1 Singular Perturbations of Complex Analytic Dynamical Systems

Robert L. Devaney *

Boston University, Boston, MA 02215, USA

Abstract. Our goal in this paper is to describe the dynamical behavior of singular perturbations of complex dynamical systems. Singular perturbations occur when a pole is introduced into the dynamics of a polynomial. In this paper, we consider the simplest possible case: start with the map $z \mapsto z^n$ where $n > 1$ and then add a pole at the origin. For simplicity, we consider the case of maps of the form $z^n + \lambda/z^n$. We then describe some of the many ways Sierpinski curve Julia sets arise in this family. We also give a classification of the dynamics on these sets and describe the intricate structure that occurs around the McMullen domain in the parameter plane for these maps. Finally, we discuss some of the major differences between the cases $n = 2$ and $n > 2$.

* Please address all correspondence to Robert L. Devaney, Department of Mathematics, Boston University, 111 Cummington Street, Boston MA 02215, or e-mail bob@bu.edu.

1.1 Introduction

Our goal in this paper is to describe the behavior of singularly perturbed complex analytic dynamical systems. By a singular perturbation we mean the following. Suppose we have a complex analytic map F_0 which, for simplicity, we will assume to be a polynomial of degree d. Consider the new map F_λ obtained by adding a pole at $a \in \mathbb{C}$ so that

$$F_\lambda(z) = F_0(z) + \frac{\lambda}{(z-a)^m}$$

where $\lambda \in \mathbb{C}$ is a parameter. The map F_λ is a singular perturbation of F_0 as soon as $\lambda \neq 0$ since the degree of the map changes from d to $d+m$. As a consequence, the dynamical behavior of F_λ is much richer than that of F_0, although some portions of the dynamics of F_0 persist depending upon the location of the pole a.

The reason for the interest in singular perturbations arises from Newton's method. Suppose we are applying Newton's method to find the roots of a family of polynomials P_λ which has a multiple root at, say, the parameter $\lambda = 0$. For example, consider the especially simple case of $P_\lambda(z) = z^2 + \lambda$. When $\lambda = 0$ this polynomial has a multiple root at 0 and the Newton iteration function is simply $N_0(z) = z/2$. However, when $\lambda \neq 0$, the Newton iteration function becomes

$$N_\lambda(z) = \frac{z^2 - \lambda}{2z} = \frac{z}{2} - \frac{\lambda}{2z}$$

and we see that, as in the family F_λ, the degree jumps as we move away from $\lambda = 0$. In addition, instead of just having a globally attracting fixed point at the origin, after the perturbation, the dynamical behavior of N_λ become much richer.

For simplicity, in this paper, we will consider the simplest possible case where $F_0(z) = z^n$ and $n \geq 2$. So the dynamics of F_0 are well understood. If $|z| < 1$, the orbit of z tends to the attracting fixed point at the origin under iteration of F_0. If $|z| > 1$, the orbit of z tends to ∞ which is also an attracting fixed point for F_0 in the Riemann sphere. On the circle $|z| = 1$, the map is well known to be chaotic; this set is the *Julia set* of F_0, which will be defined below.

For most of this paper, the singularly perturbed family will be of the form

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

where again $n \geq 2$. So the attracting fixed point at the origin is replaced by a pole, just as in the Newton's method example above.

We remark that we could equally well consider the case where the degree of perturbation is $d \neq n$, but the case $d = n$ has some special symmetries that make this case easier to understand.

The principal goal of this paper is to give a survey of some of the many recent results concerning the dynamical and parameter planes for the family F_λ. A subtheme in this paper is to show that the case where $n = 2$ is quite different from the case where $n > 2$. Indeed, this lower degree family of maps turns out to have much more complicated dynamical behavior than the case where $n > 2$. Another subtheme is to describe some of the interesting topological spaces that arise as the Julia sets of the maps F_λ.

1.2 Preliminaries

Consider the family of maps

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

where $n > 2$. The *Julia set* of F_λ, denoted by $J(F_\lambda)$, is defined to be the set of points at which the family of iterates of F_λ fails to be a normal family in the sense of Montel. There are many other equivalent definitions of the Julia set. For example, the Julia set is the closure of the set of repelling periodic points of F_λ, and, in our special case, it is also the boundary of the set of points whose orbits escape to ∞. As a consequence, the Julia set is the set of points on which F_λ behaves chaotically, since arbitrarily close to any point in $J(F_\lambda)$ there is both a repelling periodic point and a point whose orbit escapes to ∞. The complement of the Julia set is called the *Fatou set.*

There are a number of critical points for F_λ. One critical point occurs at ∞, which is always a fixed point. A second critical point occurs at the origin, which is mapped directly to ∞. A straightforward computation shows that there are $2n$ other critical points for F_λ given by $c_\lambda = \lambda^{1/2n}$. These are the *free* critical points for F_λ. Despite the fact that there are $2n$ free critical points, there are only two critical values given by $v_\lambda = \pm 2\sqrt{\lambda}$. However, there really is only one free critical orbit since, when n is even, both of the critical values are mapped to the same point. When n is odd, the orbits of the two critical values behave symmetrically under the map $z \mapsto -z$. Thus this family of maps, like the quadratic polynomial family, is a natural one-parameter family of maps. The parameter plane (the λ-plane) is then a record of the behavior of the free critical orbit, just as in the case of the Mandelbrot set. There are also $2n$ prepoles at the points $p_\lambda = (-\lambda)^{1/2n}$, i.e., $F_\lambda(p_\lambda) = 0$.

Here is one reason why the case $n = 2$ is so different from the case $n > 2$. Consider the second iterate of the critical points. We compute

$$F_\lambda^2(c_\lambda) = 2^n \lambda^{n/2} + \frac{1}{2^n \lambda^{(n/2)-1}}.$$

When $n > 2$, it follows that $F_\lambda^2(c_\lambda) \to \infty$ as $\lambda \to 0$. But when $n = 2$, the second iterate of c_λ reduces to $4\lambda + 1/4$, so $F_\lambda^2(c_\lambda) \to 1/4$ as $\lambda \to 0$. The reason this is significant will become clear in the next section.

Let C_λ be the circle given by $|z| = |\lambda|^{1/2n}$. Note that C_λ contains all of the critical points and the prepoles. A straightforward computation shows that F_λ maps C_λ $2n$ to 1 onto the straight line connecting the two critical values. We call C_λ the *critical circle* and its image the *critical line*. This is not the case for the family $z^n + \lambda/z^d$ with $n \neq d$, so this is one of the reasons why we deal only with the case $n = d$. One may also check that any other circle centered at the origin is mapped n to 1 onto an ellipse whose foci are $\pm v_\lambda$. Also, the straight ray connecting 0 to ∞ and passing through c_λ (a *critical ray*) is mapped 2 to 1 onto the straight ray connecting either v_λ or $-v_\lambda$ to ∞ and extending the critical line. Similarly, each straight ray connecting 0 to ∞ and passing through p_λ (a *prepole ray*) is mapped 1 to 1 onto the straight line passing through the origin and perpendicular to the critical line.

The maps F_λ have additional symmetries. Let ω be a primitive $2n^{\text{th}}$ root of unity. Then we have that $F_\lambda(\omega z) = \omega^n F_\lambda(z)$. Hence the orbits of points of the form $\omega^j z$ all behave "symmetrically" under iteration of F_λ. For example, if $F_\lambda^i(z) \to \infty$, then $F_\lambda^i(\omega^k z)$ also tends to ∞ for each k. If $F_\lambda^i(z)$ tends to an attracting cycle, then so does $F_\lambda^i(\omega^k z)$. However, the cycles involved may be different depending on k and, indeed, they may even have different periods. Nonetheless, all points lying on this set of attracting cycles are of the form $\omega^j z_0$ for some $z_0 \in \mathbb{C}$. Another symmetry is given by the involution $H_\lambda(z) = \sqrt{\lambda}/z$. Here we have $F_\lambda(H_\lambda(z)) = F_\lambda(z)$.

When $|z|$ is large, the term λ/z^n in the formula for F_λ is negligible, so $F_\lambda(z) \approx z^n$ near ∞. Consequently, the point at ∞ is a superattracting fixed point for F_λ and it is well known that F_λ is conjugate to $z \mapsto z^n$ in a neighborhood of ∞, so we have an immediate basin of attraction B_λ at ∞. Since F_λ has a pole of order n at 0, there is an open neighborhood of 0 that is mapped n to 1 onto a neighborhood of ∞ in B_λ. If the entire basin of ∞ is disjoint from this neighborhood around the origin, then there is a open set about 0 that is mapped n to 1 onto B_λ, and this entire set is disjoint from B_λ. This set is called the *trap door*, since any orbit that eventually enters B_λ must do so by passing through the trap door. We denote the trap door by T_λ.

Using the symmetry $F_\lambda(\omega z) = \omega^n F_\lambda(z)$, it is straightforward to check that B_λ, T_λ, and $J(F_\lambda)$ are all symmetric under $z \mapsto \omega z$. We say that these sets possess $2n$-fold symmetry. In particular, since the critical points are arranged symmetrically about the origin, it follows that if one of the critical points lies in B_λ (resp., T_λ), then all of the critical points lie in B_λ (resp., T_λ). Also, the map H_λ interchanges B_λ and T_λ and preserves both the Julia and the Fatou set.

For other components of the Fatou set, the symmetry situation is somewhat different: either a component contains $\omega^j z_0$ for a given z_0 in the Fatou set and all $j \in \mathbb{Z}$, or else such a component contains none of the $\omega^j z_0$ with $j \neq 0 \bmod 2n$. See [6] for a proof of this fact.

1.3 The Escape Trichotomy

For the well-studied family of quadratic maps $Q_c(z) = z^2 + c$ with c a complex parameter there is the well known Fundamental Dichotomy:

1. If the orbit of the only free critical point 0 tends to ∞, then the Julia set of Q_c is a Cantor set;
2. If the orbit of 0 does not tend to ∞, then the Julia set is a connected set.

In this section we discuss a similar result for F_λ that we call the Escape Trichotomy. Unlike the family of quadratic maps Q_c, there exist three different "ways" that the critical orbit for F_λ can tend to infinity. If the critical orbit tends to infinity, then all of the critical values must lie in either B_λ, T_λ, or some of preimages of T_λ. These three different possibilities lead to the three distinct types of Julia sets for F_λ that comprise the Escape Trichotomy.

Theorem (The Escape Trichotomy). *Suppose $n \geq 2$ and that the orbits of the free critical points of F_λ tend to ∞. Then:*

1. *If one of the critical values lies in B_λ, then $J(F_\lambda)$ is a Cantor set and $F_\lambda \mid J(F_\lambda)$ is a one-sided shift on $2n$ symbols. Otherwise, the preimage T_λ is disjoint from B_λ.*
2. *If one of the critical values lies in $T_\lambda \neq B_\lambda$, then $J(F_\lambda)$ is a Cantor set of simple closed curves (quasicircles), all concentric about the origin. This case does not occur when $n = 2$.*
3. *If one of the critical values lies in a preimage of B_λ that is different from T_λ, then $J(F_\lambda)$ is a Sierpinski curve.*

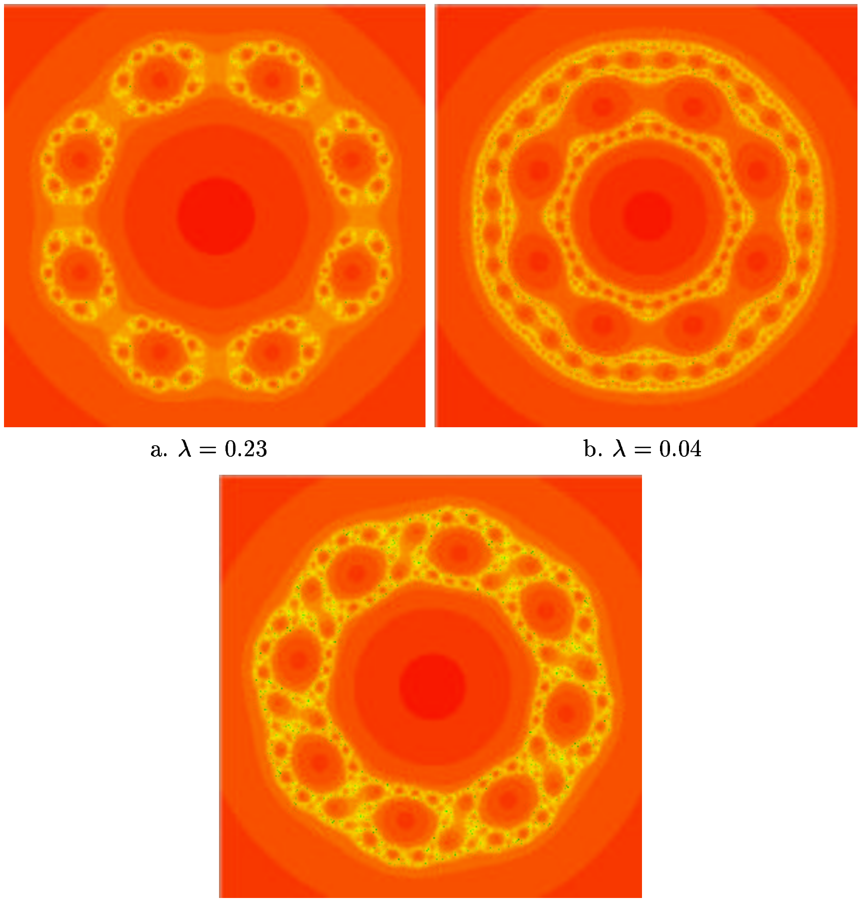
If the critical orbits never escape to ∞, then $J(F_\lambda)$ is a connected set.

Several Julia sets illustrating this trichotomy and drawn from the family where $n = 4$ are included in Figure 1.1.

A *Sierpinski curve* is a very interesting topological space. By definition, a Sierpinski curve is a planar set that is homeomorphic to the well-known Sierpinski carpet fractal. But a Sierpinski curve has an alternative topological characterization: any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by disjoint simple closed curves is known to be homeomorphic to the Sierpinski carpet [16]. Moreover, such a set is a universal planar set in the sense that it contains a homeomorphic copy of any compact, connected, one-dimensional subset of the plane.

We remark that the second part of the Escape Trichotomy was first proved by McMullen [11]. He showed that this result only holds if $n > 2$ and λ is sufficiently close to 0. Indeed, the critical values can never lie in T_λ when $n = 2$ for $|\lambda|$ small since the image of v_λ tends to $1/4$ as $\lambda \to 0$. When $n > 2$, the image of v_λ tends ∞ as $\lambda \to 0$.

In Figure 1.2, we show the λ plane in the case $n = 4$. The outer region in this image consists of λ-values for which $J(F_\lambda)$ is a Cantor set and is called



a. $\lambda = 0.23$ b. $\lambda = 0.04$

c. $\lambda = -0.12$

Fig. 1.1. Some Julia sets for $z^4 + \lambda/z^4$: if $\lambda = 0.23$, $J(F_\lambda)$ is a Cantor set; if $\lambda = 0.0006$, $J(F_\lambda)$ is a Cantor set of circles; and if $\lambda = 0.125i$, $J(F_\lambda)$ is a Sierpinski curve.

the *Cantor locus*. The central disk is the *McMullen domain* in which $J(F_\lambda)$ is a Cantor set of simple closed curves. The region between these two sets is called the *connectedness locus* as the Julia sets are always connected when λ lies in this region. The other disks in the connectedness locus correspond to *Sierpinski holes* in which the corresponding Julia sets are Sierpinski curves.

1.4 Proof of the Escape Trichotomy

In this section we provide a rough sketch of the proof of the Escape Trichotomy Theorem.

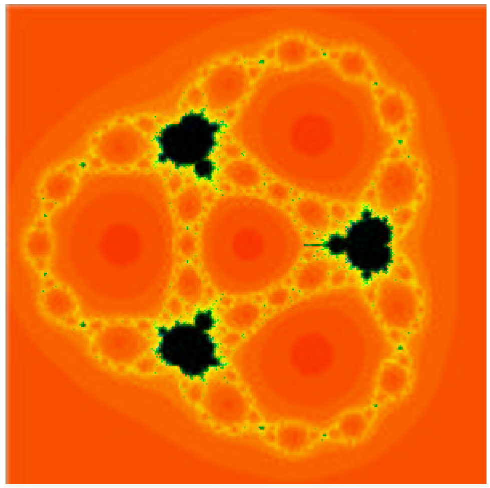


Fig. 1.2. The parameter plane when $n = 4$.

First let v_λ be one of the critical values of F_λ and assume $v_\lambda \in B_\lambda$. So $\pm v_\lambda$ both lie in B_λ. Let $\pm\gamma$ be two symmetric and disjoint curves lying in B_λ and connecting $\pm v_\lambda$ to ∞. Let μ_j be the $2n$ preimages of $\pm\gamma$ for $j = 1, \ldots, 2n$. One checks easily that each μ_j lies in B_λ, extends from 0 to ∞, and contains a single critical point. Hence B_λ and T_λ are not disjoint. Let U be a neighborhood of ∞ that contains $\pm\gamma$ and is $2n$-fold symmetric. We may choose U so that U meets each μ_j in a single arc. Let $V = H_\lambda(U)$ where we may assume that U and V are disjoint. Then both U and V lie in B_λ as do the μ_j. So $J(F_\lambda)$ lies in the complement of the union of U, V, and the μ_j, which is a collection of $2n$ simply connected sets, $I_1, \ldots, I_{2n}$. It follows easily that each of the I_j is mapped univalently onto a region that contains all of the I_k. Then standard techniques from complex dynamics show that the set of points whose orbits remain for all time in the union of the I_k is a Cantor set and the restriction of F_λ to this set is conjugate to the one-sided shift map on $2n$ symbols. This Cantor set is $J(F_\lambda)$.

For case 2, we have by assumption that B_λ and T_λ are disjoint open disks and all of the critical points lie in $F_\lambda^{-1}(T_\lambda)$. We claim that $F_\lambda^{-1}(T_\lambda)$ is an open annulus A. To see this note that $F_\lambda^{-1}(T_\lambda)$ cannot be a collection of $2n$ disjoint disks, for, in such a case, each of the disks would be mapped at least 2 to 1 onto T_λ. Therefore there would be at least $4n$ preimages of any point in T_λ, contradicting the fact that the degree of F_λ is $2n$. Hence at least two of the preimages of T_λ must intersect. But then, by symmetry, all of these preimages must intersect so the preimage of T_λ is a single open connected set. But then the Riemann-Hurwitz formula implies that A must be an annulus that contains all of the critical points.

Now A lies in the annular region that separates B_λ and T_λ and divides this region into two closed subannuli, A_{in} and A_{out}. Then F_λ takes each of these annuli as an n to 1 covering onto the entire complement of $B_\lambda \cup T_\lambda$. This implies that there are a pair of subannuli that are mapped onto A, one of which lies in each of A_{in} and the other in A_{out}. Continuing, one shows that $F_\lambda^{-j}(T_\lambda)$ consists of 2^{j-1} open, disjoint subannuli, each of which lies in the Fatou set. The complement of the union of all of these annuli (together with B_λ and T_λ) is then the Julia set which can be shown to be a Cantor set of simple closed curves.

Incidentally, the above argument indicates why this case cannot happen when $n = 2$. Since A_{in} and A_{out} are each mapped as 2 to 1 coverings of the annulus $\mathbb{C} - \{B_\lambda \cup T_\lambda\}$, it follows that the modulus of each of these subannuli is exactly half of the modulus of the bigger annulus. This implies that there is no room in $\mathbb{C} - \{B_\lambda \cup T_\lambda\}$ for any other sets, eliminating the possibility of the existence of A and further preimages of T_λ. So v_λ cannot lie in T_λ when $n = 2$.

We now describe the final case of the Escape Trichotomy where the critical values have orbits that eventually escape through the trap door, but the critical values themselves do not lie in T_λ. In this case the Julia set is a Sierpinski curve. To show this, we need to verify the five properties that characterize a Sierpinski curve. It turns out that four and a half of these preperties are trivial to show. First and second, since we are assuming that the critical orbit eventually enters the basin of ∞, we have that the Julia set is given by $\mathbb{C} - \cup F_\lambda^{-j}(B_\lambda)$. That is, $J(F_\lambda)$ is $\mathbb{C}$ with countably many disjoint, simply connected, open sets removed. Hence $J(F_\lambda)$ is compact and connected. Third, since $J(F_\lambda) \neq \mathbb{C}$, it is known that $J(F_\lambda)$ cannot contain any open sets, so $J(F_\lambda)$ is nowhere dense. Fourth, since the critical orbits all tend to ∞ and hence do not lie in or accumulate on $J(F_\lambda)$, it follows that F_λ is hyperbolic on $J(F_\lambda)$ and standard arguments show that $J(F_\lambda)$ is locally connected (see [12]). Hence $J(F_\lambda)$ fulfills the first four of the conditions to be a Sierpinski curve.

To finish proving that $J(F_\lambda)$ is a Sierpinski curve we need to show that the boundaries of B_λ as well as all of the preimages of B_λ are simple closed curves and that these boundary curves are pairwise disjoint. Now, since B_λ is a simply connected component of the Fatou set, it follows that the boundary of B_λ is locally connected. However, this boundary may have pinch points as in the case of quadratic Julia sets such as the basilica or Douady's rabbit. This is the only non-standard portion of the proof. However, assuming these boundaries are indeed simple closed curves, they cannot intersect because any such intersection point would necessarily be a critical point. This follows because, for some high power k, F_λ^k takes both of these curves to the boundary of B_λ, so the orbit of any intersection point must pass through a critical point since F_λ^k is locally 2 to 1 there. For the proof that these boundaries are simple closed curves, we refer to [6].

1.5 Classification of Escape Time Julia Sets

As can be seen in Figure 1.2, there are a large number of Sierpinski holes in the parameter planes for these maps. We say that such a hole has *escape time* κ if, for each λ in the hole, the critical orbits land in B_λ at iteration κ. A parameter λ is called the center of the Sierpinski hole if the orbit of the critical points of F_λ all land on the point at ∞ rather than tend to ∞. The following result is proved in [2], [15].

Theorem. *There is a unique center of each Sierpinski hole. Moreover, there are exactly* $(n-1)(2n)^{\kappa-3}$ *Sierpinski holes with escape time* κ *in the parameter plane.*

The proof of this result uses quasiconformal surgery techniques to show that there is a unique center of each Sierpinski hole. Given this, the equation for the centers of the holes, namely $F_\lambda^{\kappa-1}(c_\lambda) = 0$, is easily seen to reduce to a polynomial equation of degree $(n-1)(2n)^{\kappa-3}$, and so the roots of this equation are all distinct.

As an example of the above count of Sierpinski holes, when $n = 3$ there are 2 Sierpinski holes in the parameter plane with escape time 3, 12 holes with escape time 4, 432 Sierpinski holes with escape time 6, and $120,932,352$ holes with escape time 13. All of the parameters from this large collection of holes thus have Julia sets that are homeomorphic, so the natural question is: are the dynamics on these Julia sets the same?

The answer to this question is given in [8]:

Theorem. (Escape Time Conjugacy). *Let*

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n} \text{ and } F_\mu(z) = z^n + \frac{\mu}{z^n}$$

where λ *and* μ *are parameters that lie in Sierpinski holes.*

1. *If* λ *and* μ *lie in the same Sierpinski hole, then* F_λ *and* F_μ *are topologically conjugate on their Julia sets;*
2. *If* λ *and* μ *lie in Sierpinski holes with different escape times, then* F_λ *and* F_μ *are not topologically conjugate on their Julia sets;*
3. *Suppose* λ *and* μ *are centers of different Sierpinski holes that have the same escape time. Let* α *be a primitive* $(n-1)^{\text{st}}$ *root of unity. Then* F_λ *and* F_μ *are topologically conjugate on their Julia sets if and only if, for some integer* j*,*
   - $\mu = \alpha^{2j}\lambda$, *or*
   - $\mu = \alpha^{2j}\overline{\lambda}$.

   Therefore, if λ *and* μ *are parameters lying in different Sierpinski holes whose escape times are the same, then* F_λ *and* F_μ *are topologically conjugate on their Julia sets if and only if the centers of these Sierpinski holes have the above property.*

The proof of the first part of this theorem follows by quasiconformal surgery techniques. The second part follows from the fact that any conjugacy between F_λ and F_μ must take ∂B_λ to ∂B_μ, ∂T_λ to ∂T_μ, and the k^{th} preimages of ∂T_λ to the corresponding preimages of ∂T_μ. But the preimages of T_λ and T_μ containing the critical points are special: their boundaries are mapped 2 to 1 onto their images, and these are the only preimages of ∂T_λ and ∂T_μ that have this property. Hence, two such conjugate maps must have the same escape times. Finally, for part three, it suffices to consider the maps that are the centers of the corresponding holes. But these maps are "critically finite" in the sense that all of the critical orbits eventually land on the fixed point at ∞. By Thurston's Theorem [9], in the orientation preserving case, two such maps can be conjugated by a Möbius transformation. But such a conjugacy must then take ∞ to ∞ (since ∞ is the only superattracting fixed point) and 0 to 0 (since 0 is the only preimage of ∞). It follows that the conjugacy must be of the form $z \mapsto \alpha z$. Then, comparing coefficients in the conjugacy equation

$$\alpha F_\lambda(z) = F_\mu(\alpha z)$$

shows that $\alpha^{n-1} = 1$ and $\mu = \alpha^2 \lambda$. In the case of an orientation reversing conjugacy, it is easy to check that F_λ is conjugate to $F_{\overline{\lambda}}$ via $z \mapsto \overline{z}$, so this gives all of the possible conjugate centers of Sierpinski holes.

This result allows us to give a precise count of the number of different conjugacy classes of escape time Sierpinski curves, because only those holes that are symmetric under rotation by successive squares of a primitive $(n-1)^{\text{st}}$ root of unity or by complex conjugation have the same dynamics.

Theorem (Number of Conjugacy Classes.) *The number of topological conjugacy classes of escape time Sierpinski curve Julia sets with escape time κ is given by*

1. $(2n)^{\kappa-3}$ *if n is odd;*
2. $(2n)^{\kappa-3}/2 + 2^{\kappa-4}$ *if n is even.*

For example, when $n = 3$, we have seen that there are exactly 432 Sierpinski holes in this family with escape time 6, but there are only 216 different conjugacy classes of such maps. Similarly, there are $120,932,352$ Sierpinski holes with escape time 13 but only $60,466,176$ different conjugacy classes, so clearly there is a great variety of different dynamical behaviors on these escape time Sierpinski curve Julia sets.

The reason for the different number of conjugacy classes when n is even and odd comes from the fact that, when n is odd, there are no Sierpinski holes that meet the real axis (and so have no comple conjugate holes). Along the real axis there is only a pair of Mandelbrot sets and the McMullen domain. As a consequence, there are always exactly $n-1$ different Sierpinski holes with conjugate dynamics. See Figure 1.3 for a picture of the parameter plane when $n = 3$. When n is even, the situation is very different; there is always a

Cantor necklace along the negative real axis (more about this in Section 7). See Figures 1.2 and 1.5 for pictures of the parameter planes when $n = 4$ and $n = 2$.

1.6 Structure around the McMullen Domain

Recall that, when $n > 2$, if $|\lambda|$ is sufficiently small, v_λ lies in T_λ and $J(F_\lambda)$ is a Cantor set of simple closed curves. The entire region in the parameter plane for which this occurs is called the McMullen domain, $\mathcal{M}$. It is known [3] that $\mathcal{M}$ is an open, simply connected region that is bounded by a simple closed curve. In this section we describe some of the remarkable structure that surrounds $\mathcal{M}$ in the parameter plane. Since there is no McMullen domain when $n = 2$, the structure we describe below is absent in this case.

In Figure 1.3 we display the parameter plane for $n = 3$ together with two successive magnifications around $\mathcal{M}$. Note that there appears to be a collection of closed curves surrounding $\partial\mathcal{M}$ that pass through more and more Sierpinski holes as these curves approach $\partial\mathcal{M}$. Closer inspection seems to indicate that these curves also pass through small copies of Mandelbrot sets as well. This is indeed true, as the following result was shown in [2] and [7].

Theorem (Rings around the McMullen domain). *For each $n \geq 3$, the McMullen domain is surrounded by infinitely many simple closed curves $\mathcal{S}^k$ for $k = 1, 2, \ldots$ having the property that:*

1. *Each curve $\mathcal{S}^k$ surrounds $\mathcal{M}$ as well as $\mathcal{S}^{k+1}$, and the $\mathcal{S}^k$ accumulate on the boundary of the McMullen domain as $k \to \infty$.*
2. *The curve $\mathcal{S}^k$ meets the centers of τ_k^n Sierpinski holes, each with escape time $k + 2$, where*
$$\tau_k^n = (n-2)n^{k-1} + 1.$$
3. *The curve $\mathcal{S}^k$ also passes through τ_k^n centers of baby Mandelbrot sets, and these Mandelbrot sets and Sierpinski holes alternate as the parameter winds around $\mathcal{S}^k$.*

By a center of a baby Mandelbrot set, we mean the parameter drawn from the main cardioid of the Mandelbrot set for which the corresponding attracting cycle is actually superattracting, i.e., one of the critical points of F_λ is periodic. It turns out that all of these baby Mandelbrot sets are buried in the sense that they do not touch the outer boundary of the connectedness locus in the parameter plane. Then it is known that any parameter drawn from the main cardioid of this Mandelbrot set has a Julia set that is also a Sierpinski curve. These Sierpinski curves are dynamically very different from the escape time Sierpinski curves produced by the Escape Trichotomy, since each has an attracting cycle whose boundary curves are disjoint simple closed curves that are invariant under some iterate of F_λ. For an escape time Sierpinski curve, there is only one invariant boundary curve, namely ∂B_λ. For

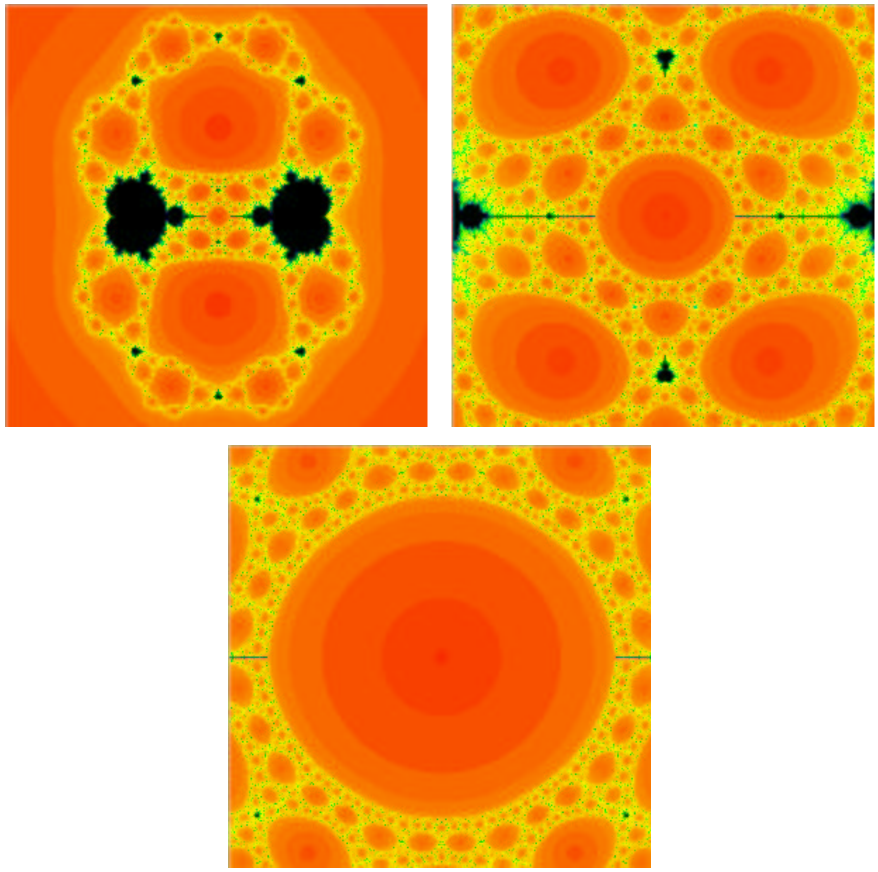


Fig. 1.3. The parameter plane for the family $z^3 + \lambda/z^3$ and several magnifications. The central disk is the McMullen domain $\mathcal{M}$. The simple closed curves in the Theorem accumulate on the boundary of $\mathcal{M}$.

a proof that parameters from the main cardioids of buried baby Mandlebrot sets also yield Sierpinski curves, see [2].

Here are some of the ideas involved in the proof of the rings around $\mathcal{M}$ theorem. When λ satisfies $|\lambda| < 2^{-2n/(n-1)}$, one checks easily that $|v_\lambda| < |c_\lambda|$. So the critical circle C_λ lies in the exterior of its image, the critical line. Then there is a preimage of C_λ, ζ_λ^1, that lies outside of C_λ and is mapped n to 1 onto C_λ. Then there is an outer preimage of ζ_λ^1, ζ_λ^2, that is mapped n to 1 to ζ_λ^1, and so forth. We thus find an infinite collection of closed curves ζ_λ^k moving outward from the critical circle in the dynamical plane and having the property that each ζ_λ^k contains exactly $n^k \cdot 2n$ points that are mapped by F_λ^k to one of the critical points in C_λ and the same number of points that are similarly mapped to a prepole in C_λ. Now F_λ takes the interior of C_λ to the complement of the critical line as an n to 1 covering map. So one can consider the map $\phi(\lambda) = F_\lambda(v_\lambda)$. This is a map that takes the parameter plane to the dynamical plane. One can show that this map is univalent on each of the $n-1$ sectors in the parameter plane bounded by the straight rays through the "spines" of the $n-1$ large Mandelbrot sets symmetrically arranged around the origin. Moreover, ϕ takes each such sector onto $\mathbb{C}$ minus a pair of half-lines.

Now consider a particular k^{th} preimage of one of the critical points lying in C_λ that lies in ζ_λ^k. Call this point u_λ. Then u_λ varies analytically with λ as λ ranges over each of the sectors. So we can consider the analytic map $\Phi(\lambda) = \phi^{-1}(u_\lambda)$. This map takes the sector in the parameter plane to itself. Then one can show using the Schwarz Lemma that Φ has a unique fixed point in this sector. This fixed point is a parameter λ^* for which $\phi(\lambda^*) = u_{\lambda^*}$, i.e., $F_{\lambda^*}(v_{\lambda^*})$ lands on the given k^{th} preimage of a critical point. Then either this critical point or its negative is fixed by $F_{\lambda^*}^{k+2}$. This produces a center of a baby Mandelbrot set for each of the given critical points on ζ_λ^k (modulo an identification as λ winds around the origin). We similarly get centers of Sierpinski holes by letting u_λ be a preimage of a prepole instead of a critical point.

1.7 Cantor Necklaces

There is another structure called *Cantor necklaces* that occurs in both the dynamical and parameter planes for these maps. A Cantor necklace is a planar set that is homeomorphic to the following set. Consider the Cantor middle-thirds set lying in the unit interval. Replace each removed open interval with a circular open disk whose diameter is the same as the length of the removed interval and whose boundary touches the two endpoints of the removed interval. The union of the Cantor set with these countably many open disks is a Cantor necklace.

To see Cantor necklaces in these families of maps, we restrict for simplicity to the case $n = 2$ (though the same proof works more or less verbatim when $n > 2$). Let the four critical points of F_λ be denoted by $c_0, \ldots, c_3$ where the c_j vary analytically with λ and, when $\lambda \in \mathbb{R}^+$, $c_0 \in \mathbb{R}^+$ and the other critcal points are arranged around the origin in counterclockwise fashion. For each λ in the connectedness locus, we may pick $\nu > 1$ so that the circle of radius ν lies in B_λ. Let β_λ be the preimage of this circle that lies in B_λ and let τ_λ be the corresponding preimage in T_λ. Then consider the following two regions I_0 and I_1. The set I_0 is the smaller of the two regions bounded by the critical rays through c_0 and c_3 and the curves β_λ and τ_λ. Let $I_1 = -I_0$. Then F_λ maps both I_0 and I_1 univalently over $I_0 \cup I_1$. Standard techniques from complex dynamics then show that the set of points whose orbits remain for all time in $I_0 \cup I_1$ is a Cantor set, just as in part 1 of the Escape Trichotomy.

Now there is a fixed point in I_0 and a preimage of this fixed point in I_1 and one checks easily that both of these points lie in ∂B_λ. Similarly, there are two preimages of the prefixed point in ∂T_λ, and pre-preimages of these points in two preimages of ∂T_λ. Continuing in this manner, we may adjoin all of these appropriate preimages of T_λ to the "endpoints" in the Cantor set and the resulting set is a Cantor necklace.

To see at least a portion of the Cantor necklace in the parameter plane when $n = 2$, consider the region D in parameter plane that is the portion of the disk of radius 2 centered at the origin that lies in the left half plane. Since the second iterate of the critical point is given by the map $\phi(\lambda) = 4\lambda + 1/4$, it follows that $\phi(D)$ is a half disk that strictly contains D. Now, for each $\lambda \in D$, we can construct the Cantor set in the dynamical plane as above. Furthermore, if $\lambda \in D$, then the set I_1 is easily seen to be contained in the disk D for each $\lambda \in D$, provided the outer radius $\nu < 2$. (The Cantor set construction holds even if λ is not in the connectedness locus.)

Now fix any point z_λ in the Cantor set. Here z_λ is a point with a given itinerary in $I_0 \cup I_1$, so z_λ varies analytically with λ. Thus we have two maps that are defined on D, the map ϕ and z_λ. The map ϕ is invertible, so one may check easily that the map $\lambda \mapsto \phi^{-1}(z_\lambda)$ takes D inside itself. By the Schwarz Lemma, this map then has a unique fixed point in D. This point is the unique parameter for which the second iterate of the critical points all land on the given point in the Cantor set. Hence we get one such parameter corresponding to each point in the Cantor set, and these points then form the Cantor set portion of the Cantor necklace in the parameter plane. To get the other part of the necklace, we adjoin the associated Sierpinski holes just as we did in the dynamical plane case.

One may extend this necklace to include parameters for which the critical orbits land on the portion of the Cantor set in I_0. See [1] for details. Figure 1.4 displays the portion of the Cantor necklace corresponding to parameters for which the critical orbits land on points in I_1.

1.8 The Case n = 2

As mentioned above, the situation in the case $n = 2$ is very different. We no longer have a McMullen domain. Rather, the following result is shown in [1]:

Theorem. *Suppose $n = 2$. Then, in every neighborhood of the origin in the parameter plane, there are infinitely many disjoint open sets $\mathcal{O}_j$, $j = 1, 2, 3, \ldots$, containing parameters having the following properties:*

1. *If $\lambda \in \mathcal{O}_j$, then the Julia set of F_λ is a Sierpinski curve, so that if $\lambda \in \mathcal{O}_j$ and $\mu \in \mathcal{O}_k$, the Julia sets of F_λ and F_μ are homeomorphic;*
2. *But if $k \neq j$, the maps F_λ and F_μ are not topologically conjugate on their respective Julia sets.*

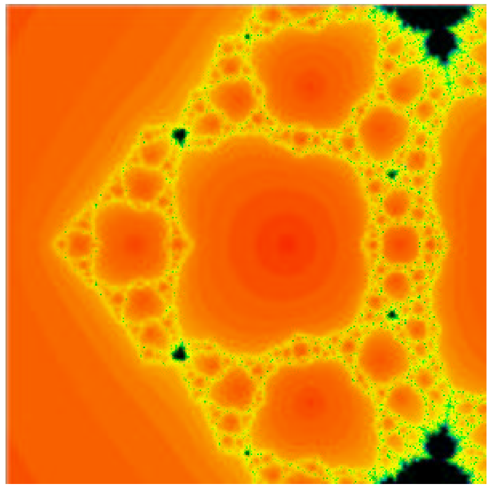


Fig. 1.4. A portion of the Cantor necklace in the parameter plane for the family $z^2 + \lambda/z^2$. Note the large Sierpinski hole along the negative real axis flanked by two smaller Sierpinski holes which are, in turn, each flanked by two smaller Sierpinski holes, etc.

In Figure 1.5 we display the parameter plane for the case $n = 2$ together with a magnification. The large central region is not a McMullen domain; rather it is a Sierpinski hole and it does not contain the origin.

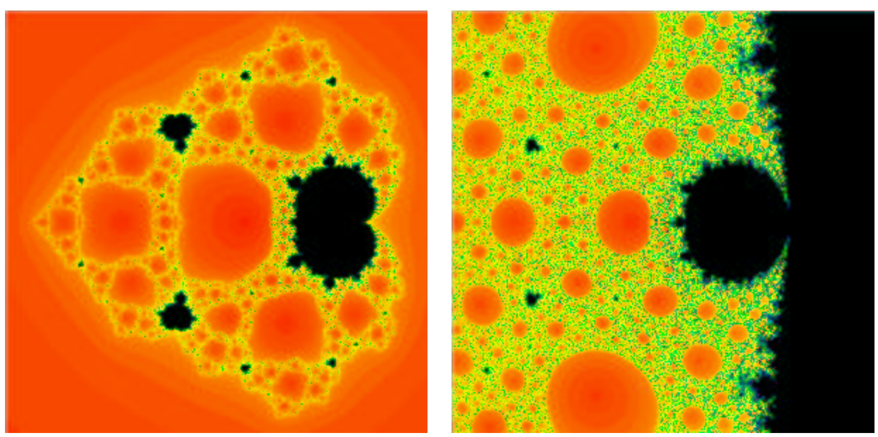


Fig. 1.5. The parameter plane for the family $z^2+\lambda/z^2$ and a magnification centered at the origin.

We sketch the proof of this. We shall show that there are infinitely many open intervals in $\mathbb{R}^-$ in any neighborhood of the origin in parameter space in which the critical orbit eventually escapes. These Sierpinski holes need not lie along $\mathbb{R}^-$; we choose this just to simplify the proof.

Recall that, when $n = 2$, the four critical points and four prepoles of F_λ all lie on the critical circle of radius $|\lambda|^{1/4}$ centered at the origin. Also, the second image of all of the critical points is given by

$$F_\lambda^2(c_\lambda) = 4\lambda + \frac{1}{4}$$

and so $\lambda \mapsto F_\lambda^2(c_\lambda)$ is an analytic function of λ that is a homeomorphism. If $-1/16 < \lambda < 0$, then one checks easily that the critical circle is mapped strictly inside itself. It follows that $J(F_\lambda)$ is a connected set and B_λ and T_λ are disjoint. In particular, the second image of the critical point lands on the real axis and lies in the complement of B_λ in $\mathbb{R}$.

We claim that there is an increasing sequence $\lambda_2, \lambda_3, \ldots$ in $\mathbb{R}^-$ with $\lambda_j \to 0$ and $F_{\lambda_j}^j(c_{\lambda_j}) = 0$ but $F_\lambda^i(c_{\lambda_j}) > 0$ for all $i < j$. To see this, note that, since $F_\lambda^2(c_\lambda) = 4\lambda + 1/4$, this quantity increases monotonically toward $1/4$ as $\lambda \to 0^-$. Now the orbit of $1/4$ remains in $\mathbb{R}^+$ for all iterations of F_0 and decreases monotonically to 0. Hence, given N, for λ sufficiently small, $F_\lambda^j(c_\lambda)$ also lies in $\mathbb{R}^+$ for $2 \leq j \leq N$ and moreover this finite sequence is decreasing. Now suppose $\beta < \alpha < 0$. We have $F_\beta(x) < F_\alpha(x)$ for all $x \in \mathbb{R}^+$. Also, $F_\beta^2(c_\beta) < F_\alpha^2(c_\alpha) < 1/4$. Hence $F_\beta^j(c_\beta) < F_\alpha^j(c_\alpha)$ for all j for which $F_\beta^j(c_\beta) \in \mathbb{R}^+$. The result then follows by continuity of F_λ with respect to λ.

Thus we have infinitely many Sierpinski holes in the parameter plane converging to 0 in $\mathbb{R}^-$. Since the escape times of these Sierpinski holes are all different, it follows that any two parameters drawn from diferent holes in this collection have non-conjugate dynamics (as shown in Section 5).

Note that $\lambda_2 = -1/16$. Using the previous observation, we may find open intervals I_j about λ_j for $j = 2, 3, \ldots$ having the property that, if $\lambda \in I_j$, then $F_\lambda^j(c_\lambda) \in T_\lambda$, and so $F_\lambda^{j+1}(c_\lambda) \in B_\lambda$. Therefore, $F_\lambda^n(c_\lambda) \to \infty$ as $n \to \infty$, and the Escape Trichotomy then shows that $J(F_\lambda)$ is a Sierpinski curve.

1.9 Julia Sets Converging to the Unit Disk

There is another way that the case $n = 2$ differs sharply from the case $n > 2$. Here the situation involves the structure of the Julia sets of F_λ when λ is close to 0. As we saw above, when $n > 2$ and $|\lambda|$ is small enough, λ lies in the McMullen domain and so the corresponding Julia sets are always Cantor sets of concentric simple closed curves. But when $n = 2$, the Julia sets vary wildly; often, but not always, they are Sierpinski curves. For example, in Figure 1.5, note that there is a copy of the Mandelbrot set whose "tail" actually extends to the origin. Whenever λ is chosen in this set, $J(F_\lambda)$ contains small pieces

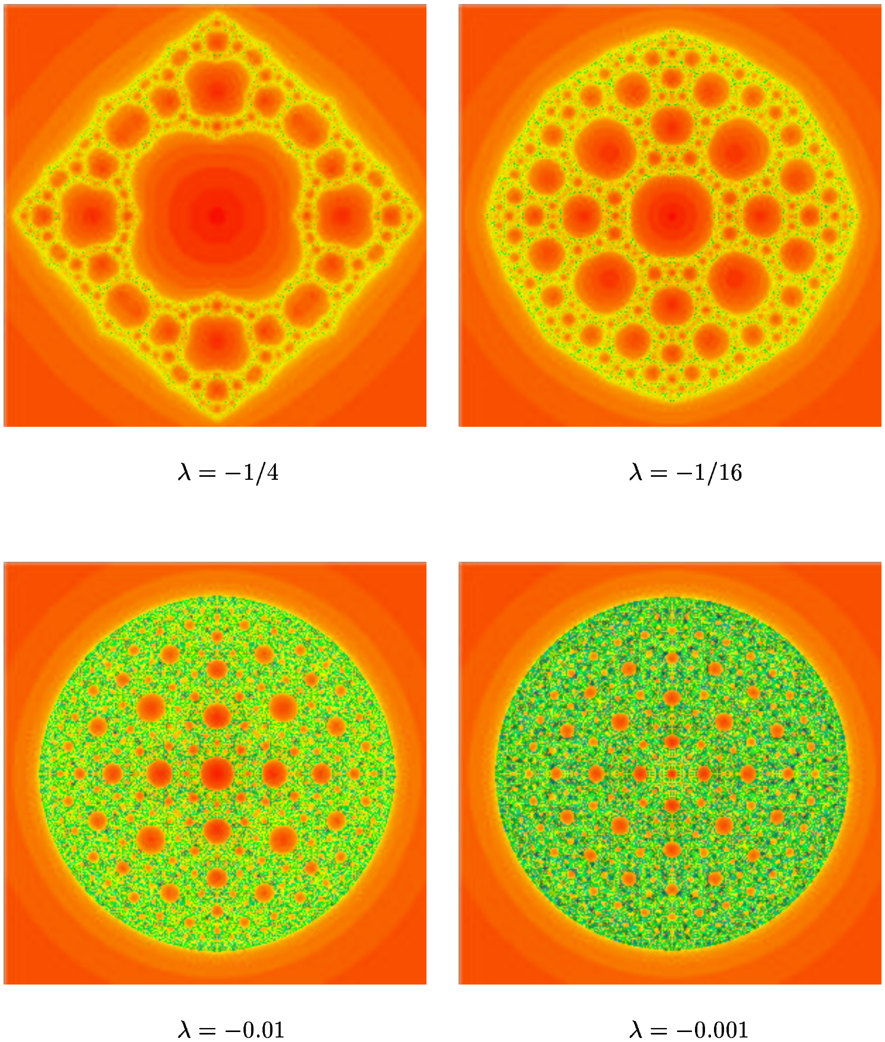


Fig. 1.6. Sierpinski curve Julia sets for various negative values of λ in the case $n = 2$. All of these sets are homeomorphic, but the dynamical behavior on each is very different.

that are homeomorphic to the corresponding Julia set from the quadratic family $z^2 + c$ together with infinitely many preimages of ∂B_λ (as well as other buried points).

In Figure 1.6, note that as the parameter approaches 0, the size of the preimages of the trap door seems to decrease and the Julia set seems to grow "larger." This is indeed the case, for in [4] it was shown:

Theorem. *When $n = 2$, as $\lambda \to 0$, $J(F_\lambda)$ converges to the closed unit disk in the Hausdorff topology. On the other hand, when $n > 2$, this is not the case, as the Fatou set always contains an annular component that contains a round annulus of some fixed width for all λ lying in some disk about the origin.*

This first part of this result is somewhat surprising, since it is well known that Julia sets can never contain open sets unless the Julia set is the entire Riemann sphere. So here we find Julia sets getting arbitrarily close to the unit disk as $\lambda \to 0$. Of course, when $\lambda = 0$, there is an "implosion" and the Julia set is equal to the unit circle, not the unit disk.

The reason why these Julia sets converge to the closed unit disk $\mathbb{D}$ is as follows. Suppose that this is not the case. Then, given any sufficiently small $\epsilon > 0$, we may find a sequence of parameters $\lambda_j \to 0$ and another sequence of points $z_j \in \mathbb{D}$ such that $J(F_{\lambda_j}) \cap B_{2\epsilon}(z_j) = \emptyset$ for each j. Here $B_{2\epsilon}(z_j)$ is the disk of radius 2ϵ about z_j. Since $\mathbb{D}$ is compact, there is a subsequence of the z_j that converges to some nonzero point $z^* \in \mathbb{D}$. For infinitely many parameters in the corresponding subsequence, we then have $J(F_{\lambda_j}) \cap B_\epsilon(z^*) = \emptyset$. Hence we may assume at the outset that we are dealing with a subsequence $\lambda_j \to 0$ such that $J(F_{\lambda_j}) \cap B_\epsilon(z^*) = \emptyset$.

Now consider the circle of radius $|z^*|$ centered at the origin. This circle meets $B_\epsilon(z^*)$ in an arc γ of length ℓ. Choose k so that $2^k\ell \geq 2\pi$. Since $\lambda_j \to 0$, we may choose j large enough so that $|F^i_{\lambda_j}(z) - z^{2^i}|$ is very small for $1 \leq i \leq k$, provided z lies outside the circle of radius $|z^*|/2$ centered at the origin. In particular, it follows that $F^k_{\lambda_j}(\gamma)$ is a curve whose argument increases by at least 2π, i.e., the curve $F^k_{\lambda_j}(\gamma)$ wraps at least once around the origin. As a consequence, the curve $F^k_{\lambda_j}(\gamma)$ must meet the Cantor necklace in the dynamical plane. But the boundary of this necklace lies in $J(F_{\lambda_j})$. Hence $J(F_{\lambda_j})$ must intersect this boundary. Since the Julia set is backward invariant, it follows that $J(F_{\lambda_j})$ must intersect $B_\epsilon(z^*)$. This then yields a contradiction, and so the result follows.

References

1. Blanchard, P., Devaney, R. L., Look, D. M., Seal, P., and Shapiro, Y. Sierpinski Curve Julia Sets and Singular Perturbations of Complex Polynomials. *Ergodic Theory and Dynamical Systems.* **25** (2005), 1047-1055.

2. Devaney, R. L. The McMullen Domain: Satellite Mandelbrot Sets and Sierpinski Holes. *Conformal Geometry and Dynamics.* **11** (2007), 164-190.
3. Devaney, R. L. Structure of the McMullen Domain in the Parameter Planes for Rational Maps. *Fundmenta Mathematicae.* **185** (2005), 267-285.
4. Devaney, R. L. and Garijo, A. Julia Sets Converging to the Unit Disk. *Proceedings of the AMS.* **136** (2008), 981-988.
5. Devaney, R. L. and Look, D. M. Buried Sierpinski Curve Julia Sets. *Discrete and Continuous Dynamical Systems.* **13** (2005), 1035-1046.
6. Devaney, R. L., Look, D. M. and Uminsky, D. The Escape Trichotomy for Singularly Perturbed Rational Maps. *Indiana University mathematics Journal.* **54** (2005), 267-285.
7. Devaney, R. L. and Marotta, S. The McMullen Domain: Rings Around the Boundary. *Transactions of the AMS.* **359** (2007), 3251-3273.
8. Devaney, R. L. and Pilgrim, K. M. Dynamic Classification of Escape Time Sierpinski Curve Julia Sets. To appear.
9. Douady, A. and Hubbard, J. A Proof of Thurston's Topological Characterization of Rational Functions. *Acta. Math.* **171** (1993), 263-297.
10. Douady, A. and Hubbard, J. Etude Dynamique des Polynômes Complexes. Partie I, *Publ. Math. D'Orsay* **84-02** (1984). Partie II, *Publ. Math. D'Orsay* **85-04** (1985).
11. McMullen, C. Automorphisms of Rational Maps. *Holomorphic Functions and Moduli*. Vol. 1. Math. Sci. Res. Inst. Publ. **10**. Springer, New York (1988).
12. Milnor, J. *Dynamics in One Complex Variable.* Princeton University Press, (2006).
13. Milnor, J. and Tan Lei. A "Sierpinski Carpet" as Julia Set. Appendix F in Geometry and Dynamics of Quadratic Rational Maps. *Experiment. Math.* **2** (1993), 37-83.
14. Morosawa, S., Nishimura, Y., Taniguchi, M., and Ueda, T. *Holomorphic Dynamics.* Cambridge University Press (2000).
15. Roesch, P. On Capture Zones for the Family $f_\lambda(z) = z^2 + \lambda/z^2$. In *Dynamics on the Riemann Sphere*, European Mathematical Society (2006), 121-130.
16. Whyburn, G. T. Topological Characterization of the Sierpinski Curve. *Fund. Math.* **45** (1958), 320-324.
17. Yongcheng, Y. On the Julia Sets of Quadratic Rational Maps. *Complex Variables.* **18** (1992), 141-147.