Julia Sets Converging to the Unit Disk

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Abstract

We consider the family of rational maps $F_\lambda(z) = z^n + \lambda/z^d$ where $n, d \geq 2$ and λ is small. If λ is equal to 0 the limiting map is $F_0(z) = z^n$ and the Julia set is the unit circle. We investigate the behavior of the Julia sets of F_λ when λ tends 0, obtaining two very different cases depending on n and d. The first case occurs when $n = d = 2$; here the Julia sets of F_λ converge as sets to the closed unit disk. In the second case, when one of n or d is larger than 2, there is always an annulus of some fixed size in the complement of the Julia set, no matter how small $|\lambda|$ is.

In this paper we consider the family of rational maps

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

where $n, d \geq 2$ and $|\lambda|$ is small but nonzero. Our goal is to investigate the behavior of the Julia sets for this family of maps as $\lambda \to 0$.

Clearly, as $\lambda \to 0$, the maps F_λ converge to $z \mapsto z^n$. The Julia set of this limiting map is well understood: it is the unit circle. But the situation for F_λ is quite different since the degree of F_λ jumps to $n + d$ as soon as $\lambda \neq 0$.

There are two very different cases to consider when $\lambda \neq 0$. The first case occurs when n and d are not both equal to 2. In this case, it is known by

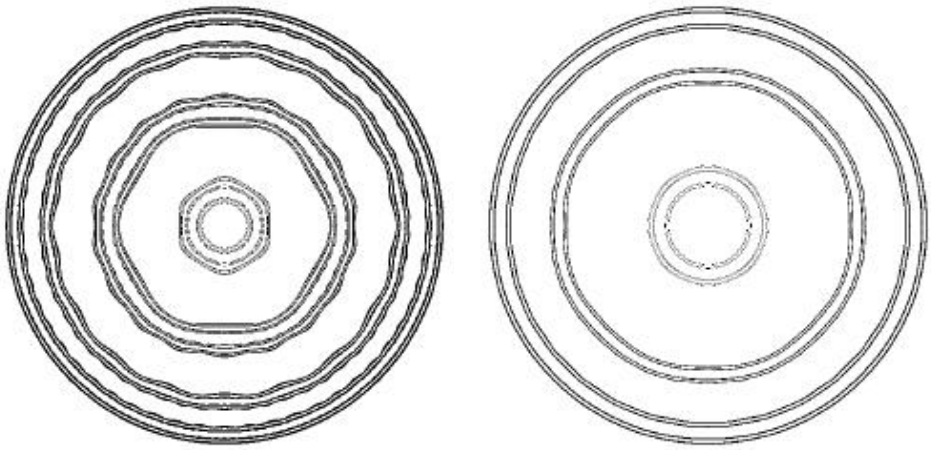

Figure 1: The Julia sets for $z^3 - 0.001/z^3$ and $z^4 - 0.001/z^4$ are both Cantor sets of circles.

work of McMullen [8] that the Julia set of F_λ for λ sufficiently small is a Cantor set of simple closed curves, each of which surrounds the origin. See Figure 1. So, in particular, the Julia set is not connected when $|\lambda|$ is small. Indeed, there is a unique open disk $\mathcal{M}$ surrounding the origin in the λ-plane such that the Julia set of F_λ is always a Cantor set of simple closed curves when $\lambda \in \mathcal{M} - \{0\}$ [7]. Moreover, all such maps with $\lambda \in \mathcal{M} - \{0\}$ are the same dynamically in the sense that any two such maps are conjugate on their Julia sets.

When $n = d = 2$, the situation is very different. It is known [1] that there are no Julia sets of the above type in this case; in fact, all Julia sets of F_λ for $|\lambda|$ small are connected sets. See Figure 2. Also, it is known that there are uncountably many dynamically different types of Julia sets in any neighborhood of the origin in parameter space. Indeed, in any neighborhood of 0, there are infinitely many parameters for which the Julia set is a Sierpinski curve, but the dynamics of any two such maps are non-conjugate.

How these two different types of Julia sets evolve as $\lambda \to 0$ is also quite different. In Figure 1, note the large white annular regions between the closed curves in the Julia sets. These annuli lie in the complement of the Julia set. As n or d grows, one can find even larger annuli in the complement of the Julia set, no matter how small $|\lambda|$ is. On the other hand, in Figure 2, the complementary regions in the Julia sets all seem to be disks whose radii

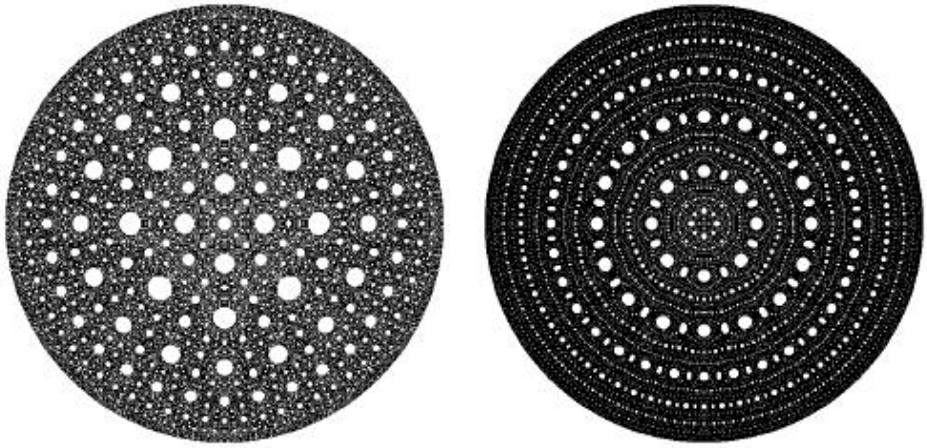

Figure 2: The Julia sets for $n = 2$ and $\lambda = -0.001$ and $\lambda = -0.00001$

decrease with $|\lambda|$. This is indeed the case. We shall prove that, when $n = d = 2$, the Julia sets of F_λ converge as sets to the closed unit disk. On the other hand, when one of n or d is larger than 2, there is always an annulus of some fixed size in the complement of the Julia set, no matter how small $|\lambda|$ is. More precisely, the main result in this paper is:

Theorem

1. *Suppose $n = d = 2$. If λ_j is a sequence of parameters converging to 0, then the Julia sets of F_{λ_j} converge as sets to the closed unit disk.*

2. *If one of n or d is not equal to 2, this is not the case. Specifically, for a given punctured neighborhood U of $0 \in \mathcal{M}$, there exists $\delta > 0$ such that, for each $\lambda \in U$, there is a round annulus (i.e., bounded by circles) in the complement of the Julia set inside the unit circle whose internal and external radii differ by at least δ.*

1 Julia Sets Converging to the Unit Disk

In this section we restrict attention to the family of maps

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2}$$

where $\lambda \neq 0$. Note that the point at ∞ is a superattracting fixed point. We denote the immediate basin of ∞ by B_λ. The origin is a pole of order two, so there is a neighborhood of 0 that is mapped onto a neighborhood of ∞ in B_λ. If this neighborhood is disjoint from B_λ, we denote the preimage of B_λ containing 0 by T_λ. This set is known as the *trap door*, since any orbit that eventually enters B_λ must do so by passing through the trap door.

The *Julia set* of F_λ, denoted by $J(F_\lambda)$, is known to be the boundary of the set of points whose orbits escape to ∞. See [1].

One computes easily that the map F_λ has four critical points given by $c_\lambda = \lambda^{1/4}$. There are only two critical values, however, namely $v_\lambda = 2\lambda^{1/2}$. Furthermore, the second iterates of all of the critical points all land on the same point, namely $\omega_\lambda = 1/4 + 4\lambda$. Thus, there is essentially only one critical orbit for this family. Moreover, $J(F_\lambda), B_\lambda$, and T_λ are all symmetric under rotation by a fourth root of unity.

The following Theorem is proved in [1]:

Theorem.

1. *Suppose one (and hence both) of the critical values lie in B_λ. Then $J(F_\lambda)$ is a Cantor set. Otherwise, $J(F_\lambda)$ is connected.*

2. *Suppose that, for some $k \geq 2$, $F_\lambda^k(c_\lambda)$ lies in B_λ but $F_\lambda^{k-1}(c_\lambda)$ does not lie in B_λ. Then the Julia set of F_λ is a Sierpinski curve.*

A *Sierpinski curve* is any planar set that is homeomorphic to the well known Sierpinski carpet fractal. Equivalently, any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any pair of complementary domains are bounded by simple closed curves that are pairwise disjoint is known to be homeomorphic to the Sierpinski carpet [10]. In [5] it was shown that, in any neighborhood of the origin in the λ-plane, there are infinitely many parameters for which $J(F_\lambda)$ is a Sierpinski curve, but the dynamical behavior of F_λ is different for each of these parameters.

Suppose now that $J(F_\lambda)$ is a connected set. Then it has been proved in [3] that, if $|\lambda| < 1/16$, then the Julia set always contains an invariant *Cantor necklace*. A Cantor necklace is a set that is homeomorphic to the following subset of the plane: Place the Cantor middle thirds set on the x-axis. Then adjoin a circle of radius $1/3^j$ in place of each of the 2^j removed intervals at the j^{th} level of the construction of the Cantor middle thirds set. The union of the Cantor set and the adjoined circles is the model for the Cantor necklace.

We remark that this result holds for any λ for which $J(F_\lambda)$ is connected, not just those with $|\lambda| < 1/16$. The only difference is that the boundaries of the open regions now need not be simple closed curves — they may just be the boundary of a disk (which need not be a simple closed curve).

In the Julia set of F_λ, the invariant Cantor necklace has the following properties: the simple closed curve corresponding to largest circle in the model is the boundary of the trap door. All of the closed curves corresponding to the circles at level j correspond to the boundaries of preimages of ∂B_λ that map to this set after j iterations. The Cantor set portion of the necklace is an invariant set on which F_λ is hyperbolic and, in fact, conjugate to the one-sided shift map on two symbols. The two extreme points in this set correspond to the fixed point and its negative, both of which lie in ∂B_λ. Hence the Cantor necklace stretches completely "across" $J(F_\lambda)$. Moreover, it is known that the Cantor necklace is located in a particular subset of the Julia set. Specifically, let $c_0(\lambda)$ be the critical point of F_λ that lies in the sector $0 \leq \operatorname{Arg} z < \pi/2$ when $0 \leq \operatorname{Arg} \lambda < 2\pi$. Let c_j be the other critical points arranged in the counterclockwise direction around the origin as j increases. Let I_0 denote the sector bounded by the two straight lines connecting the origin to ∞ and passing through c_0 and c_1. Let I_1 be the negative of this sector. Then, as shown in [3], the Cantor set portion of the necklace is constrained to lie in this region for all λ with $0 \leq \operatorname{Arg} \lambda < 2\pi$.

We now prove that when $n = d = 2$, the Julia sets of F_λ converge to the closed unit disk $\mathbb{D}$ as $\lambda \to 0$. By converges to the unit disk we mean the following:

Theorem. *Let $\epsilon > 0$ and denote the disk of radius ϵ centered at z by $B_\epsilon(z)$. There exists $\mu > 0$ such that, for any λ with $0 < |\lambda| \leq \mu$, $J(F_\lambda) \cap B_\epsilon(z) \neq \emptyset$ for all $z \in \mathbb{D}$.*

Proof: Suppose that this is not the case. Then, given any $\epsilon > 0$, we may find a sequence of parameters $\lambda_j \to 0$ and another sequence of points $z_j \in \mathbb{D}$ such that $J(F_{\lambda_j}) \cap B_{2\epsilon}(z_j) = \emptyset$ for each j. Since $\mathbb{D}$ is compact, there is a subsequence of the z_j that converges to some point $z^* \in \mathbb{D}$. For each parameter in the corresponding subsequence, we then have $J(F_{\lambda_j}) \cap B_\epsilon(z^*) = \emptyset$. Hence we may assume at the outset that we are dealing with a subsequence $\lambda_j \to 0$ such that $J(F_{\lambda_j}) \cap B_\epsilon(z^*) = \emptyset$.

Now consider the circle of radius $|z^*|$ centered at the origin. This circle meets $B_\epsilon(z^*)$ in an arc γ of length ℓ. Choose k so that $2^k \ell \geq 2\pi$.

Since $\lambda_j \to 0$, we may choose j large enough so that $|F^i_{\lambda_j}(z) - z^{2^i}|$ is very small for $1 \leq i \leq k$, provided z lies outside the circle of radius $|z^*|/2$ centered at the origin. In particular, it follows that $F^k_{\lambda_j}(\gamma)$ is a curve whose argument increases by approximately 2π, i.e., the curve $F^k_{\lambda_j}(\gamma)$ wraps at least once around the origin. As a consequence, the curve $F^k_{\lambda_j}(\gamma)$ must meet the Cantor necklace in the dynamical plane. But this necklace lies in $J(F_{\lambda_j})$. Hence $J(F_{\lambda_j})$ must intersect this curve. Since the Julia set is backward invariant, it follows that $J(F_{\lambda_j})$ must intersect $B_\epsilon(z^*)$. This then yields a contradiction, and so the result follows.

□

2 The McMullen Domain Case

In this section we discuss the size of the Julia sets for $F_\lambda(z) = z^n + \lambda/z^d$ in the case where $n, d \geq 2$ but not both $n = d = 2$ and $|\lambda|$ is small. As we discussed above, in this case the Julia set is a Cantor set of simple closed curves, each of which surrounds the origin. So the Fatou set is a countable collection of annuli together with the disks B_λ and T_λ. Following [2], we say that an annulus of the form $0 < r_1 < |z| < r_2$ is a *round annulus.* Such an annulus therefore contains a disk whose diameter is $r_2 - r_1$. Our goal is to prove:

Theorem. *Given n and d, there exists $\mu, \delta > 0$ such that, for all λ with $|\lambda| \leq \mu$, one of the complementary annuli in the Fatou set of F_λ contains a round annulus whose inner and outer radii differ by at least δ.*

As a consequence of this result, we have a very different situation when $\lambda \to 0$ in this case: the Julia sets no longer converge to the unit disk as in the case $n = d = 2$.

For clarity, we shall restrict at first to the case $n = 3$; we discuss the rather straightforward modifications necessary to handle the more general case at the end of this section. So, for now, let $F_\lambda(z) = z^3 + \lambda/z^3$ where λ belongs to the McMullen domain.

Let $\mathcal{A}$ denote the open annulus separating $\overline{T_\lambda}$ from $\overline{B_\lambda}$. There are three disjoint open annuli lying in $\mathcal{A}$. The first is the preimage of T_λ under F_λ which we denote by E_0. This is the *escape annulus.* The region lying between $\overline{B_\lambda}$ and $\overline{E_0}$ is an open annulus that we denote by A_0, and the region lying between

$\overline{T_\lambda}$ and $\overline{E_0}$ is also an open annulus that we denote by A_0'. Note that each of these three annuli is bounded by a simple closed curve that is mapped to either ∂B_λ or ∂T_λ. So these annuli are disjoint but their closures fill the region between B_λ and T_λ.

Let $m = m_\lambda$ denote the modulus of $\mathcal{A}$. Since the trap door shrinks to the origin as $\lambda \to 0$, it follows that $m_\lambda \to \infty$ as $\lambda \to 0$. Now F_λ maps E_0 onto T_λ whereas F_λ takes both A_0 and A_0' onto $\mathcal{A}$ as a three-to-one covering map. Hence the modulus of both A_0 and A_0' is $m/3$. The complementary annulus, namely E_0, thus also has modulus $m/3$.

Since F_λ takes A_0 onto $\mathcal{A}$ as a three-to-one covering, there are preimages of A_0, E_0, and A_0' in A_0. Denote these preimages by A_1, E_1, and A_1' respectively. Then the modulus of each of these annuli is $m/3^2$. Now we continue inductively. Since F_λ maps A_j as a three-to-one covering over $A_j \cup E_j \cup A_j'$, there are three disjoint annuli in A_j called A_{j+1}, E_{j+1}, and A_{j+1}' respectively; these annuli are mapped to their predecessors. Hence the modulus of each of these annuli is $m/3^{j+2}$. We have that F_λ^{j+1} maps E_j onto T_λ, so each E_j lies in the Fatou set. See Figure 3. We shall show that, for each sufficiently small λ, there is an E_j that contains a round annulus which contains disks whose diameter is uniformly bounded away from 0.

We now complete the proof of the Theorem in the case $n = 3$. Since $m = m_\lambda \to \infty$ as $\lambda \to 0$, we may find a $j \geq 0$ that depends on λ such that

$$1 < \operatorname{mod} A_j = \operatorname{mod} E_j \leq 3.$$

Let $\alpha = \alpha(\lambda)$ denote the modulus of A_j and E_j. So $1 < \alpha(\lambda) \leq 3$ for all sufficiently small λ. We emphasize that j increases as $\lambda \to 0$, but there always exist A_j and E_j with moduli between 1 and 3.

Given any ϵ with $0 < \epsilon < 1$, let x_0 satisfy

$$\frac{1}{2\pi} \log\left(\frac{1-\epsilon}{x_0}\right) = \alpha$$

so that $x_0 = (1-\epsilon)\exp(-2\pi\alpha)$. Similarly, let x_1 satisfy

$$\frac{1}{2\pi} \log\left(\frac{1+\epsilon}{x_1}\right) = \alpha$$

so that $x_1 = (1+\epsilon)\exp(-2\pi\alpha)$. Also let y_0 and y_1 satisfy

$$\frac{1}{2\pi} \log\left(\frac{1-\epsilon}{y_0}\right) = 2\alpha$$

and

$$\frac{1}{2\pi}\log\left(\frac{1+\epsilon}{y_1}\right) = 2\alpha$$

respectively, so that $y_0 = (1-\epsilon)\exp(-4\pi\alpha)$ and $y_1 = (1+\epsilon)\exp(-2\pi\alpha)$. We assume here that ϵ is small enough so that

$$\exp(-4\pi\alpha)(1+\epsilon) < \exp(-2\pi\alpha)(1-\epsilon) < \exp(-2\pi\alpha)(1+\epsilon) < 1-\epsilon.$$

Therefore we have

$$y_0 < y_1 < x_0 < x_1 < 1-\epsilon$$

for sufficiently small λ. See Figure 3 for a sketch of the construction above. Observe that, since $1 < \alpha \leq 3$, we have that each of the x_i and y_i lie in a compact interval in the open interval $(0,1)$ and, in particular, these points are strictly bounded away from 0.

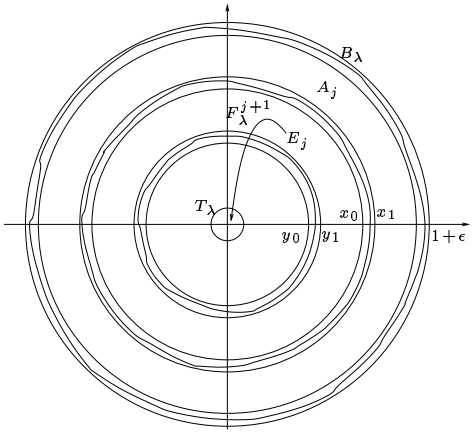

Figure 3: A sketch of the relevant objects in the McMullen domain case.

Since ∂B_λ tends to the unit circle as $\lambda \to 0$, we may choose $\mu > 0$ so that, if $|\lambda| \leq \mu$, then we have that ∂B_λ lies in the round annulus given by $1-\epsilon < |z| < 1+\epsilon$.

Now consider the position of A_j. Since $\operatorname{mod} A_j = \alpha$ and the outer boundary of A_j is ∂B_λ (and so lies outside the circle $|z| = 1-\epsilon$), we have that the inner boundary of A_j cannot lie strictly inside the circle $|z| = x_0$. Indeed, if that were the case, we would have

$$\operatorname{mod} A_j > \frac{1}{2\pi}\log\left(\frac{1-\epsilon}{x_0}\right) = \alpha$$

which is a contradiction. On the other hand, the inner boundary of A_j cannot lie strictly outside the circle $|z| = x_1$ for otherwise we would have

$$\operatorname{mod} A_j < \frac{1}{2\pi} \log \left(\frac{1+\epsilon}{x_1} \right) = \alpha,$$

again a contradiction.

Now let X_j be the open annulus $A_j \cup E_j$ together with the simple closed curve separating them. Note that $\operatorname{mod} X = 2\alpha$. The inner boundary of X_j cannot lie strictly inside the circle $|z| = y_0$ for we would then have

$$\operatorname{mod}(X) > \frac{1}{2\pi} \log \left(\frac{1-\epsilon}{y_0} \right) = 2\alpha = \operatorname{mod} X.$$

Now consider the annulus E_j. By the above, the inner boundary of E_j cannot lie strictly inside the circle $|z| = y_0$. Also, by the above, the outer boundary of E_j cannot lie strictly outside the circle $|z| = x_1$ since the inner boundary of A_j has this property. By [2], E_j must contain a round annulus of modulus at least $\alpha - 1/2 > 1/2$. Therefore this round annulus must therefore be contained in the annulus given by $y_0 < |z| < x_1$. Hence the difference between the inner and outer radii of this round annulus is bounded away from 0. This provides the δ in the Theorem.

□

Finally, we complete the proof in the McMullen domain case for all values $n, d \geq 2$ but not both $n = d = 2$ and $|\lambda|$ small. In this general case all the dynamical objects are defined in the same way as in the previous Theorem, but now the moduli of the rings depend on n and d.

As before, let $m = m_\lambda$ be the modulus of $\mathcal{A}$. Now F_λ maps A_0 onto $\mathcal{A}$ as an n-to-1 covering map and maps A_0' onto $\mathcal{A}$ as an d-to-1 covering map. Hence the moduli of A_0 and A_0' are m/n and m/d, respectively. The complementary annulus, denoted by E_0, has modulus $m(1-1/n-1/d)$, since

$$\operatorname{mod}(\mathcal{A}) = \operatorname{mod}(A_0') + \operatorname{mod}(E_0) + \operatorname{mod}(A_0).$$

Inductively, we denote by A_{j+1}, E_{j+1} and A_{j+1}' the preimage of A_j, E_j and A_j' in A_j. Since F_λ maps A_j as a n-to-1 covering map over $A_j \cup E_j \cup A_j'$, we have that the moduli of A_{j+1}, E_{j+1} and A_{j+1}' are $\operatorname{mod}(A_j)/n, \operatorname{mod}(E_j)/n$ and $\operatorname{mod}(A_j')/n$, respectively. Taking this into account, we obtain that

$$\begin{aligned}
\mathrm{mod}\,(A_{j+1}) &= \frac{\mathrm{mod}\,(A_0)}{n^{j+1}} = \frac{m}{n^{j+2}} \\
\mathrm{mod}\,(E_{j+1}) &= \frac{\mathrm{mod}\,(E_0)}{n^{j+1}} = \frac{m}{n^{j+2}}\left(1-\frac{1}{n}-\frac{1}{d}\right) \\
\mathrm{mod}\,(A'_{j+1}) &= \frac{\mathrm{mod}\,(A'_0)}{n^{j+1}} = \frac{m}{dn^{j+1}}\,,
\end{aligned}$$

for each $j \geq 0$. As before, $m = m_\lambda \to \infty$ when λ tends to 0, so we can choose j such that

$$1 < \mathrm{mod}\,(E_j) \leq n\left(1-\frac{1}{n}-\frac{1}{d}\right).$$

Note that, for this value of j, we have that $1 < \mathrm{mod}\,(A_j) \leq n$, since $0 < (1-\frac{1}{n}-\frac{1}{d}) < 1$ for $n, d \geq 2$ but not both $n = d = 2$.

We denote by $\alpha = \alpha(\lambda)$ and $\beta = \beta(\lambda)$ the modulus of A_j and E_j, respectively. Given any ϵ with $0 < \epsilon < 1$, we define the following four auxiliary values $x_0 = (1-\epsilon)\exp(-2\pi\alpha)$, $x_1 = (1+\epsilon)\exp(-2\pi\alpha)$, $y_0 = (1-\epsilon)\exp(-2\pi(\alpha+\beta))$ and $x_0 = (1+\epsilon)\exp(-2\pi(\alpha+\beta))$. Using the same argument as in the previous Theorem, E_j contains a round annulus of modulus at least $\beta - 1/2 > 1/2$. Therefore this round annulus must be contained in the round annulus given by $y_0 < |z| < x_1$.

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