My Favorite Planar Fractal

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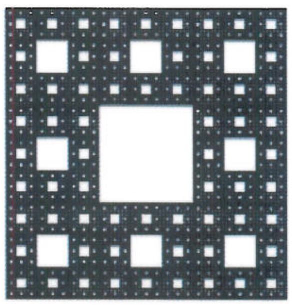
Unlike most people who adore the Sierpinski triangle, my favorite planar fractal is a Sierpinski curve. By definition, a Sierpinski curve is any planar set that is homeomorphic to the well-known Sierpinski carpet fractal (see the figure below). Here homeomorphic means that there is a one-to-one, onto, continuous map with a continuous inverse that takes this set to the carpet. So a Sierpinski curve is just a continuous deformation of the carpet.



There are three reasons for my fascination with this set. First, a Sierpinski curve is a "universal plane continuum." Roughly speaking, this means that any compact plane curve, no matter how intricate, can be homeomorphically manipulated to fit inside the carpet, [2]. So the carpet is a "dictionary" of all possible such planar curves. The second reason is that, by a theorem of Whyburn [3], there is a topological characterization of this set: any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any pair of complementary domains are bounded by simple closed curves that are pairwise disjoint is homeomorphic to the carpet.

Then my third reason for loving this set is that Sierpinski curves arise all the time as Julia sets of complex functions. To illustrate this, let's concentrate on the family of rational maps given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

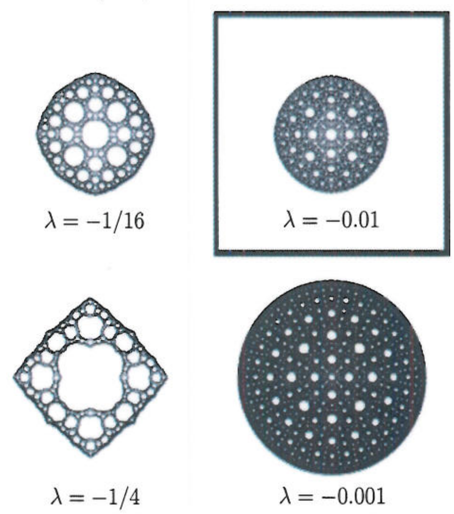
where $n \geq 2$ and $\lambda \in \mathbb{C}$. Let F_λ^j denote the j^{th} iterate of F_λ. Then the set of points $F_\lambda^j(z), j = 0, 1, 2, \ldots$, is the orbit of the point z. For these maps, there are $2n$ critical points given by $\lambda^{1/2n}$, but, just as in the case of quadratic polynomials, there really is only one critical orbit because all the orbits of critical points behave symmetrically under iteration of F_λ.

Out near ∞, F_λ is essentially given by z^n, so all points far from the origin tend to ∞ under iteration. So we have an immediate basin of attraction of ∞. When $|\lambda|$ is small enough, this is an open disk surrounding ∞ in which all orbits just tend to ∞. Then, there are infinitely many disjoint preimages of this basin. The *Julia set* of F_λ, denoted by $J(F_\lambda)$, is, by definition, the boundary of the set of points that escape to ∞. Equivalently, the Julia set is also the closure of the set of repelling periodic points. So, arbitaraily close to any point in the Julia set, we have both escaping and periodic points, so the Julia set is the place where chaos occurs for these maps.

So how do Sierpinski curves arise as Julia sets for F_λ? There are many ways [1]. One way is that, if the orbit of the critical points enter the immediate basin of ∞ at iteration 3 or later, then $J(F_\lambda)$ is a Sierpinski curve. Several such Julia sets are displayed in the figure below. The parameter plane (the λ-plane) for these maps then contains infinitely many open disks called Sierpinski holes, and each parameter in these holes has a Julia set that is a Sierpinski curve.

Another way is the following: it turns out that there are infinitely many small copies of the Mandelbrot set in the parameter plane for these maps. Those Mandelbrot sets that do not extend to the boundary of the parameter plane have main cardioids, and all parameters in these main cardioids have Julia sets that are again Sierpinski curves. And finally, it is known that there are uncountably many closed curves surrounding the origin in the λ-plane on which, again, all parameters have Sierpinski curve Julia sets.

So all of the above Julia sets are the same topologically. However, only parameters drawn from Sierpinski holes or main cardioids that are symmetrically located under complex conjugation or rotation by an $n-1^{\text{st}}$ root of unity have the same dynamical behavior. The same also holds true on each of the infinitely many curves in the dynamical plane described above. So we find a wealth of Sierpinski curve Julia sets, all with very different dynamics. Understanding all of these different dynamical behaviors is a wide open problem.



Sierpinski curve Julia sets for various values of λ when $n = 2$.

References

1. R.L. Devaney, Singular Perturbations of complex polynomials, *Bulletin AMS* (to appear).
2. W. Sierpinski, Sur une courbe cantorienne qui contient une image biunivoque et continue de toute courbe donne, *C. r. hebd. Seanc. Acad. Sci., Paris* **162** (1916), 629–632.
3. G.T. Whyburn, Topological characterization of the Sierpinski curve, *Fundamenta Mathematicae* **45** (1958), 320–324.