n$ as well, although we will not touch upon this subject in this paper.

Different types of complex analytic functions—polynomials, rational maps, entire transcendental maps, and meromorphic functions—can lead to very different dynamical behaviors. For simplicity we will initially concentrate on polynomials, since many of the basic properties and definitions we describe for polynomials extend to other kinds of functions. We will also sketch some of the different behaviors that arise in other maps towards the end of this paper.

Let P be a polynomial in the complex plane. The goal is to understand the behavior of this function when it is iterated. So let the second iterate of P be $P^2 = P \circ P$ and, inductively, let the n^{th} iterate of P be $P^n = P \circ P^{n-1}$. Given $z \in \mathbb{C}$, then the question is: what happens to the orbit of z, i.e., the sequence of points $z, P(z), P^2(z), \ldots$. Many different types of orbits can occur. For example, the orbit of z could be periodic of period n; that is, for some $n > 0$ we have $P^n(z) = z$. Or it could be eventually periodic, meaning that $P^{j+n}(z) = P^j(z)$ for some $n, j > 0$. The orbit could also tend to ∞ in the plane. And, as we shall see later, there are various other possibilities for the behavior of these orbits.

One of the most important objects in complex dynamics is the Julia set of P which we denote by $J(P)$. This set has several equivalent definitions. Since P is a polynomial, there is an open set surrounding ∞ in the Riemann sphere that consists of points whose orbits simply tend to ∞. This leads to a definition of the Julia set from a geometric point of view: $J(P)$ is the boundary of the set of points whose orbits tend to ∞. From a dynamical systems point of view, the Julia set is also the closure of the set of repelling periodic points. Here a repelling (resp., attracting) periodic point is a point z for which $P^n(z) = z$ and $|(P^n)'(z)| > 1$ (resp., $|(P^n)'(z)| < 1$).

These two equivalent definitions imply that the Julia set is the chaotic set, for arbitrarily close to any point in the Julia set, there are points whose orbits tend to ∞ as well as periodic points whose orbits return to themselves. This is sensitive dependence on initial conditions, the hallmark of chaotic behavior. The complement of the Julia set is called the Fatou set; this is the set where the dynamical behavior is usually quite tame.

From a complex analysis point of view, $J(P)$ is also the set of points in $\mathbb{C}$ at which the family of iterates of P fails to be a normal family in the sense of Montel. This means that, by Montel's Theorem, any neighborhood of a point in $J(P)$, no matter how small, is eventually mapped over the entire complex plane (minus at most one point), which provides us with another way to see that the map P is extremely sensitive to initial conditions on its Julia set.

There are other types of periodic points that will come up later in this paper. A periodic point z of period n is super-attracting if $(P^n)'(z) = 0$. The periodic point is neutral if $(P^n)'(z) = e^{2\pi i\theta}$. When θ is rational, the periodic point is called parabolic (or rationally neutral) and the nearby dynamics are completely understood. But when θ is irrational the periodic point is irrationally neutral, and there are certain θ-values where we still have no idea what happens near z. Finally, a periodic point of period one is called a fixed point.

Before going into more detail about the mathematics of complex dynamics, let's first pause and turn back the clock to see how this field emerged one hundred years ago.

3 Complex Dynamics through 1942

The early study of complex dynamics is dominated by French mathematicians. Nonetheless significant early developments (and perhaps even its origins) occurred elsewhere in Germany, Poland, Italy, Japan and, in the year of the MAA's birth, the United States. In order to set the stage for a discussion of the works of Fatou and Julia—as well as the work of the Americans—we will briefly discuss the origins of the field.

Those curious to know more about the beginnings of complex dynamics should see [1]. To find find out more about the events discussed in this section, also see [2] and [3].

3.1 The Origins

Beginning in 1883 the French mathematician Gabriel Kœnigs wrote a series of papers outlining the local theory of the iteration of a complex analytic function. He proved fundamental results involving the existence of repelling and attracting fixed points and developed a surprisingly robust local theory describing the dynamics of iteration on a neighborhood of an attracting (but not super-attracting) fixed point. Other French mathematicians, including mathematicians on whose dissertation committees he served, soon filled in details regarding the local behavior of super-attracting and rationally neutral fixed points.

One of Kœnigs' primary tools was the Schröder functional equation given by $S \circ f = f'(p) \cdot S$. Given a function f with an attracting (but not super-attracting) fixed point at p, Kœnigs rigorously demonstrated in [29] that an invertible function S exists on a neighborhood of p satisfying the Schröder equation. Since $S \circ f = f'(p) \cdot S$ implies that $f^n(z) = S^{-1}((f'(p))^n \cdot S(z))$, solving the Schröder equation models iteration near p via the linear mapping $z \mapsto f'(p) \cdot z$ on a neighborhood of the origin. That is, in a neighborhood of p, f is analytically conjugate to this linear map. One of the major foci of post-Kœnigs study of iteration was the solution of related functional equations.

Kœnigs, however, was not the first to consider iteration of complex functions in a dynamical context. In 1870–1871, the German mathematician and logician Ernst Schröder (of the Schröder-Bernstein Theorem) wrote two papers [40] and [41] on iterative algorithms for solving equations. His interest was piqued by the Newton's method algorithm $z_{n+1} = z_n - \dfrac{f(z_n)}{f'(z_n)}$ used to approximate solutions to $f(z) = 0$. When things go well, Newton's method generates a sequence $\{z_n\}$ converging to a root of f.

Schröder's curiosity about Newton's method led him into a brief but insightful study of iteration on the complex plane where he discovered the phenomena of attracting fixed points. Viewing Newton's method as the complex function $N_f(z) = z - \dfrac{f(z)}{f'(z)}$, he discovered that a possibly complex root p of f is also a super-attracting fixed point of N_f. This not only explained to Schröder why Newton's method works, but led him to generalize Newton's method and create a family of root-solving algorithms that continues to draw interest today.

Schröder also became interested in the Schröder functional equation. Although he could not solve it for arbitrary f, as Kœnigs did roughly fifteen years later, he used a Schröder equation based on the trigonometric identity $\dfrac{2z}{1+z^2} = -i \tan(2 \cdot \arctan(iz))$ to iterate $N_q(z) = \dfrac{1+z^2}{2z}$, the Newton's method function for the quadratic $q(z) = z^2 - 1$. He showed that on the left (resp., right) half-plane $N_q^n(z) \to -1$ (resp., 1). He observed sensitive dependence on initial conditions on the imaginary axis and called attention to behavior we would now term chaotic.

In the late 1870s Arthur Cayley independently used very different methods to obtain this same result in [8], but his examination did not involve general principles of iteration, as did Schröder's. Buoyed by their successful examinations of Newton's method for the quadratic, Cayley and Schröder each attempted without any success to the find the convergence regions for Newton's method for higher degree polynomials. Both remarked that the obstacles to such a study were quite formidable.

3.2 The Announcement of the 1918 Grand Prize

While neither Kœnigs nor his immediate successors were able to describe iteration in the case where a periodic point p is irrationally neutral, Kœnigs' greatest frustration appeared to be his inability to extend his study beyond the local behavior of iteration near a fixed point, a fact he explicitly lamented at the conclusion of his 1884 paper on iteration.

Things had not improved by 1897, when Leau expressed a similar frustration that he could not find the full domains of solutions to functional equations such as the Schröder equation. Kœnigs wondered aloud whether it was possible to expand the study beyond a neighborhood of a fixed point. Leau thought such an attempt "impractical."

The reasons for Kœnigs and Leau's failure to move their focus beyond the local are in large measure historical. In France, at least, set theory and point set topology were in their infancy. Important tools in analysis had yet to be invented: for example, Montel's theory of normal families, which would prove instrumental to the successes of Fatou and Julia, would not be unveiled for almost another decade. It would then be another ten years before its applicability to complex dynamics would be understood, and even then Julia (and to a lesser extent Fatou) kept this insight under wraps for a bit longer.

In Kœnigs' inability to extend the study of iteration beyond the fixed point are the seeds of the works of Fatou and Julia. Not only did both mathematicians adopt the terminology and techniques of Kœnigs, including the study of functional equations,[1] but his failure to extend knowledge of iteration beyond a neighborhood of a fixed point became a primary motivation for their studies.

Fatou's first published work regarding complex dynamics [13] appeared in 1906, and one of its accomplishments is a description of the global properties

[1] Julia, however, intentionally postponed his use of functional equations until after the appearance of his 1918 monograph.

of iteration for the family of functions $z \mapsto \dfrac{z^k}{z^k + 2}$. He continued to study the iteration of complex functions for several years before publishing again on the subject in 1917 (see [3] for more details about this). There were also other French mathematicians who studied iteration in the early 1900s, most notably Samuel Lattès, who published several papers between 1903–1918 on iteration focusing on the iteration of functions with more than one (possibly complex) variable.

The desire to extend the study of iteration beyond the neighborhood of a fixed point became central in late 1915 when the French Academy of Sciences announced that the 1918 Grand Prize in Mathematics would be devoted to the study of iteration of complex functions. The Academy cited Kœnigs' work and suggested that entrants might want to focus on the iteration of rational complex functions of a single variable. With the prize came 3000 francs, a tidy sum in its time.

While Fatou had already begun a study of iteration almost ten years before the Grand Prize was announced, Julia had little if any previous experience in complex dynamics and was almost certainly inspired by the announcement.

3.3 The Awarding of the Prize

When the contest was announced in 1915, Julia was in the midst of a long recovery from a terrible and disfiguring war injury that he later customarily covered with a nose patch.

Julia had entered the war as an exceptionally promising 21 year-old mathematics student at the École Normale Supérieure and suffered his wound in January 1915 in the battle of the *Chemin des Dames*. His recuperation was long and painful, and the severity of his injuries made it difficult for him to do mathematics for quite some time. However, as his recuperation progressed, he took up mathematics again to resounding success. He read mathematics deeply and in 1917 completed his doctoral thesis, which also earned him the Academy's 1917 Bordin Prize. At some point in late 1916 or early 1917, he decided to enter the competition for the 1918 Grand Prix, and by spring 1917 his work was well underway.

Meanwhile, Fatou had also been hard at work. The deadline for official entries was December 31, 1917, but results were often announced before formally submitting an entry, and in May 1917 Fatou published [14] which contains several preliminary findings that grew out of his 1906 publication.

It is not known if either mathematician had suspected the other was planning to enter the contest before this, but the results Fatou put forth in 1917 evidently startled Julia, who had already independently achieved many of them. At this point, Julia made the tactical decision to submit his own preliminary results to the Academy through a series of sealed letters that would remain unopened until Julia decided otherwise. There was nothing unusual in this, and the Academy even had a special registry dedicated to processing sealed submissions.

By the end of May it seems that neither mathematician had thought to apply Montel's theory of normal families. That changed on June 4 with a short publication by Montel [33]. During its course, Montel applied his theory of normal families to a sequence of functions. Although neither the application nor the sequence had anything to do with iteration, it would be difficult for either Fatou and Julia to look at the sequence that Montel expressed as "$f_1(z), f_2(z), \ldots, f_n(z), \ldots$" and not think of iteration. This publication evidently opened both men's eyes to the potential of normal families. However, it seems that neither knew that the other had had the same insight, at least initially.

Over the next few months both men found the theory of normal families powerful, and each, mostly likely operating in ignorance—but perhaps in suspicion—of what the other was doing, established a series of now fundamental results including the partitioning of the sphere into domains of normality (the Fatou set) and non-normality (the Julia set). While Julia submitted his preliminary findings to the Academy via the sealed letters, Fatou readied the short publication [15] announcing additional preliminary results that appeared on December 17.

Since he had submitted his letters prior to the appearance of Fatou's December 17 announcement, Julia no doubt felt he had established and deserved priority for the results they contained, and so on December 24 he asked the Academy in [23] for a formal priority judgment. Following established procedures, on December 31, the deadline for the contest, the Academy ruled in Julia's favor, saying he had indeed communicated his results before the appearance of Fatou's December 17 announcement.

It is unclear, however, what advantage Julia gained by his tactics—unless his goal was to drive Fatou out of the contest—since Fatou decided not to submit an entry. It was a curious decision on Fatou's part, but he evidently kept his own council, and the reasoning behind it remains a mystery.

Michèle Audin argues in [3] that, had he entered, the Academy would have split the prize between Julia and Fatou, and at the meeting in late December 1918 when the results of the Grand Prize were announced, Fatou did receive a 2000 franc prize for his work in analysis throughout his career. Clearly, the Academy wanted to recognize Fatou, but perhaps they would have chosen the same route and still awarded the Grand Prize to Julia even if Fatou had submitted an entry.

The events surrounding the prize proved controversial, and Audin presents a strong case that they were polarizing. For example, Montel and Lebesgue seemed to have had great sympathy for Fatou. On the other hand, other Parisian mathematicians, particularly Émile Picard, championed Julia.

To complicate matters further, the personalities of Fatou and Julia were quite different. Although Fatou came from a prominent naval family, he suffered from ill health (and perhaps anxiety) much of his life, and consequently did not serve in the military. Despite his friendship with Montel and Lebesgue, he worked as an astronomer at the Paris Observatory rather than as academician in a department of mathematics. Fatou was by nature reticent, and Léon Bloch, a physicist and friend of Fatou, said that Fatou found it difficult to speak in front of an audience, which suggests, perhaps, a reason why he sought work at the Paris Observatory rather than a teaching position. However, this may not be entirely accurate since Fatou evidently applied to Collège de France in the early 1920s.

Perhaps Fatou's career choice and lack of military service in the time of war made him a bit of an outsider. Julia on the other hand was a war hero, and during his recuperation from his battle wounds was often visited by Picard and Georges Humbert (the latter of whom issued the Academy's priority judgement in favor of Julia). Moreover, Julia was a rising young star fifteen years Fatou's junior. Many young French intellectuals had died in the war, Picard's elder son included, and to many in the older generation, Julia represented the future.

It is important to keep in mind, however, that there is no evidence that any of this came into play during the priority judgement or the awarding of the prize. Much of Humbert's report cannot be debated. It stated that Julia's results stemming from the theory of normal families were submitted first, which is true. It claims that the results from Fatou's publication are by and large present in Julia's sealed envelopes, also true. The only matter than can be debated is Humbert's claim that Julia's results are at times more general.

3.4 The Work of Fatou and Julia

Julia's prize entry [24] is an almost 200-page monograph concerning the iteration of rational complex functions of a single variable that was published in 1918. Fatou's monograph [16], well over 200 pages, also focuses almost exclusively on rational functions and was published in three parts beginning in 1919. One assumes that at least part of Fatou's monograph was originally intended to be submitted for the Grand Prize.

Fatou and Julia's monographs collectively form the bedrock of contemporary complex dynamics. Partitioning the plane into domains of normality and non-normality, they exploited the deep connections between Montel's theory of normal families and complex dynamics. Fatou and Julia each understood the topological structure of the Julia and Fatou sets, as well as the dynamics of iteration on each, including the fact that the forward orbit of a neighborhood of a point in the Julia set encompasses the entire sphere, with the exception of at most two points.

Likewise, they each showed that the domain of normality contains zero, one, two or infinitely many components.[2] This last result helps explain the difficulty that Schröder and Cayley had in extending their analysis of Newton's method to the cubic: since each of the three roots of a cubic corresponds to a separate component of the Fatou set, there must be infinitely many, and their methods were simply not up to the task of understanding this.

Fatou and Julia each offered proofs that fractal Julia sets were the norm, not the exception, and explored many now famous examples. One fascinating aspect of their work was their ability to understand what now famous Julia sets looked like. Despite lacking the computational means to visualize such sets, they were able to explain what they perceived using existing examples from mathematics such as the Koch snowflake, which Helge von Koch introduced in 1906. Julia's schematic of the Julia set for $z \mapsto (-z^3+3z)/2$ (which bears some similarity to the left image in Figure 1), is based on the Koch curves, and Fatou invoked Koch as well.

Fatou explored hypothetical regions that he called singular domains, that is, components of the Fatou set on which the family of iterates of f forms a normal family but is not contained in a domain of attraction for a peri-

[2] At the time of Julia's submission, Lattès' example of a function whose Julia set encompasses the entire Riemann sphere was unknown to Julia, although he speculated that such functions quite possibly exist. Once Lattès' result was known, both Fatou and Julia seemed rather nonplussed by it.

odic orbit. He was perfectly candid that he did know whether such regions even exist; nonetheless he established a limit upon them. The reader might recognize these regions as Siegel disks or Herman rings;[3] to Fatou, however, their ultimate character was unknown, and he was careful not to speculate what they might look like. In contrast, Julia doubted the existence of such regions, and in a brief 1919 follow-up to his monograph outlined a proof that Siegel disks could not exist [25]. In the mid-1930s, Julia realized that his argument contained an error, yet this did not seem to shake his confidence that his claim was correct.

While the studies of each man are remarkably similar there are differences. Most striking among them perhaps is Fatou's openness about the possible existence of singular domains whose existence Julia denied. Interestingly, Fatou even remarked upon Julia's denial and seemed not to take it as gospel.

There were also differences in style: Julia wrote in an austere axiomatic style while Fatou's account was looping and discursive, often revisiting ideas, much as a novelist might return to a character many times to better depict her maturation.

3.5 Complex Dynamics in the US: 1915–1917

Perhaps the most stubborn problem Julia and Fatou encountered involved iteration around an irrationally neutral fixed point p, that is, one whose derivative is $f'(p) = e^{2\pi i\theta}$ with θ irrational. We know now that such a fixed point could be in the Fatou set, in which case a Siegel disk exists, or p could be in the Julia set, $J(f)$. The only result that either mathematician stated regarding this case was Julia's mistaken proof that Siegel disks do not exist.

Unbeknownst to them, however, a mathematician in the United States, George Pfeiffer, had already proved a substantial result. In April 1917 he published the paper [35] in the *Transactions of the American Mathematical Society* in which he found conditions on the derivative $f'(p)$ of an irrationally neutral fixed point p that precluded the existence of a convergent solution S to the Schröder equation $S \circ f = f'(p) \cdot S$. In other words, he found conditions which imply that an irrationally neutral fixed point is in $J(f)$.

[3] A Siegel disk is a component of the Fatou set on which the map is conjugate to rotation of a disk and will be discussed in more detail later in this section and in §4. A Herman ring is a component of the Fatou set on which the map is conjugate to a rotation of an annulus. Herman rings will be discussed in §5.

Pfeiffer had already announced this result in presentations to the AMS in October 1915 and April 1916, as well as in a footnote in the 1915 paper [34] on conformal arcs published in the *American Journal of Mathematics.*

His 1917 paper cited the work of Kœnigs as well as others who investigated the iteration of complex functions and the associated functional equations, and explicitly noted that his was the first to produce any definitive result in the case where the derivative of the fixed point $f'(p)$ was an irrational root of unity.

Pfeiffer constructed a function f with an irrationally neutral fixed point at p whose derivative $f'(p)$ satisfies a convoluted recursion relation. Next, he deduced a function $S = \sum_{k=0}^{\infty} s_k(z-p)^k$ which algebraically satisfies $S \circ f = f'(p) \cdot S$ by assuming such a function exists, and then solving for its coefficients. He showed that the denominators of the s_k become quite small as $k \to \infty$ forcing the coefficients to grow quite large which causes S to diverge on any neighborhood of p. In other words, Pfeiffer constructed a function S with small divisors. He remarked in his paper that he had received a helpful (but unspecified) suggestion from George David Birkhoff. Birkhoff was no doubt familiar with small divisors problems in celestial mechanics, and perhaps he gave Pfeiffer advice on treating them.

Pfeiffer observed that he became interested in Schröder equation via the lectures of another American mathematician, Edward Kasner,[4] a founding member of the MAA who taught Pfeiffer at Columbia. Kasner's lectures involved conformal invariants, which link to the Schröder equation, though not in the context of complex dynamics.

In 1918 Pfeiffer published a follow-up [36] to his 1917 paper concerning a related functional equation, $g^2 = f$, where f is given, but was known more as a teacher than a researcher, although he did serve as an editor for the *Annals* in the 1920s. Pfeiffer later taught at Princeton before settling in at Columbia where he taught until his death in 1943.

It is probably not surprising that Julia and Fatou worked in ignorance of Pfeiffer's results. The war no doubt made the transportation of American journals and mathematical ideas problematic, and it is not clear that they even looked to America for help.

[4]Kasner is perhaps most famous for his association with the words "googol" and "googolplex", which he coined, he says, after asking his nephews, who were young children at the time, what they might call a very large number. Others might know him in conjunction with his co-editing of *Mathematics and the Imagination* with James R. Newman.

Nonetheless, there was a burgeoning interest in iteration in America in 1915 which, in addition to Pfeiffer's announcements, saw the publication of papers involving iteration by two other American mathematicians. As was the case with Pfeiffer's announcements, these works also predated the French Academy of Sciences' December 1915 Grand Prize announcement.

The first paper [4] was written by another founding member of the MAA, Albert A. Bennett, and the other [37] by the mathematician, Joseph Fels Ritt, who later published periodically in the MAA circle of magazines and became a life-long member of the MAA in the early 1920s.

Bennett's paper, appearing in the *Annals* in September 1915, came before Ritt's and represents the first American research paper to look at the iteration of complex functions. While Bennett's paper does not contain any important new results, its greatest benefit was an introduction to US readers of the results of Kœnigs and others. Bennett followed up this paper with [5] the next year on the iteration of functions of several variables.

While Pfeiffer traced his interest in functional equations to problems arising out of lectures by Kasner, it is not clear what sparked Bennett's interest in iteration. A 1914 letter written by Oswald Veblen to Birkhoff discussed a conversation he had with Bennett while they were both in Paris in which he had urged Bennett to seek new mathematical directions. Perhaps Bennett's paper is the fruit of that discussion.

Bennett went on to a distinguished mathematical career at Brown after teaching at Princeton. He also served the MAA in several capacities including a member of the Council (equivalent to the current-day Board of Governors), Vice-President, Trustee, and Editor-in-Chief of the *Monthly.* In 1967 he wrote a history of the pre-World War II MAA [6] that appeared in the Fiftieth Anniversary Issue of the *Monthly.* He died in 1971.

Ritt's paper appeared in the *Annals* in December 1915 and concerned the so-called Babbage functional equation, $f^n = f$ for a real function f, an equation that the British logician and mathematician George Babbage examined in early 1800s. This paper was the first of several by Ritt to concern iteration, some of which made lasting contributions to the field, especially his 1923 paper [39] in the *Transactions* on complex permutable functions, that is, functions f and g which satisfy $f \circ g = g \circ f$ which Fatou and Julia had also studied. Setting $g = f^n$, it follows that $f^n \circ f = f \circ f^n$, so permutable functions are linked to the process of iteration.

Ritt, an important American mathematician who enjoyed a long career at Columbia until his death in 1951, was a student of Kasner. He published his first results on the iteration of complex functions in France in early 1918 in the same journal that Fatou published his preliminary results [38]. Since Ritt's interest in iteration stems back to 1915 prior to the announcement of the 1918 Grand Prize, one wonders if he considered submitting an entry to it.

While the American interest in iteration waned, it did not disappear. As we will see, a paper fundamental to the study of complex dynamics was published in the *Annals* in 1942, although the author was not an American mathematician.

3.6 1920-1942

Following their great monographs on the iteration of rational complex functions, Julia and Fatou each studied dynamics well into the 1920s, writing hundreds of pages and over forty publications between them. While none of these works had the majesty of their monographs, there were important works among them.

Beginning in 1919 Julia applied techniques involving normal families honed in his study of iteration to the so-called "curves of Julia," which result from examining the values a function f takes along an angle whose vertex is an isolated essential singularity [26]. In 1922 Fatou examined the dynamics of a particular kind of algebraic function in [17], and in 1926 published a foundational work on the iteration of transcendental functions [20], each of which opened new lines of inquiry. In the early 1920s, both wrote important papers on permutable functions [18], [27] (another topic introduced by Kœnigs!) and explored the iteration of functions of more than one variable [19], [28].

There were others abroad who were inspired by their studies. The renowned Japanese complex analyst, Kiyoshi Oka, became intrigued by complex dynamics in the late 1920s, and even travelled to Paris where he began a long, still unpublished paper on permutable functions that drew upon the studies of Ritt, Fatou and Julia.[5] In Germany, Hubert Cremer steeped himself in Fatou and Julia's monographs and in 1924 gave a presentation at the Mathematics Colloquium at the University of Berlin that introduced their ideas to a German audience.

[5] This paper is available on the web at `http://www.lib.nara-wu.ac.jp/oka/ikou/s19/p000-1.html`. It is speculated, but not documented, that Oka met and perhaps studied with Julia.

Cremer's interest in the subject grew. Beginning in 1927, he took up the study of irrationally neutral fixed points, which he continued through a series of papers over the next decade. Unlike Fatou and Julia, he read and acknowledged Pfeiffer's work, and Cremer's best known result is actually a refinement of Pfeiffer's discovery of (to use the contemporary point of a view) irrationally neutral points p that belong to the Julia set [9].

The conditions that Pfeiffer placed on $f'(p)$ defy concise explanation. It was Cremer's genius to find conditions that can be easily expressed: Let f be a rational function of a single complex variable of degree s with an irrationally neutral fixed point at p. If

$$\liminf_{n=1,2,\ldots} \sqrt[s^n]{|(f'(p))^n - 1|} = 0,$$

then a convergent solution to the associated Schröder equation $S \circ f = f'(p) \cdot S$ does not exist.

A few years later, in connection with his interest in maps of annuli, Cremer obtained another important result in [10], namely, that if a singular domain (a component of the Fatou set that is not part of a domain of attraction) exists for a rational function, its degree of connection is at most two. Moreover, he also showed that doubly connected singular domains could not exist if f was entire. While he did not prove such domains exist, his result serves as an anticipation of Herman rings.

Like Fatou, Cremer remained agnostic towards the existence of Siegel disks or Herman rings throughout his study but seemed skeptical of the validity of Julia's proof that Siegel disks could not exist.

It seems reasonable to assume that Cremer tried to show the existence of irrationally neutral fixed points that were not in the Julia set, that is, that Siegel disks exist. If so, he was unsuccessful, but his work suggests an explicit connection between number theory and the center problem, one that was also implicit in Pfeiffer's paper: Cremer showed that the lim-inf conditions that he imposed upon $f'(p) = e^{2\pi i\theta}$ (stated above) forces θ to be a Liouville number, which are said to be well-approximated by rational numbers.

It turns out that in order for a Siegel disk to exist, the conditions on $f'(p)$ need to be flipped: if θ is "highly irrational," that is, the continued fraction expansion of θ consists of a collection of integers that are bounded above, then a Siegel disk surrounding p exists and p is in the Fatou set.

Indeed, in 1942 Karl Ludwig Siegel, a German mathematician who came to Princeton to escape Nazi Germany, published a remarkably important paper whose slenderness—six pages—belies its impact [43]. Siegel showed that if θ is highly irrational, then a convergent solution to the Schröder equation $S \circ f = f'(p) \cdot S$ exists. In other words, iteration around such an irrationally neutral fixed point is conjugate to an irrational rotation by $2\pi\theta$.

Siegel's construction of S relies on delicate bounds on the coefficients of S whose denominators are quite small. Siegel's solution thus represents a successful resolution a small divisors problem—itself an important achievement. Indeed, Jürgen Moser, who after the war was a student of Siegel's back in Germany, found inspiration in Siegel's work for his own studies of what was to become known as KAM theory.[6]

Siegel's result closed a door on a phase in the development of complex dynamics that began with Fatou's 1906 paper. The center problem in complex dynamics was arguably the most obvious of the problems Fatou and Julia left unresolved in their monographs. While research in the iteration of complex dynamics never completely stopped—soon after the war Paul Charles Rosenbloom published a short paper on fixed points of entire functions and Irvine Noel Baker began his own exploration of entire functions in 1955—it is safe to say that the subject no longer received the attention it had prior to the war, nor would it for quite some time.

4 The Renaissance of Complex Dynamics

While there certainly was some work going on in the field of complex dynamics in the period 1942-79, nothing compares to what happened in 1979. At that time, Benoit Mandelbrot was working at the IBM Thomas J. Watson Research Center, home to some of the most powerful computers of the day. Interestingly, Benoit Mandelbrot had an uncle, Szolem Mandelbrojt, who was also a mathematician. Szolem was a student of Jacques Hadamard and later succeeded him as a Professor at the Collège de France. Mandelbrojt worked in the field of complex analysis and was familiar with the work of Julia and Fatou. He eventually informed Benoit Mandelbrot about the interesting objects that Julia and Fatou had thought about so many years earlier, and so Mandelbrot decided to have a look at these objects using computer graphics. What he saw astounded him (as well as the rest of the mathematical community).

[6] For more about Siegel's solution and its connection to KAM theory, see [2].

Mandelbrot decided to concentrate for simplicity on quadratic polynomials. It is well known that any such quadratic map is dynamically equivalent to one of the form $P_c(z) = z^2 + c$ where c is a complex parameter. Now, when $c = 0$, the Julia set of z^2 is the unit circle; all points outside the unit circle have orbits that tend to ∞, while all points inside the unit circle have orbits that tend to 0, which is therefore an attracting fixed point. Similarly, the Julia set of $z^2 - 2$ is the interval $[-2, 2]$ on the real axis in $\mathbb{C}$, though this is a little harder to prove. It turns out that these are the only two "computable" Julia sets for $z^2 + c$; all other Julia sets for $z^2 + c$ are fractals.

Without going into details, a fractal object is a set that is everywhere self-similar (if you zoom in on the set, you see the same structure over and over again) and that also has the property that its "fractal" dimension (usually the Hausdorff dimension) exceeds its topological dimension. For Julia sets of $z^2 + c$, the topological dimension is just 1 if this set is connected and it is 0 otherwise, but when $c \neq 0, -2$, the fractal dimension is often not an integer. For example, the Julia set for $c = -1$ is the "basilica" and for $c = -.12 + .75i$ it is the "Douady rabbit." See Figure 1. Zooming in to the rabbit shows that the rabbit's ears have ears, and those sub-ears have ears, etc., etc. That is self-similarity. Mandelbrot, the father of fractal geometry, was intrigued.

Mandelbrot plunged more deeply into the quadratic case. Julia and Fatou knew that the Julia set of $z^2 + c$ was either a connected set or else a Cantor set, i.e., a totally disconnected set. There are no quadratic Julia sets that consist of 2 or 20 or 200 components; either the Julia set is one piece or it consists of uncountably many pieces, each of which is a point. And Julia and Fatou also knew that, amazingly, it was the orbit of 0 that determines this: if $P_c^n(0) \to \infty$, then $J(P_c)$ is a Cantor set, but if the orbit of 0 behaves otherwise, $J(P_c)$ is a connected set. The reason that the orbit of 0 determines this is that 0 is the only critical point for $z^2 + c$ and the fate of the "critical orbits" essentially determines everything in complex dynamics, something Fatou and Julia both understood well. (For higher degree polynomials, there are usually more critical orbits and so the structure for these maps is more "complex.")

So Mandelbrot decided to draw the picture of all those c-values in the complex plane for which the orbit of 0 does not tend to ∞. What astonishingly comes out is one of the most famous and most beautiful objects in all of mathematics, the set that now bears his name, the Mandelbrot set. See Figure 2.

The black bulbs visible in the Mandelbrot set each contain parameters for

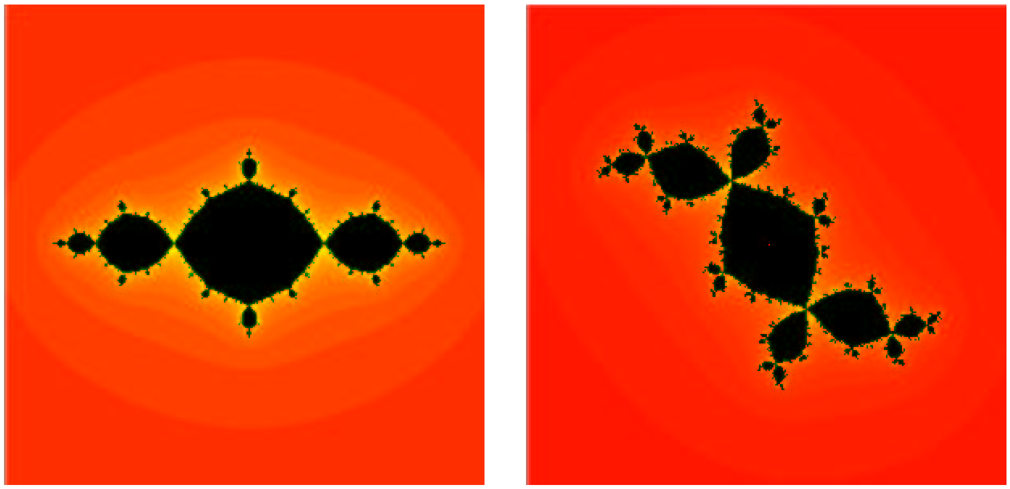


Figure 1: The Julia sets for $z^2 - 1$ (the basilica) and $z^2 - 0.12 + .75i$ (the Douady rabbit). Colored points have orbits that escape to ∞ while black points have orbits that tend to a periodic orbit of period 2 in the basilica case and of period 3 in the rabbit case. So the Julia sets here are the boundaries of the black and colored regions.

which there is an attracting periodic orbit of some given period. For c-values in the large main cardioid, each P_c has an attracting fixed point, and the corresponding Julia set is a simple closed curve. The large bulb to the left of the main cardioid is actually an open disk of radius $1/4$ centered at $c = -1$ and c-values here give rise to an attracting cycle of period 2 (the basilica is the Julia set that arises when c is at the center of this disk). And the two large disks above and below the main cardioid correspond to parameters for which there is an attracting cycle of period 3; the Douady rabbit sits at the center of the northern period-3 bulb.

After the appearance of the Mandelbrot set, many mathematicians jumped in and continued the work of Fatou and Julia. Luckily, the areas of mathematics known as dynamical systems and complex analysis had made important strides forward during the prior fifty years, and many new tools were therefore available to extend the earlier results. One of the most important new results was Sullivan's No-Wandering Domains Theorem [45] from 1985. In this paper Sullivan showed that any component of the Fatou set must be

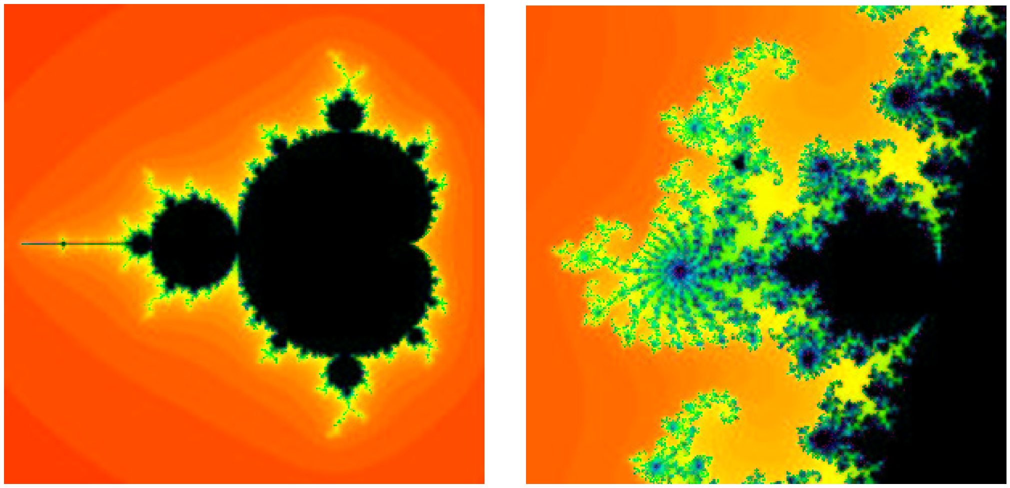


Figure 2: The Mandelbrot set and a magnification. Colored points are c-values for which the orbits of 0 escape to ∞; black points are c-values for which this does not happen. So the Mandelbrot set is the black region in these images.

eventually periodic in the case of polynomials or rational maps.[7] In particular, it then follows that there are only three types of Fatou components in the polynomial case:

1. Attracting basins, in which all points tend to a particular attracting periodic orbit which therefore lies in the Fatou set;

2. Parabolic basins, in which all points tend to a periodic orbit of period n that now lies in the Julia set and for which the derivative of P^n is of the form $\exp(2\pi i(p/q))$;

3. Siegel disks.

Along the boundaries of the bulbs in the Mandelbrot set are the c-values for which P_c has a cycle that is neutral, i.e., the derivative of P^n is of the form $\exp(2\pi i\theta)$. If θ is rational, then we are in the case of a parabolic

[7] Although it is a natural to ask if wandering domains exist, neither Fatou nor Julia seem to have raised this question in their published works.

basin. If θ is "highly irrational" we are in the Siegel disk case. See [32] for the precise technical definitions of highly irrational. But there are certain irrational values of θ, for example the ones Pfeiffer and Cremer found, for which we do not have a Siegel disk. What happens here dynamically is still not understood. Think about this: if the quadratic function $z^2 + c$ has a fixed point whose multiplier is a not-so-irrational number, we still do not know what happens near this fixed point. This is one of the major open problems in complex dynamics.

Another important contribution in the 1980s was made by Douady and Hubbard [12]. It is well known that the basin of attraction B_c of ∞ in the Riemann sphere is an open disk when c lies in the Mandelbrot set. Hence, by the Riemann Mapping Theorem, there is an analytic homeomorphism $\phi_c : B_c \to \mathbb{D}$ that takes ∞ to 0 and for which $\phi_c'(0) = a > 0$. Douady and Hubbard showed that this map actually conjugates the map z^2 on the disk $\mathbb{D}$ to P_c in B_c. That is, $\phi_c(P_c(z)) = (\phi_c(z))^2$. This implies that P_c behaves dynamically on B_c just like z^2 does in $\mathbb{D}$. Since z^2 interchanges the straight rays lying in the unit disk and given by $\exp(2\pi i t)$, the curves that are mapped to these straight rays by ϕ_c are also interchanged by P_c. These curves are called external rays of angle θ and we denote them by $\gamma_\theta(t)$. If the limit as $t \to 1$ of $\gamma_\theta(t)$ is a unique point in $J(P_c)$, then we say that the external ray γ_θ lands at this limit point. And, if all such external rays land, then we essentially know the dynamics on the Julia set since the straight rays are permuted by z^2. This may not happen, however. For example, if the Julia set is a locally connected set, then all of the external rays do indeed land. But if $J(P_c)$ is not locally connected, then some rays may only accumulate on a portion of $J(P_c)$. This is what may happen when we have those not-so-irrational parameters in the boundary of the Mandelbrot set.

More importantly, the same external ray construction can be carried over to the parameter plane. Let Φ be the map defined on the complement of the Mandelbrot set in the c-plane given by $\Phi(c) = \phi_c(c)$. Douady and Hubbard [12] also show that Φ is an analytic homeomorphism onto $\mathbb{D}$. The main open problem involving the Mandelbrot set is then:

Conjecture: The boundary of the Mandelbrot set is a locally connected set.

If this conjecture is true, then all of the corresponding external rays in the parameter plane land at a unique points in the boundary of the Mandelbrot set. In this case, we would then get a complete map of the Mandelbrot set that tells us everything about its structure: how the bulbs containing periodic cycles are arranged; how the parameters along the "antennas" on the bulbs are situated, etc. Despite the fact that we are dealing here with the relatively simple map $z^2 + c$, this conjecture seems to be a long way from being resolved. In particular, a recent result of Buff and Chéritat [7] shows that certain Julia sets for quadratic polynomials that contain those not-so-irrational fixed points can have positive Lebesgue measure. This means that things are even more complicated than most complex dynamicists had thought back in the 1980's. Furthermore, a result of Shishikura [42] states that the Hausdorff dimension of the boundary of the Mandelbrot set is 2, so this boundary is also an extremely complicated object.

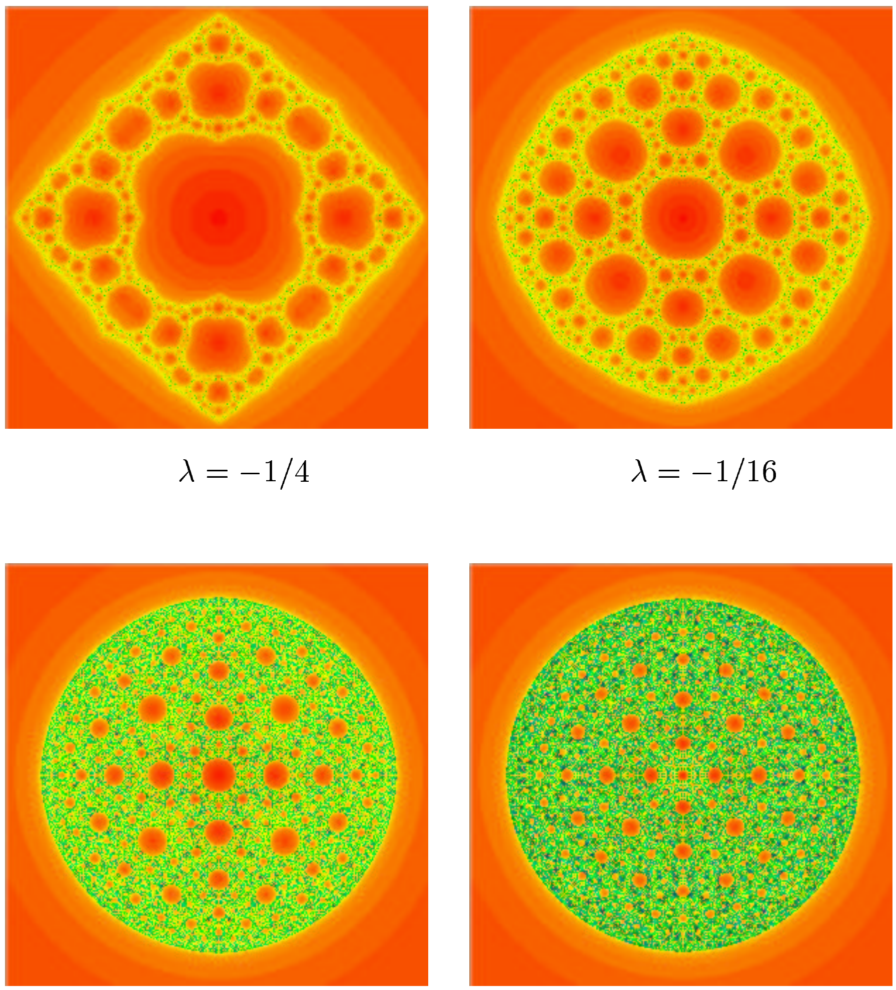
5 Rational Maps

Complex rational maps are naturally more complicated than polynomials, primarily because there often is no basin of attraction at ∞ and there are usually many more critical orbits. So, to simplify matters, we concentrate here on the family of degree $2n$ rational maps given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

where we assume $n \geq 2$. It turns out that ∞ is an attracting fixed point in $\overline{\mathbb{C}}$ since $F_\lambda \approx z^n$ when $|z|$ is large, so we do have an immediate basin B_λ of ∞. Also, one checks easily that the critical points are given by $\lambda^{1/2n}$. However, there are only two critical values given by $\pm 2\sqrt{\lambda}$. And, just as in the case of $z^2 + c$, there really is only one critical orbit (up to symmetry). This follows from the fact that, if n is even, both critical values then map to the same point, whereas, if n is odd, the map is symmetric under $z \mapsto -z$, so the orbits of both $\pm 2\sqrt{\lambda}$ behave symmetrically.

The Fatou set for a rational map can now contain a kind of set that does not occur with a polynomial, namely a Herman ring. First discovered by Herman in 1979 [22] right around the time the Mandelbrot set was first observed (although, as noted above, Cremer anticipated Herman rings in the early 1930s), these regions are (eventually) periodic annular regions of period n in which all points rotate around distinct simple closed curves under a given irrational rotation. The reasons these types of Fatou domains do not occur for polynomials is that there has to be a pole inside one of these Herman rings; otherwise, all points inside these annuli would be mapped to corresponding



$\lambda = -1/4$ $\lambda = -1/16$

$\lambda = -0.01$ $\lambda = -0.001$

Figure 3: Various Sierpinski curve Julia sets drawn from the family $z^2 + \lambda/z^2$; all of these sets are homeomorphic, but it is known that the dynamical behavior on each of these sets is very different. The red regions are the preimages of B_λ.

points inside the image annuli. So the iterates would form a normal family in these disks.

Unlike quadratic polynomials where there were only two types of Julia sets depending on the escape behavior of the critical orbit, there is now an "escape trichotomy" for this family. Since we have a basin of attraction B_λ at ∞ and a pole at 0, there is a neighborhood of 0 that is mapped to B_λ. If this neighborhood is disjoint from B_λ we call it the trap door T_λ since any orbit that eventually enters B_λ must do so by passing through T_λ. Then there are three possible types of Julia sets depending upon the behavior of the critical orbit:

1. If the critical values lie in B_λ (if one does, the other must also due to the $z \mapsto -z$ symmetry), then the Julia set of F_λ is a Cantor set;

2. If the critical values lie in T_λ, then $J(F_\lambda)$ is a Cantor set of simple closed curves surrounding the origin [31];

3. In all other cases $J(F_\lambda)$ is a connected set, and if the critical values do not lie in B_λ or T_λ but the critical orbit eventually enters B_λ, then $J(F_\lambda)$ is a Sierpinski curve.

A Sierpinski curve is a planar set that is homeomorphic to the well-known Sierpinski carpet fractal. See Figure 3 for several Sierpinski curve Julia sets that arise in the family $z^2 + \lambda/z^2$. These sets are very important from a topological point of view in that they form a dictionary of all possible plane curves. More precisely, given any one-dimensional plane continuum, this curve can be homeomorphically manipulated so that it can be embedded in the Sierpinski carpet [44].

6 Entire Functions

Now we turn to the very different case of entire transcendental functions where the possibilities for Fatou sets (as well as Julia sets, as we will soon see) become even richer. Wandering domains are now possible. These are Fatou domains that are never eventually periodic. For example, the map $z \mapsto z + 2\pi \sin z$ has a wandering domain. The vertical lines given by $\operatorname{Re} z = 2k\pi$ for $k \in \mathbb{Z}$ are easily seen to be invariant under this map and each lies in the Julia set. However, neighborhoods of the critical points given by $\pi/2 + 2k\pi$ lie in the Fatou set and all wander off to ∞.

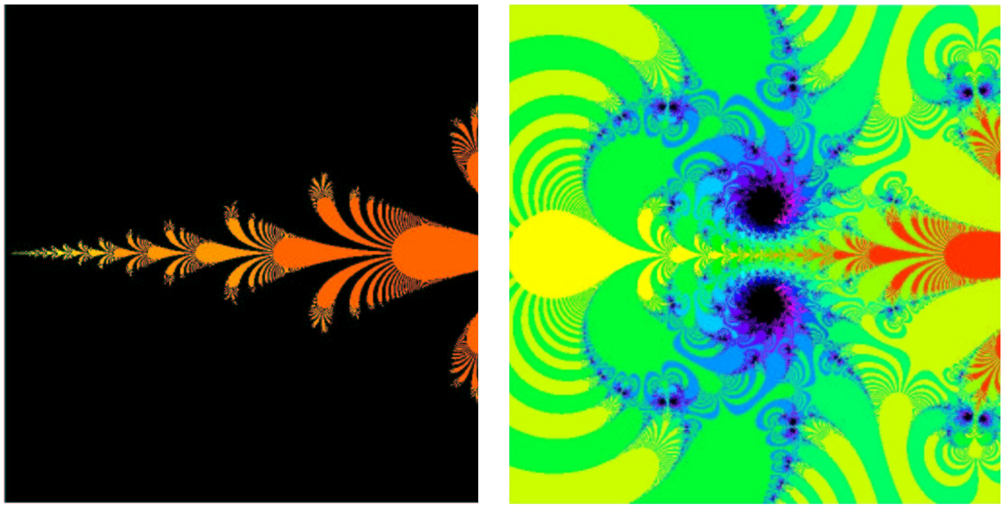


Figure 4: A small piece of the Cantor bouquet for E_λ with $\lambda < 1/e$ and the ensuing explosion when $\lambda > 1/e$. Colored points again have orbits that escape to ∞ and so, in this case, lie in the Julia set.

Another new possibility that arises for the Fatou set is a Baker domain. These are open sets extending to ∞ in which all orbits tend to ∞. But, unlike the quadratic polynomial case, there is no longer an open disk that completely surrounds the point at ∞. Since ∞ is an essential singularity there are points in the Julia set that are arbitrarily close to ∞. An example of this arises in the map $z \mapsto z + e^{-z} + 1$, where points in the right-half plane tend to ∞. This was shown by Fatou in 1926 in [20].

The analogue of quadratic polynomials in the entire transcendental case is the exponential function $E_\lambda(z) = \lambda \exp(z)$. There are no critical points for this function, but there is what is known as an asymptotic value, namely 0. This is the omitted value for the exponential maps and, moreover, any curve tending to ∞ in the far left half-plane is mapped to a curve that limits on the asymptotic value. As a consequence, 0 plays the same role as the critical points do in the case of polynomials or rational maps. But now, a very different phenomenon occurs. A theorem of Goldberg, Keen, and Sullivan [21], [45] says that, if the orbit of the asymptotic value 0 tends to ∞, then $J(E_\lambda)$ is now the entire complex plane.

Now consider the family E_λ where $\lambda \in \mathbb{R}^+$. The graphs of E_λ along the real axis show that, if $\lambda > 1/e$, then the orbit of 0 (in fact, all orbits on the real axis) tend to ∞. So, in this case, $J(E_\lambda) = \mathbb{C}$. But, if $\lambda < 1/e$, there exist one attracting and one repelling fixed point in $\mathbb{R}^+$. Moreover, it is easy to check that all points to the left of the repelling fixed point in $\mathbb{R}^+$ are contracted into a disk lying to the left of this half-plane. So, by the Contraction Mapping Principle, these points all have orbits that tend to the attracting fixed point, and so this half-plane lies in the Fatou set. In fact, when $\lambda \leq 1/e$, it is known that $J(E_\lambda)$ is a Cantor bouquet, i.e., a collection of infinitely many disjoint smooth curves with endpoints that extend to ∞ in the right half-plane.

So there is an amazing explosion in the Julia sets for these maps when λ passes through $1/e$. See Figure 4. When $\lambda \leq 1/e$, the Julia set lies in the right half-plane, but when $\lambda > 1/e$, suddenly $J(E_\lambda) = \mathbb{C}$. No new periodic points are born as λ passes through $1/e$; rather, all of the repelling periodic points move continuously and suddenly become dense in the complex plane. What a change! See Figure 4.

7 The Future of Complex Dynamics

The natural questions is: where is the field of complex dynamics heading in the next century? Already there have been many excursions into areas outside of polynomial dynamics, like the study of rational and entire maps alluded to above. But much more is beginning to happen and likely to expand in the future. This includes the study of other complex maps (like meromorphic functions) as well as higher dimensional complex analytic maps. Another recent topic of interest is algebraic dynamics where questions involving algebraic aspects (rather than the dynamical behavior) of iterated functions arise. And much more is on the horizon. The beauty of this expansion includes the fact that many distinct areas of mathematics now enter the picture, including dynamical systems, complex analysis, topology, number theory, and algebraic geometry.

And one final note. Complex dynamics is a field that is quite accessible to undergraduate students. After all, the primary topic of interest is the simplest nonlinear function, $z^2 + c$. Many undergrads can begin by studying the quadratic family and then move on to investigate other families of functions (their own choice: cubics, quartics, trigonometric functions, etc.) The beauty of this is, while the complete understanding of these maps will certainly be elusive, nonetheless, in many cases, the students become the first mathematicians to see the interesting behavior in their chosen family of interest. This definitely sparks their interest in research-level mathematics..

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