Cantor and Sierpinski, Julia and Fatou: Complex Topology Meets Complex Dynamics

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1 Introduction

Topologists have enjoyed pondering the exotic properties of fascinating objects such as indecomposable continua, Sierpinski curves, and Cantor bouquets for almost one hundred years, while complex dynamicists have only recently begun to enjoy the beauty and intricacy of fractal objects known as the Julia sets. Recent developments, however, have brought both of these fields closer together, as many of the extraordinary spaces from planar topology have now been shown to arise as Julia sets. In this article we describe a few of these topological incursions into complex dynamics.

2 Julia sets

Let $F : \mathbb{C} \to \mathbb{C}$ be a complex analytic function, and let F^n denote F composed with itself n times, the nth iterate of F. For a point $z \in \mathbb{C}$, the *orbit* of z is the sequence $z, F(z), F^2(z), \ldots$. Of interest in dynamics is the fate of these orbits: Is this fate predictable or is it not?

In complex dynamics, the predictable set is the *Fatou set*; points in this set have the property that all nearby orbits behave "similarly." Thanks to work of Julia and Fatou in the 1910s and Sullivan in the 1980s, the dynamics of F on the Fatou set is completely understood. There are only a few types of behaviors associated with such points: Most often, points in the Fatou set simply tend to an attracting periodic orbit, although there are several other well understood though less commonly encountered possibilities.

The *Julia set* is the complement of the Fatou set: It consists of points for which nearby orbits behave in vastly different manners. This is the "chaotic" set for such maps. By a classical theorem of Montel, if z is a point in the Julia set of F and U is any neighborhood of z, then the union of the forward images of U contains the entire plane (with the exception of at most one point). So F depends quite sensitively on initial conditions on its Julia set in the sense that a small error in specifying the initial point can lead to huge changes in the fate of the orbit. There are other equivalent definitions of the Julia set. For example, the Julia set is also the closure of the set of repelling periodic points for F. From a dynamical systems point of view, all of the interesting behavior of a complex analytic function occurs on its Julia set, and it is this set that contains the interesting topology.

As a simple example, consider the function $F(z) = z^2$. The behavior of all orbits of this function is easy to describe. If $|z| < 1$, then $|F(z)| < |z|$ and so all orbits that begin inside the unit circle simply tend to 0, which is an attracting fixed point. If $|z| > 1$, then all orbits increase in magnitude and tend to ∞. Finally, if z lies on the unit circle, then the images of any small neighborhood of this point under F^n eventually cover the entire plane, except (possibly) the origin. As a consequence, the Julia set of z^2 is the unit circle, and the Fatou set contains all other points in $\mathbb{C}$. The reader should be forewarned that very few other Julia sets are as simple to understand. Often, these Julia sets are extremely complicated fractal sets with equally complicated topology, as we shall see.

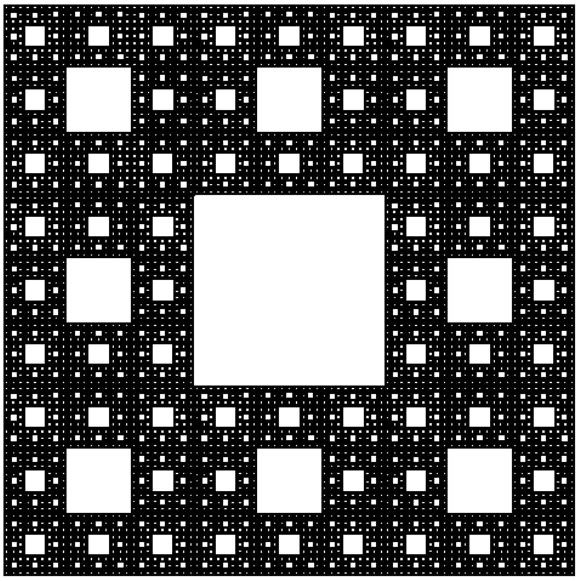

Figure 1: The Sierpinski carpet.

3 Sierpinski Curves

A *Sierpinski curve* is a planar set that is homeomorphic to the well known Sierpinski carpet fractal C (see Figure 1). The Sierpinski carpet is constructed as follows. Start with the unit square in the plane and divide it into nine equal subsquares. Then remove the open middle square, leaving eight closed subsquares. Now repeat this process, removing the open middle third from each of the eight subsquares, leaving 64 smaller squares. When this process is repeated ad infinitum, the resulting set is the Sierpinski carpet.

While this set may at first look rather tame, in fact its topology is quite rich: the Sierpinski carpet contains a homeomorphic copy of any compact, connected one (topological) dimensional planar set, no matter how complicated that set is. In this sense the Sierpinski curve is a "universal" planar set.

Note that all of the open squares removed during the construction of C have boundaries that are pairwise disjoint. Indeed, the lines $x = 1/2$ and $y = 1/2$ meet C in a Cantor middle-thirds set, with the endpoints of this Cantor set providing the intersections of the boundaries of removed squares. In addition, it is easy to check that the carpet is compact, connected, locally connected, and nowhere dense in the plane. In fact, these properties characterize Sierpinski curves, for any planar set that shares all of these properties is homeomorphic to the Sierpinski carpet [11] and hence is also a universal planar set.

4 Sierpinski Curve Julia Sets

Sierpinski curves arise as Julia sets of certain rational functions. The first example of this was given by Milnor and Tan Lei [10]. A more accessible collection of such Julia sets may be found in the family of rational functions given by

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2}$$

where $\lambda \neq 0$ is a complex parameter. It is known [2] that, for this family, there are infinitely many open sets in the λ-plane in any neighborhood of $\lambda = 0$ that have the property that the Julia set of F_λ is a Sierpinski curve whenever λ lies in one of these sets. Hence all of these Julia sets are homeomorphic, so that, from a topological point of view, all of these Julia sets are the same. However, from the point of view of dynamical systems, the dynamics on these Julia sets are quite different: Dynamicists say that two maps whose parameters lie in different open sets are not topologically conjugate.

For a rough idea of the construction of these Julia sets, note that if $|z|$ is sufficiently large, then $|\lambda/z^2|$ is small, so F_λ is essentially given by $z \mapsto z^2$. As a consequence, any orbit sufficiently far from the origin simply tends to ∞. The open set about the point at ∞ consisting of all points whose

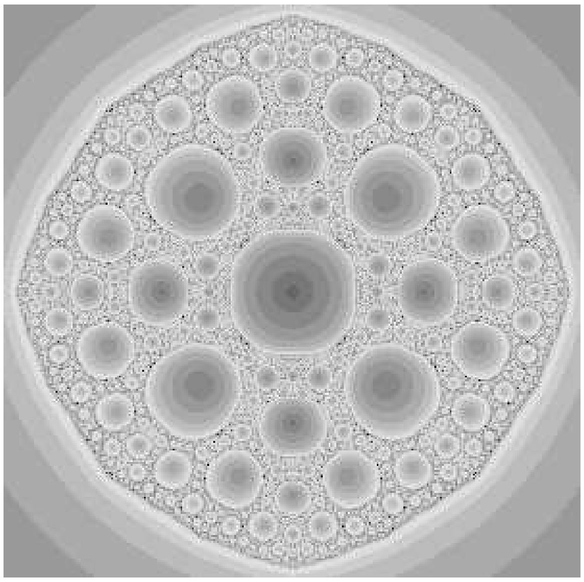

Figure 2: The Julia set for $F(z) = z^2 - 1/16z^2$. Colors indicate the number of iterations to enter a neighborhood of ∞, with shades of red indicating fastest entrance, followed by yellow, green, and blue. The boundary of the colored region is the Julia set.

orbits tend to ∞ is called the *basin of attraction* of ∞. As in the case of z^2, provided that $|\lambda|$ is small, the boundary of this basin is a simple closed curve surrounding the origin. Inside this curve, the dynamical behavior is much more complicated.

For definiteness, let us fix $\lambda = -1/16$ and denote the corresponding map by F. Clearly, F has a pole at 0. There are four pre-poles for this function, at the points $\pm 1/2$ and $\pm i/2$, and there are also four critical points for F at points of the form $\omega/2$, where ω is any fourth root of -1. Note that $F(\omega/2) = \pm i/2$, so that $F^2(\omega/2) = 0$, and so all four critical points are mapped to the pole after two iterations. This is what makes the case $\lambda = -1/16$ so special. The Julia set of F is shown in Figure 2.

Let B denote the basin of attraction of ∞. As above, B is bounded by a simple closed curve. There is an open set T about the pole at 0 that is mapped in two to one fashion onto B; we call T the *trap door* since any orbit that enters it "falls through" it and ends up in the basin of ∞. The only preimages of a point in the basin lie either in B or in T since the rational map F has degree four. One checks easily that the boundaries of T and B are disjoint.

Now consider the preimages of T. The preimage of the real axis under F is just the real and imaginary axes. Thus the four preimages of T are open sets surrounding the prepoles on these axes, and the boundaries of each of these sets are disjoint from one another as well as from the boundaries of T and B. These are the four large red regions surrounding T that intersect the axes.

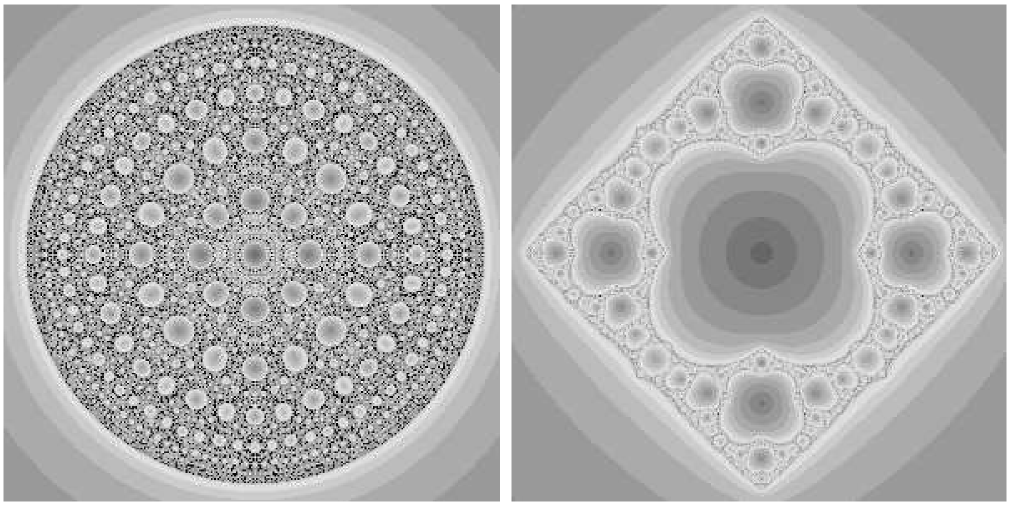

Figure 3: The Julia sets for $z^2 - 0.003/z^2$ and $z^2 - 0.32/z^2$.

Next consider the preimages of these sets. The four critical points fall into the trap door at iteration two, and they are surrounded by open sets that have the same property. These are the largest red regions intersecting the rays $\theta = \pm\pi/4$ and $\theta = \pm 3\pi/4$. There are eight other smaller open sets that are mapped onto the trap door by F^2, and each of these preimages is bounded by a simple closed curve which is disjoint from those previously constructed.

Continuing in this manner yields the set of points whose orbits eventually enter B. These are the analogues of the removed open squares in the Sierpinski carpet. It is known that the union of these sets forms the Fatou set for F; the Julia set is its complement. See [2] for more details.

The Sierpinski curve Julia sets of several other members of the family F_λ are shown in Figure 3. Each is homeomorphic to the Julia set for $\lambda = -1/16$, but the dynamical behavior on each set is quite different.

The well known fractal called the Sierpinski triangle (or gasket) also arises as a Julia set, this time for the related function $G(z) = z^2 + \lambda/z$ where $\lambda \approx -0.593$. See Figure 4. Though this set shares the same first name as the Sierpinski curve, it is both topologically and dynamically quite different. For example, note that the boundaries of B, T, and the preimages of T are not pairwise disjoint in this case.

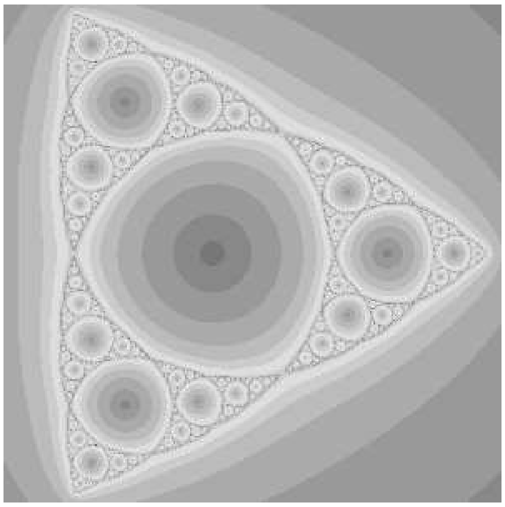

Figure 4: The Sierpinski triangle Julia set for $z^2 - 0.593/z$.

5 Cantor Bouquets and the Complex Exponential

Another type of interesting Julia set is a *Cantor bouquet*. Roughly speaking, a Cantor bouquet is an uncountable collection of disjoint continuous curves tending to ∞ in a certain direction in the plane, each of which has a distinguished endpoint. More precisely, following Aarts and Oversteegen [1], a Cantor bouquet is any planar set that is homeomorphic to a *straight brush*. To define this set, let $\mathcal{B}$ be a subset of $[0, \infty) \times \mathcal{I}$ where $\mathcal{I}$ is a dense subset of the irrational numbers. The set $\mathcal{B}$ is a straight brush if it has the following three properties:

1. Hairiness. For each point $(x, \alpha) \in \mathcal{B}$, there is a $t_\alpha \in [0, \infty)$ such that $\{t \mid (t, \alpha) \in \mathcal{B}\} = [t_\alpha, \infty)$. The point (t_α, α) is the *endpoint* of the *hair* given by $[t_\alpha, \infty) \times \{\alpha\}$.

2. Endpoint density. For each $(x, \alpha) \in \mathcal{B}$, there exist a pair of sequences $\{\beta_n\}$ and $\{\gamma_n\}$ in $\mathcal{I}$ converging to α from both above and below and such that the corresponding sequences of endpoints t_{β_n} and t_{γ_n} converge to x.

3. Closed. The set $\mathcal{B}$ is a closed subset of $\mathbb{R}^2$.

To see a Cantor bouquet in complex dynamics, consider the complex exponential function $E_\lambda(z) = \lambda \exp(z)$ where $0 < \lambda < 1/e$. For such a value of λ, the graph of the real exponential $\lambda \exp(x)$ meets the diagonal line $y = x$ at two points, an attracting fixed point at q_λ and a repelling fixed point at p_λ. Note that $E'_\lambda(-\log \lambda) = 1$, so that $q_\lambda < -\log \lambda < p_\lambda$. See Figure 5.

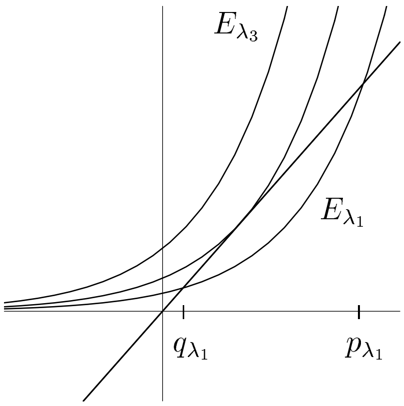

Figure 5: The graphs of E_{λ_j} for $\lambda_1 < 1/e, \lambda_2 = 1/e$, and $\lambda_3 > 1/e$.

In $\mathbb{C}$, consider the vertical line $\operatorname{Re} z = -\log \lambda$. The exponential wraps this line around a circle centered at the origin and lying to the left of $x = -\log \lambda$, since $E_\lambda(-\log \lambda) = 1 < -\log \lambda$. All points to the left of this line are therefore contracted inside this circle, and so, by the Contraction Mapping Principle, all orbits originating in the half plane $H = \mathrm{Re}\, z < -\log\lambda$ must tend to the attracting fixed point q_λ. As a consequence, all of these points lie in the Fatou set.

The Julia set is known to be the complement of the basin of attraction of q_λ. To construct the basin, we ask which points lie in the various preimages of H under E_λ. Any point lying on a horizontal line of the form $y = (2n+1)\pi$ is mapped by E_λ to the negative real axis, so these points lie in the basin. There is then an open set about this line to the right of H that is shaped like a finger pointing to ∞ and mapped by E_λ onto H. The complement of these open sets consists of infinitely many closed "C"-shaped regions extending to ∞ in the right half plane. Each of these regions is mapped in one-to-one fashion onto the half plane forming the complement of H in $\mathbb{C}$. Hence we may remove infinitely many smaller subfingers from each of these regions; these are the subfingers that map onto the fingers about each line $y = (2n+1)\pi$ and hence into H after two iterations of E_λ. Continuing in this fashion, we remove infinitely many subfingers at each iteration of E_λ. In the limit, the set of points which do not fall into H after some iterate of E_λ is the Julia set of E_λ, $J(E_\lambda)$, and this set is known to be a Cantor bouquet [5].

Here is the main dynamical property of $J(E_\lambda)$. All points in $J(E_\lambda)$ except the endpoints have orbits that tend to ∞. Hence all of the bounded orbits in $J(E_\lambda)$ must lie in the set of endpoints. But the bounded orbits must include the set of repelling periodic orbits, and this set is known to be dense in the Julia set. Therefore the endpoints of the hairs accumulate on all points in the bouquet, and this shows why the endpoint density property holds.

For the interesting topology, consider the following facts.

1. Mayer [7] has shown that the set of all endpoints of $J(E_\lambda)$ together with the point at ∞ in the Riemann sphere forms a connected set. However, if we remove just one point from that set, namely the point at ∞, the resulting set is not only disconnected, but is actually **totally** disconnected!

2. McMullen [8] has shown that $J(E_\lambda)$ has Hausdorff dimension 2 but Lebesgue measure 0. There are other Cantor bouquets that have quite different measure theoretic properties. For example, the Julia sets of $\cos z$ and $\lambda \sin z$ (with $|\lambda| < 1$) are also Cantor bouquets, but these sets have infinite Lebesgue measure (and Hausdorff dimension 2).

3. Karpinska [6] has shown that the Julia set consists of two disjoint subsets: the "small" set consisting of the endpoints alone, and the "large" set consisting of the complement, namely the "tails" of all the hairs. The set of tails turns out to have Hausdorff dimension 1, while the apparently much smaller subset consisting of just the endpoints has Hausdorff dimension 2!

6 The Exploding Exponential

The Julia set of E_λ undergoes a remarkable change as λ passes through the value $1/e$. When $\lambda = 1/e$ the graph of E_λ is tangent to the diagonal line at $x = 1$, so that the two fixed points q_λ and p_λ coalesce to become one *neutral* fixed point. For $\lambda > 1/e$ the fixed points disappear from the real line. Dynamicists call this simple transition a saddle-node bifurcation (although in this low dimensional setting there is no saddle point apparent anywhere).

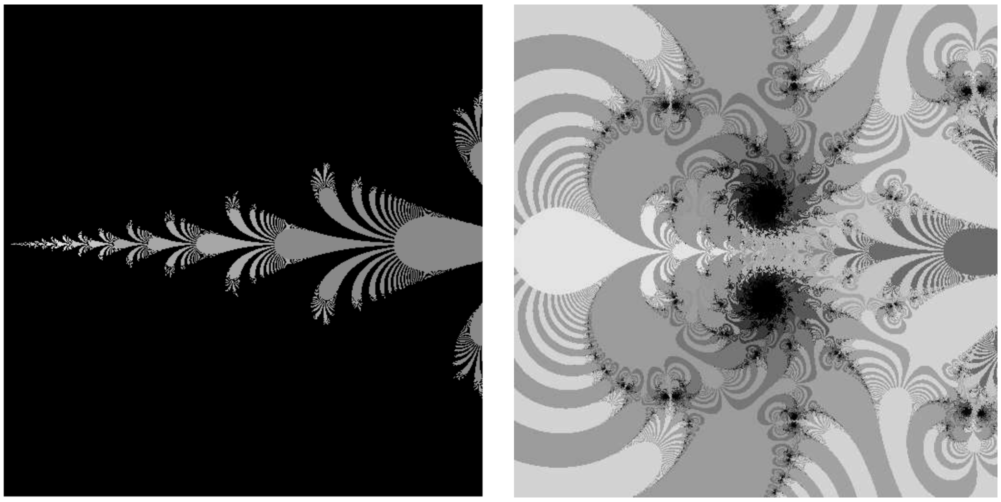

Figure 6: The tip of the Cantor bouquet for E_λ with $\lambda < 1/e$ and the ensuing explosion when $\lambda > 1/e$. Note the remnants of the bouquet.

In the plane, however, this change is much more dramatic. Suddenly, for $\lambda > 1/e$, the Julia set becomes the entire plane. Chaotic behavior is everywhere. Repelling periodic points are now dense in $\mathbb{C}$. Formerly, these periodic points all resided on the endpoints of the Cantor bouquet. As λ changes, no new repelling periodic points are born or disappear; all of them simply move around continuously. When $\lambda \leq 1/e$, all of these periodic points lie to the right of the half plane H, but as soon as λ increases beyond $1/e$, they become dense in $\mathbb{C}$. See Figure 6.

The cause of this change is the fate of the orbit of 0, the omitted value for the exponential. When $\lambda \leq 1/e$, the orbit of 0 tends to a fixed point on the real line, but when $\lambda > 1/e$, this orbit now tends to ∞. When this occurs, it is known that $J(E_\lambda) = \mathbb{C}$.

7 Indecomposable Continua

One reason that the Julia set of E_λ explodes for $\lambda > 1/e$ is the fact that the set of repelling periodic points suddenly becomes dense in $\mathbb{C}$. A second, more topological reason is that, as λ increases through $1/e$, infinitely many of the hairs suddenly become another kind of interesting topological object, namely *indecomposable continua.* An indecomposable continuum is a compact, connected set that cannot be written as the union of two compact, connected, proper subsets. This union is not a disjoint union, by the way.

For readers not familiar with these sets, try for just a moment to think of such an indecomposable set. The closed unit interval is not indecomposable, since it may be written, for example, as $[0, 2/3] \cup [1/3, 1]$. The unit circle is not indecomposable for it is the union of its upper and lower (closed) semicircles. Neither is a sphere or a torus or even the Sierpinski carpet or the Cantor bouquet (with the point at ∞ added to make it connected) described earlier.

The simplest example of an indecomposable continnuum is the *Knaster continuum.* This set is constructed as follows. Start with the Cantor middle-thirds set on the real line in $\mathbb{R}^2$. This set is symmetric about $x = 1/2$, so we can join any symmetric pair of points in the Cantor set by a semicircle in the upper half plane centered at $x = 1/2$. Now look in the lower half plane. Points in the right hand portion of the Cantor set between 2/3 and 1 may be connected by semicircles lying in the lower half plane, this time centered about 5/6. This leaves the left half of the Cantor set. This portion may also be cut in half and symmetric pairs of points in the right portion may now be joined by semicircles. Continuing in this fashion, in the limit we get a set that is known to be indecomposable. See Figure 7.

To get a feeling for why this set is indecomposable, suppose we try to

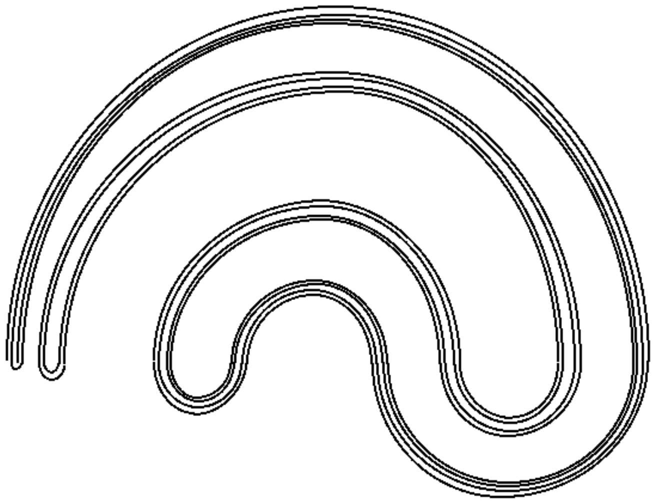

Figure 7: The barest outline of the Knaster continuum.

break this set into its left and right halves as we did with the unit interval. Then then resulting sets are clearly no longer connected. Similarly, dividing the set into its upper and lower portions also causes the resulting sets to be disconnected. If we consider a closed subset that contains the curve starting at 0, then we find that this curve passes through, in succession, the points in the Cantor set lying at $x = 1, 2/3, 1/3, 2/9, 7/9, \ldots$, and ultimately all of the endpoints of the Cantor set. Hence the closure of this curve is the entire Knaster continuum, and so this subset is not proper.

Topologically, this set contains much more. There are infinitely many disjoint curves in this set, and each of them is dense. Only the aforementioned curve through 0 has an endpoint, however, and this is the only curve that is "accessible" from the exterior.

8 Back to the Julia Set

To see how indecomposable continua arise for the complex exponential, consider what happens on the real line. For $\lambda \leq 1/e$, there is a hair in the Julia set given by the interval $[p_\lambda, \infty)$ lying along the real line. When λ exceeds $1/e$, this hair suddenly fills the entire real line. But there is more. Consider the line $y = \pi$ (or $y = -\pi$). This line is mapped to the real axis, and so, by adjoining the point at $-\infty$ (the preimage of 0) to these lines, we get a hair that is even longer. But there is a preimage of the line $y = \pi$ contained in the strip $0 < y < \pi$; this is another "C"-shaped curve that tends to ∞ tangentially to $y = \pi$ and $y = 0$. And this curve has a preimage in the strip, and this preimage has a preimage, and so forth. See Figure 8. If we compactify the picture by compressing everything into the strip $-1 \leq x \leq 1$, say, and then adjoining the endpoints as we travel around the hair, we obtain a curve that can be shown to accumulate everywhere on itself, just as in the case of the Knaster continuum. The closure of this set is then known to be an indecomposable continuum.

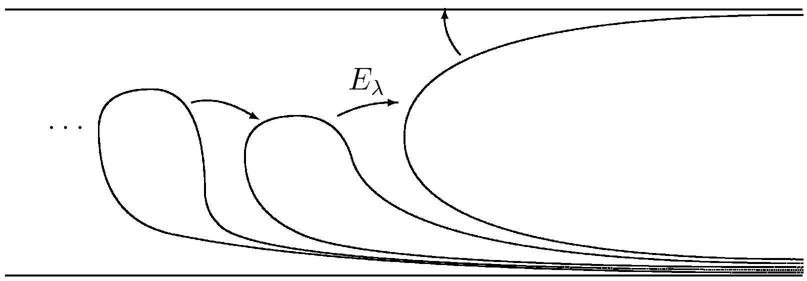

Figure 8: The hair in the region $0 \leq \operatorname{Im} z \leq \pi$.

Open problems abound in this setting. The above construction gives an indecomposable continuum in the Julia set for each value of $\lambda > 1/e$. Are each of these sets homeomorphic? Probably not. It is entirely possible to have a "continuum" of topologically different indecomposable continua.

Beyond the hair that lies on the real axis, uncountably many other hairs also explode in a similar manner as we pass through the bifurcation. See [3]. Again, what is the topology of these sets? How does this topology depend on λ? These too are open questions.

9 Acknowledgements

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