Dynamical Convergence of Polynomials to the Exponential

Clara Bodelón
Robert L. Devaney *
Michael Hayes
Gareth Roberts
Boston University

Lisa R. Goldberg
Barra, Inc.

John H. Hubbard
Cornell University

January 5, 2001

*Please address all correspondence to Robert L. Devaney, Department of Mathematics, Boston University, Boston MA 02215, or email bob@math.bu.edu.

Abstract

In this paper we investigate the relationship between the dynamics of the polynomial maps $P_{d,\lambda}(z) = (1 + z/d)^d$ and the exponential family $E_\lambda(z) = \lambda e^z$. We show that the hyperbolic components of the parameter planes for the polynomials converge to those for the exponential family as the degree d tends to infinity. We also show that certain "hairs" in the parameter plane for the exponential are limits of corresponding external rays for the polynomial families. For parameter values on the hairs, the Julia sets for the corresponding exponentials are the entire plane whereas, for polynomial parameters on the external rays, the Julia sets are Cantor sets.

Keywords Julia set, complex exponential, external ray, hairs.

AMS Subject Classification: 58F23; 30D05.

1 Introduction

Our goal in this paper is to relate the dynamics of the family of complex polynomial maps $P_{d,\lambda}(z) = (1 + z/d)^d$ to the complex exponential family $E_\lambda(z) = \lambda e^z$. Of course, the polynomials $P_{d,\lambda}$ converge uniformly on compact sets to the exponential E_λ as $d \to \infty$. We show in this paper that the polynomials converge to the exponential in a dynamical sense as well.

The $P_{d,\lambda}$ are the degree d analogues of the well-studied quadratic family $Q_c(z) = z^2 + c$, since, like Q_c, each $P_{d,\lambda}$ has a unique critical point (at $-d$) and a unique critical value (at 0). In complex dynamics, the orbit of the critical value determines much of the dynamics. For example, for each $P_{d,\lambda}$, the *filled Julia set* $K_{d,\lambda}$ is given by $\{z \in \mathbb{C} \mid \lim_{n\to\infty} P^n_{d,\lambda}(z) \not\to \infty\}$. The Julia set, $J(P_{d,\lambda})$, is the boundary of the filled Julia set. Just as in the quadratic case, it is known that both the Julia set and the filled Julia set of $P_{d,\lambda}$ are connected if the orbit of the critical value is bounded, whereas the filled Julia set is a Cantor set if the critical orbit is unbounded. Another important property of the critical orbit is the fact that, if $P_{d,\lambda}$ has an attracting cycle, then the orbit of 0 must tend to this cycle. As a consequence, $P_{d,\lambda}$ can have at most one attracting cycle.

For the exponential family, there is no critical point. However, 0 is an asymptotic (omitted) value and its orbit plays a similar role to the orbit of 0 for the polynomial family. For example, if E_λ admits an attracting cycle, then the orbit of 0 must tend to this cycle just as in the polynomial case. Consequently, there can be at most one attracting cycle for E_λ. While there is no analogue of the dichotomy on the topology of the filled Julia sets based on the fate of the orbit of 0 under E_λ, it is known that the set of bounded orbits under E_λ, as well as the set of unbounded orbits, are both dense in the plane when $E_\lambda^n(0)$ tends to ∞. If, on the other hand, E_λ has an attracting cycle, then the basin of attraction of this cycle is open and dence in the plane, so the set of unbounded orbits is nowhere dense.

In case $K_{d,\lambda}$ is connected, there is a natural uniformization of the exterior of the filled Julia set just as in the quadratic case. In the exterior of $K_{d,\lambda}$, $P_{d,\lambda}$ is conjugate to the map $z \mapsto z^d$ in the exterior of the unit disk. Hence the dynamics of $P_{d,\lambda}$ are completely understood outside $K_{d,\lambda}$. The images of the straight rays preserved by $z \mapsto z^d$ under the above conjugacy are called the external rays of $K_{d,\lambda}$.

When the exponential map has an attracting cycle, the analogue of the filled Julia set is the basin of attraction of this cycle. This basin is open and dense in the plane. It is known that the complement [7] of this basin consists of an uncountable set of curves or hairs homeomorphic to a closed half-line. On these hairs, the orbits of all points with the possible exception of the endpoint tend to ∞.

It is natural to ask about the relation between the filled Julia sets of the $P_{d,\lambda}$ and the basins of attraction of E_λ in cases where both admit an attracting cycle. We will show below that the basins of the polynomial family converge to that of the exponential as $d \to \infty$. Moreover, we will show that the external rays with a given symbolic dynamics converge as $d \to \infty$ to particular hairs for the exponential. Thus we have a type of "dynamical" convergence of the polynomial family to the exponential inside and outside the filled Julia sets.

We also consider in this paper the bifurcation sets for the polynomial and

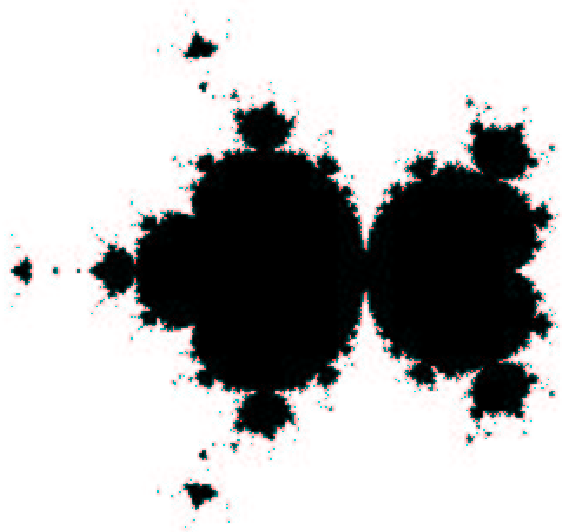

Figure 1: Degree 4 bifurcation set.

exponential families. For each $d \geq 2$, the d^{th}-*bifurcation set* $B_d = \{\lambda \in \mathbb{C} \mid 0 \in K_{d,\lambda}\}$. That is, the d^{th} bifurcation set for the polynomial family consists of those parameter values λ for which the orbit of 0 is bounded. Analogously, the exponential bifurcation set consists of those parameter values λ for which $E_\lambda^n(0)$ is bounded. The most important subset of these bifurcation sets consists of the hyperbolic components wherein the corresponding maps have an attracting cycle of some period. We will show below that the hyperbolic components of the polynomial family converge to similar components for the exponential family.

In Figures 1–3, we display several of these bifurcations sets.

In the quadratic case, the analogue of B_d is the Mandelbrot set. It is well known that the exterior of the Mandelbrot set may be uniformized. That is, there is an analytic isomorphism from the exterior of the unit disk to the exterior of the Mandelbrot set. The images of the straight rays under this isomorphism are called the external rays of the Mandelbrot set, and the symbolic dynamics associated to these rays gives much information regarding the structure of the Mandelbrot set. We will show below that a similar construction works for the B_d so that the external rays of B_d are well-defined.

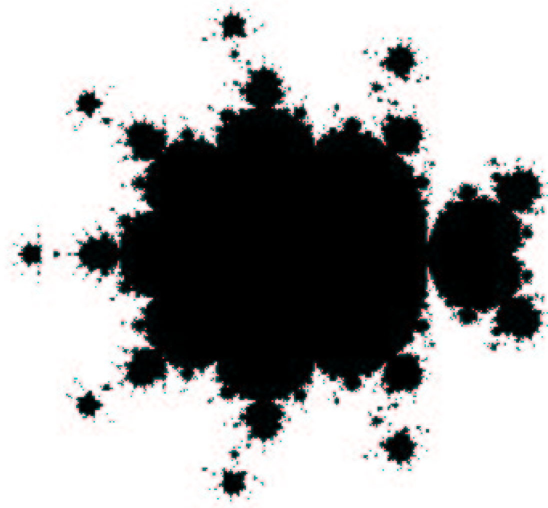

Figure 2: Degree 8 bifurcation set.

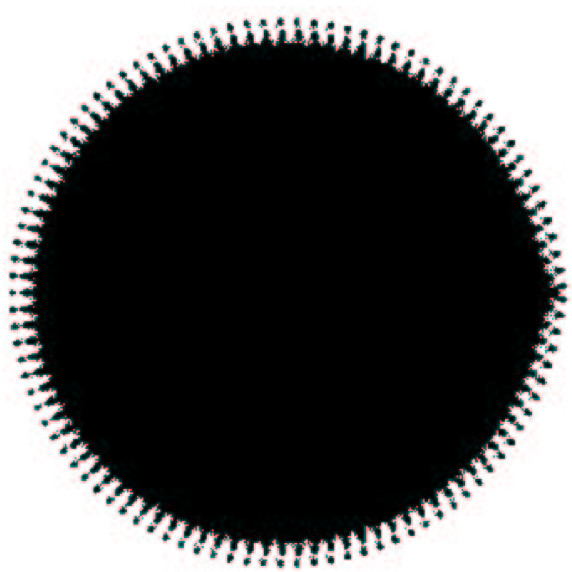

Figure 3: Degree 100 bifurcation set.

Figure 4: More details of the degree 100 bifurcation set.

For the exponential family, the complement of the hyperbolic components includes an uncountable collection of curves (also called hairs) on which the orbit of 0 tends to infinity. As in the dynamical plane, we will show that certain of the external rays for B_d tend to these hairs as $d \to \infty$. Thus we have a similar type of convergence in both the interior and exterior of the B_d as we found for the $K_{d,\lambda}$.

2 The Parameter Plane for $P_{d,\lambda}$

In this section we discuss some properties of the parameter planes for the polynomial family $P_{d,\lambda}$ and their relation to the exponential parameter plane. Recall that the filled Julia set $K_{d,\lambda}$ is the set of points with bounded orbits under $P_{d,\lambda}$. Also, the d^{th}-bifurcation set, B_d, consists of the set of λ-values for which $P^n_{d,\lambda}(0)$ is bounded.

Lemma 2.1 *If $\lambda \in B_d$, then $K_{d,\lambda}$ is connected. Otherwise, $K_{d,\lambda}$ is homeomorphic to a Cantor set.*

Proof: Each $P_{d,\lambda}$ has a single critical value (at 0). Therefore the lemma is a special case of Theorem 9.9 in [Bl].

Definition 2.2 *A polynomial or entire map is hyperbolic if every critical point is attracted by an attracting periodic cycle.*

Let B_d° denote the interior of B_d. A component W of B_d° is hyperbolic if $P_{d,\lambda}$ is hyperbolic for some, and therefore for all $\lambda \in W$. Let

$$C_k^d = \{\lambda \in B_d^\circ \mid P_{d,\lambda} \text{ has an attracting cycle of period } k\}$$

and set

$$C_k = \{\lambda \mid E_\lambda \text{ has an attracting cycle of period } k\}.$$

Recall that $P_{d,\lambda}$ can have at most one finite attracting cycle. For a given d and $\lambda \in C_k^d$, the critical point $-d$ must lie in a component of the basin of attraction of the cycle that contains a single periodic point on the attracting periodic cycle. Denote this point by $z_0 = z_0(\lambda)$ and set $z_i(\lambda) = P_{d,\lambda}^i(z_0)$ for $i = 1, \ldots, k-1$. Define the eigenvalue map $\chi_d \colon C_k^d \to D$ by $\chi_d(\lambda) = (P_{d,\lambda}^k)'(z_0(\lambda))$, where D is the open unit disk. The map χ_d is analytic since $z_0(\lambda)$ is analytic in λ.

We first describe the attracting fixed point regions C_1^d.

Proposition 2.3 *1. C_1^d is bounded by the curve $\zeta \to \zeta/(1+\frac{\zeta}{d-\zeta})^{d-1}$, where $|\zeta| = 1$. This curve is cardiod-like. $\chi_d^{-1} : D \to C_1^d$ is given by $\chi_d^{-1}(\zeta) = \zeta/(1+\frac{\zeta}{d-\zeta})^{d-1}$.*

2. $\lim_{d\to\infty} \chi_d^{-1} = \chi^{-1} : D \to C_1$ where $\chi^{-1}(\zeta) = \zeta e^{-\zeta}$ and χ is the eigenvalue map for the E_λ family.

Proof: The conditions $\lambda \in C_1^d, \chi_d(\lambda) = \zeta$ imply

$$\lambda(1+\frac{z}{d})^d = z$$

and

$$\lambda(1+\frac{z}{d})^{d-1} = \zeta.$$

Therefore,

$$\zeta = \frac{z}{1+z/d} \quad \text{or} \quad z = \frac{\zeta}{1-\zeta/d}$$

and it follows that

$$\lambda = \chi_d^{-1}(\zeta) = \frac{\zeta}{(1+\frac{\zeta}{d-\zeta})^{d-1}}$$

is an inverse for χ_d.

To prove the second statement, fix $\zeta = re^{i\theta} \in \overline{D}$ and choose a branch of the logarithm defined on $\mathbb{C} - \{\mathbb{R}^- \cup \{0\}\}$. Then if $\zeta = re^{i\theta}$

$$\begin{aligned}
\lim_{d\to\infty} \log\Big(1+\frac{\zeta}{d-\zeta}\Big)^{d-1} &= \lim_{d\to\infty}(d-1)\log\Big(\frac{d}{d-\zeta}\Big) \\
&= \lim_{d\to\infty}(d-1)[\ln d - \frac{1}{2}\ln(d^2-2rd\cos\theta+r^2)] \\
&\quad + i\lim_{d\to\infty}(d-1)\tan^{-1}(\frac{r\sin\theta}{d-r\cos\theta}) \\
&= r\cos\theta + ir\sin\theta \\
&= \zeta.
\end{aligned}$$

Therefore,

$$\lim_{d\to\infty}(1+\frac{\zeta}{d-\zeta})^{d-1} = e^{\zeta},$$

and the second statement follows immediately.

Next, we describe the sets C_k^d, for $k > 1$. Unlike the case $k = 1$, the eigenvalue map is never an isomorphism in this case.

Proposition 2.4 *Any connected component W of C_k^d is simply connected.*

Proof: Let $F_d^n(\lambda) = P_{d,\lambda}^n(-d)$. The family of functions F_d^n is a family of entire functions. Let W be a connected component of C_k^d, and let $\gamma \subset W$ be a simple closed curve bounding a region D. We will show that $D \subset W$.

Since W is open, we can choose a neighborhood U of γ in W. Since $U \subset W$, it follows that the functions F_d^{nk} converge to some periodic point $z_i(\lambda)$ on U. Since the F_d^{nk} are entire functions, they are a normal family of functions on D and therefore must converge to the analytic function $z_i(\lambda)$. By the Maximum Principle, $|\chi_d(\lambda)|$ will take its maximum value on γ, the boundary of D, which lies in C_k^d. Thus $|\chi_d(\lambda)| < 1$ on D and $D \subset W$ as desired.

We now consider the covering properties of the eigenvalue map χ_d. To simplify the proof of the next lemma, we introduce the polynomials $Q_{d,c}(z) =$

$z^d + c$, $d \geq 2$, $z \in \mathbb{C}$. Each $Q_{d,c}$ is affine conjugate to $P_{d,\lambda}$ where $\lambda = dc^{d-1}$ via the conjugacy $z \mapsto \frac{cz}{d} + c$, provided $c \neq 0$. In other words, the $(d-1)$-fold covering map $\lambda = \Pi(c) = dc^{d-1}$ from c-space to λ-space is affine conjugacy class preserving.

Let $\tilde{B}_d = \{c \in \mathbb{C} \mid 0 \in K_{d,c}\}$ where $K_{d,c}$ is the filled Julia set of $Q_{d,c}$. The map

$$\Pi|\tilde{B}_d \colon \tilde{B}_d - \{0\} \to B_d - \{0\}$$

is a $(d-1)$-fold covering map.

Lemma 2.5 *For a fixed $k \geq 2$ and d, suppose that $P^k_{d,\lambda_0}(-d) = -d$. Then $\lambda_0 \neq 0$ is a simple zero of the polynomial $G(\lambda) = P^k_{d,\lambda}(-d) + d$.*

Proof. This proof was shown to us by A. Gleason. We have that λ_0 is a root of the equation $P^k_{d,\lambda}(-d) = -d$ if and only if the critical point $-d$ is periodic for P_{d,λ_0} with period some multiple of k. Since each $P_{d,\lambda}$ is affine conjugate to $(d-1)$ distinct $Q_{d,c}$'s, it is equivalent to show that each root c_0 of the polynomial $R(c) = Q^k_{d,c}(0)$ is simple. This is easier since $R(c)$ has integer coefficients and thus R has a discriminant Δ which is an integer.

The discriminant of $\overline{R} = R \bmod d$ is $\overline{\Delta} = \Delta \bmod d$. Thus, if $\Delta = 0$, then $\overline{\Delta} = 0$, so $\overline{R}$ has a multiple root and a root in common with its derivative.

However,

$$\begin{aligned} \frac{d}{dc} R(c) &= d\Big(Q^{k-1}_{d,c}(0)\Big)^{d-1} \cdot \frac{d}{dc}\Big(Q^{k-1}_{d,c}(0)\Big) + 1 \\ &= 1 \bmod d \end{aligned}$$

so the derivative of $\overline{R}$ has no roots and, in particular, none in common with $\overline{R}$.

Theorem 2.6 *Let W be a connected component of C^d_k with $k \geq 2$. Then the eigenvalue map $\chi_d \colon W \to D$ is a $(d-1)$-fold covering map ramified above* 0.

Proof: Let $\lambda_0 \in W$. Then P_{d,λ_0} has an attracting cycle of period k. Let z_0 be a point on this cycle and suppose that $\chi_d(\lambda_0) = (P^k_{d,\lambda_0})'(z_0) = \alpha \neq 0$. Then we can find a neighborhood U of z_0 and an analytic conjugacy $\Phi : U \to \overline{D}$ such that

$$\Phi \circ P^k_{d,\lambda_0}(z) = \alpha \cdot \Phi(z).$$

Choose an $\epsilon > 0$ such that the disk of radius ϵ about α, $B_\epsilon(\alpha)$, is contained in $D - \{0\}$. We will find a neighborhood N of λ_0 in W so that the eigenvalue map $\chi_d : N \to B_\epsilon(\alpha)$ is a homeomorphism. This will show that $\chi_d | W - \chi_d^{-1}(0)$ is a covering map.

Let $\beta \in B_\epsilon(\alpha)$. We first produce a quasiconformal conjugacy between $z \to \alpha z$ and $z \to \beta z$. Let A denote the annulus $\{z : |\alpha| \leq |z| \leq 1\}$ and let B denote the annulus $\{z : |\beta| \leq |z| \leq 1\}$. We define $\Psi : A \to B$ by $\Psi(re^{i\theta}) = r_1 e^{i\theta_1}$ where

$$\begin{cases} r_1 &= 1 + \frac{1-|\beta|}{1-|\alpha|}(r-1) \\ \theta_1 &= \theta + \ln r \left(\frac{\arg\beta - \arg\alpha}{\ln|\alpha|} \right) \end{cases}$$

Note that Ψ is the identity map on the boundary of D and takes $r = |\alpha|$ to $r = |\beta|$ with $\Psi(\alpha) = \beta$. Also, Ψ maps the circles $r = c$ in A to circles centered at the origin in B.

We now extend Ψ to D in the natural way. Given a nonzero $z \in D - A$, there is a smallest positive integer n for which $\alpha^{-n} z \in A - \{r = 1\}$. Define $\Psi(z) = \beta^n \Psi(\alpha^{-n} z)$. In other words, pull z back to A under the map $z \to \alpha^{-1} z$, apply $\Psi : A \to B$, then push forward by the map $z \to \beta z$. Finally, set $\Psi(0) = 0$. By construction, Ψ is a conjugacy between the maps $z \to \alpha z$ and $z \to \beta z$. Note that Ψ is differentiable everywhere except for $z = 0$ and that Ψ is quasiconformal, but not in general conformal.

Next let $\nu_0 = \Psi^*(\sigma_0)$ denote the pullback of the standard complex structure σ_0 on D via Ψ. Note that the map $z \to \alpha z$ preserves ν_0. This follows since $\Psi(\alpha z) = \beta \Psi(z)$ and the conformal map $z \to \beta z$ preserves the standard structure σ_0.

Remark: We can think of ν_0 as a function that assigns a "family" of ellipses, or an infinitesimal ellipse, to each z by specifying the ratio of the major to minor axis and the argument of the major axis.

Now we return to P^k_{d,λ_0} and U. We will define a new ellipse field $\nu \neq \sigma_0$ on $\mathbb{C}$ that is preserved by P_{d,λ_0}. We define ν on U by $\nu(z) = \Phi^*(\nu_0(\Phi(z)))$. In other words, pull back ν_0 to U via Φ. Note that P^k_{d,λ_0} preserves ν since P^k_{d,λ_0} is analytically conjugate to the map $z \to \alpha z$ via Φ, and $z \to \alpha z$ preserves ν_0.

Next, we extend ν to all of the basin of the attracting cycle in the natural way. If $z \notin U$, but in the basin of attraction, then let n be the smallest positive integer such that $P^n_{d,\lambda_0}(z) \in U$. Set $\mu(z) = (P^{-n}_{d,\lambda_0})^*(\nu(P^n_{d,\lambda_0}(z)))$. For $z \notin$ basin of attraction, we set $\nu(z) = \sigma_0(z)$, the standard complex structure. (This occurs on the boundary of the basin of attraction and the basin of attraction of ∞.) Hence ν is a complex structure on all of $\overline{\mathbb{C}}$ which is preserved under P_{d,λ_0}. This is clear on the basin of attraction by construction and follows outside the basin since P_{d,λ_0} is analytic. Also, ν has bounded dilatation on $\overline{\mathbb{C}}$, since it's given by the pullback of an analytic function.

We have constructed an ellipse field on $\overline{\mathbb{C}}$ other than σ_0 that is preserved by P_{d,λ_0}. We now apply the Measurable Riemann Mapping Theorem to produce a unique quasiconformal homeomorphism h that satisfies $h(0) = 0, h(-d) = -d$, and $h(\infty) = \infty$ and $\nu = h^*(\sigma_0)$. That is, h straightens the ellipses of ν. Let $F = h \circ P_{d,\lambda_0} \circ h^{-1}$. Since F preserves σ_0, F is an analytic map that is quasiconformally conjugate to P_{d,λ_0}. Moreover, F has degree d and has only one critical point at $z = -d$. Therefore, F is a polynomial that is affine conjugate to P_{d,λ_0} and of the form $\mu(1 + z/d)^d$.

It is clear by construction that F has an attracting cycle of period k containing the point $h(z_0)$. To show that this attracting cycle has eigenvalue β, we use the fact that $F^k|h(U)$ is conjugate to $P^k_{d,\lambda_0}|U$ which in turn is conjugate to the map $z \to \beta z$. Letting $H : D \to h(U)$ be given by $H(z) = h \circ (\Psi \circ \Phi)^{-1}$, we see that H is a conjugacy between F^k and $z \to \beta z$ which preserves the standard complex structure σ_0 and is therefore an analytic conjugacy. It follows that F has eigenvalue β at the cycle containing $h(z_0)$. See Figure 5.

By the uniqueness of the Measurable Riemann Mapping Theorem, this construction yields a map $\mu : B_\epsilon(\alpha) \to W$ which produces a $P_{d,\lambda}$ with a given multiplier. (The map μ is just a local inverse for χ_d.) Since the Measurable Riemann Mapping Theorem depends continuously on parameters, it follows that μ is continuous. Clearly, μ is one-to-one and onto its image. Thus, letting $N = \mu(B_\epsilon(\alpha))$, we have shown that the eigenvalue map $\chi_d : N \to B_\epsilon(\alpha)$ is a homeomorphism.

All that remains is to compute the ramification index above $0 \in D$. Suppose that $\chi_d(\lambda_0) = 0$ so that $z = -d$ is on the attracting cycle of period k for P_{d,λ_0}. The periodic points $z_i(\lambda)$ are functions of λ for λ near λ_0 and $\chi_d(\lambda) = \prod_i P'_{d,\lambda}(z_i(\lambda)) = \prod_i \lambda(1 + (z_i(\lambda))/d)^{d-1}$. The multiplicity of the zero λ_0 for χ_d is the sum of the multiplicities of the zeroes of the $P'_{d,\lambda}(z_i(\lambda))$. Clearly these are all zero except for the index j where $z_j(\lambda_0) = -d$. To calculate this multiplicity, we apply the Implicit Function Theorem to $F(\lambda, z) = P^k_{d,\lambda}(z) - z$ about the point $(\lambda_0, -d)$. We have

$$F_z(\lambda_0, -d) = \chi_d(\lambda_0) - 1 = -1$$

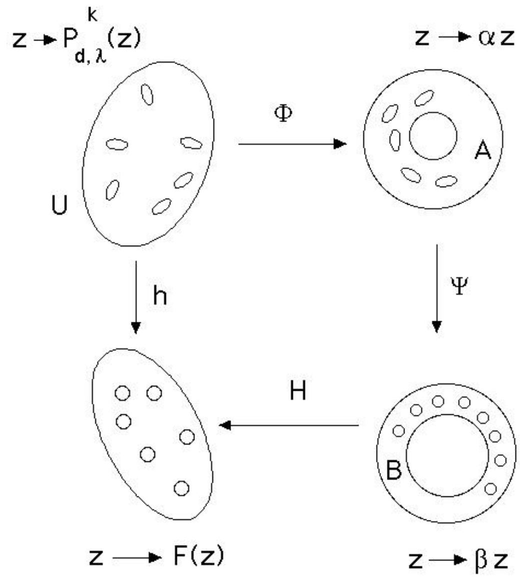

Figure 5: Construction of the ellipse field

and $F_\lambda \neq 0$ by Lemma 2.5. Therefore $z_j'(\lambda_0) \neq 0$ so that as a function of λ, $z_j(\lambda) + d$ has a simple root at $\lambda = \lambda_0$. Therefore, $P'_{d,\lambda}(z_j(\lambda))$ has multiplicity $d-1$ at the zero $\lambda = \lambda_0$. This completes the proof.

Proposition 2.7 *For each $k \geq 2$, there are $(d^{k-1}-1)/(d-1)$ hyperbolic components in the interior of B_d whose period divides k.*

Proof. By Theorem 2.6 each hyperbolic component W of the interior B_d contains a unique λ_W such that $\chi_d(\lambda_W) = 0$. In other words, the critical point $-d$ is periodic, and therefore, the number of hyperbolic components of the interior of B_d whose period divides k is equal to the number of distinct roots of the polynomial $P^k_{d,\lambda}(-d) + d$ in λ.

By induction, $P^k_{d,\lambda}(-d)$ has degree $(d^{k-1}-1)/(d-1)$, and by Lemma 2.5 all the roots of $P_{d,\lambda}(-d) + d$ are simple.

Finally, we show that the hyperbolic components of B_d converge to the corresponding hyperbolic components of the exponential family.

Theorem 2.8 *1. If E_λ has an attracting periodic point of period k, then* *there exists a D such that $P_{d,\lambda}$ has an attracting periodic point of period k for all $d > D$.*

2. *Suppose that for a fixed λ, $P_{d,\lambda}$ has an attracting periodic point of period k for infinitely many d. Then E_λ has a periodic point of period k which is either attracting or indifferent.*

Proof: The first item follows since $P_{d,\lambda}^k$ converges to E_λ^k uniformly on compact sets. For the second item, we need the following lemma.

Lemma 2.9 *Suppose $z_1, \ldots, z_n$ is an attracting periodic orbit for $P_{d,\lambda}$. Then there exists an i for which $|z_i| \leq \frac{d}{d-1}$.*

Proof: We have
$$P'_{d,\lambda}(z) = \frac{P_{d,\lambda}(z)}{1+\frac{z}{d}}.$$
Hence
$$\left|\prod_{i=1}^{n} P'_{d,\lambda}(z_i)\right| = \left|\frac{\prod z_{i+1}}{\prod(1+\frac{z_i}{d})}\right| = \prod\left|\frac{z_i}{1+\frac{z_i}{d}}\right|.$$
Since this term is less than 1, there exists i for which
$$|z_i| \leq |1+\frac{z_i}{d}| \leq 1+\frac{|z_i|}{d}.$$
Hence $|z_i| \leq \frac{d}{d-1}$.

To complete the proof of Theorem 2.8, let $z_i^{(d)}$ lie on a periodic orbit of period k for $P_{d,\lambda}$ and let $\hat{z}$ be an accumulation point of the $z_i^{(d)}$. This point exists by Lemma 2.9. Since $P_{d,\lambda}$ converges to E_λ uniformly on compact sets, we have $E_\lambda^k(\hat{z}) = \hat{z}$. If $|(E_\lambda^k)'(\hat{z})| > 1$, we may find a neighborhood U of $\hat{z}$ such that $|(E_\lambda^k)'(z)| > \xi > 1$ on U. But then, if $z_i^{(d)} \in U$, we have $|(P_{d_i,\lambda}^k)'(z_i^{(d)})| > 1$ for d sufficiently large which is a contradiction.

3 The Dynamical Plane for $P_{d,\lambda}$

We now discuss the dynamical plane for $P_{d,\lambda}(z) = \lambda(1+z/d)^d$. Much of the material in this section is similar in spirit to that for the exponential family described in [7]. Recall that $P_{d,\lambda}$ has a unique critical point at $-d$ with critical value 0.

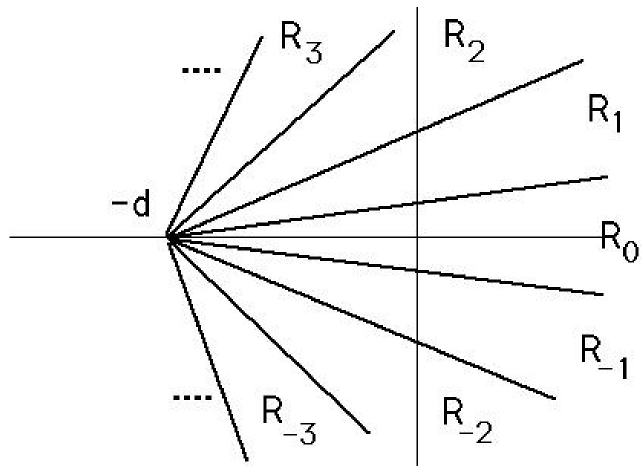

Figure 6: Construction of the wedges for $P_{d,\lambda}$

3.1 A Markov partition for $P_{d,\lambda}$

Throughout this section we will exclude the case $\lambda \in \mathbb{R}'$. This will allow us to construct a continuously varying Markov partition for the polynomial family.

The proof of the following proposition is straightforward.

Proposition 3.1 *1. $P_{d,\lambda}$ maps rays emanating from $-d$ to rays emanating from 0.*

2. $P_{d,\lambda}$ maps the circle $|z+d| = \rho$ to the circle $|z| = |\lambda|(\rho/d)^d$.

We first define a series of wedges as fundamental domains for the Markov partition. Consider the preimages of $\mathbb{R}^-$ under $P_{d,\lambda}$. By Proposition 3.1, these are rays emanating from $-d$ and there are exactly d of them. Denote the d wedges bounded by these rays by $R(j)$ where $R(0)$ is the unique wedge containing 0 and the remaining wedges are indexed consecutively in a counterclockwise direction. We use negative indices for those wedges which are completely contained in the lower half-plane $\text{Im}\, z < 0$. For simplicity, we will not index any wedge which contains an infinite segment of $\mathbb{R}^-$ in its closure. See Figure 6.

We define the itinerary of a point under iteration of $P_{d,\lambda}$ exactly as for E_λ, replacing the strips with the wedges. An itinerary will be called regular

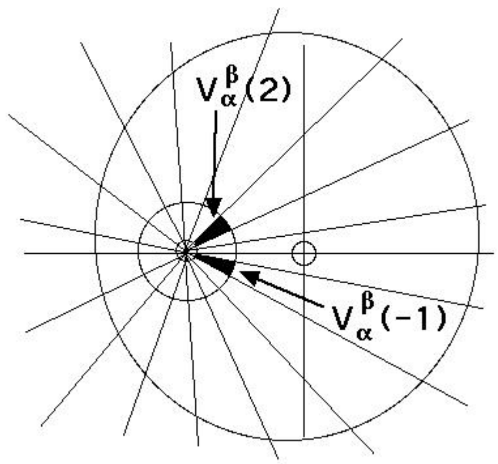

Figure 7: Construction of the $V_\alpha^\beta(s_i)$

if it does not contain a zero in any of its entries. Let Σ_d denote the set of all sequences of integers $s = s_0 s_1 s_2 \ldots$ where $0 < |s_j| < [(d-1)/2]$. While this restriction may exclude one or two of the $R(j)'s$ as a fundamental domain, it will not affect the conclusions that follow.

3.2 Invariant Cantor Sets for $P_{d,\lambda}$

By Proposition 3.1, for a fixed d and λ, we may choose α and β such that the image of the circle $|z+d| = \alpha$ under $P_{d,\lambda}$ is completely contained in $R(0)$ and the image of $|z+d| = \beta$ contains the two circles $|z+d| = \beta$ and $|z| = |\lambda|(\alpha/d)^d$ in its interior. We let

$$V_\alpha^\beta(s_i) = \{z \in R(s_i) \mid \alpha \le |z+d| \le \beta\}.$$

Our choice of α and β guarantees that $P_{d,\lambda}(V_\alpha^\beta(s_i))$ covers $V_\alpha^\beta(s_j)$ for any i and j (assuming $0 < |s_i| < [(d-1)/2]$). See Figure refwedges-1.

We define

$$V_\alpha^\beta = \bigcup_{0<|s_i|<[(d-1)/2]} V_\alpha^\beta(s_i).$$

Note that V_α^β depends on d and λ. Let $\Lambda_{\lambda,d}$ be the set of points whose orbits under $P_{d,\lambda}$ remain in V_α^β for all time. The inverse map

$$P^{-1}_{d,\lambda,s_j}(z) = d|z/\lambda|^{1/d} \exp^{i(Arg z - Arg\lambda + 2\pi s_j)/d} - d$$

is analytic on V_α^β and takes values strictly inside the wedge $V_\alpha^\beta(s_j)$. Hence P_{d,λ,s_j}^{-1} is a strict contraction in the Poincaré metric on V_α^β. As a consequence, for any $z \in V_\alpha^\beta$, the sequence $P_{d,\lambda,s_0}^{-1} \circ \ldots \circ P_{d,\lambda,s_n}^{-1}(z)$ tends to a limit in V_α^β which is independent of z. This limit point has itinerary $s_0 s_1 s_2 \ldots$.

For any sequence $s = s_0 s_1 s_2 \ldots \in \Sigma_d$, we therefore can define $\Phi(s) = \lim_{n\to\infty} P_{d,\lambda,s_0}^{-1} \circ \ldots \circ P_{d,\lambda,s_n}^{-1}(z)$ for any $z \in V_\alpha^\beta$. Standard arguments then show that Φ is a homeomorphism which gives a conjugacy between $P_{d,\lambda}$ and the shift map on Σ_d. Given d and λ, we define $z_{d,\lambda}(s)$ to be the unique point in $\Lambda_{\lambda,d}$ whose itinerary under $P_{d,\lambda}$ is s. The following proposition now follows immediately.

Proposition 3.2 *Let $s = s_0 s_1 s_2 \ldots$ be a regular sequence in Σ_d. Then there is a unique point $z_{d,\lambda}(s)$ in $\Lambda_{\lambda,d}$ whose itinerary under $P_{d,\lambda}$ is s. This point lies in the Julia set $J(P_{d,\lambda})$. Moreover, if s is a repeating sequence, then $z_{d,\lambda}(s)$ is a repelling periodic point.*

The dynamics of $P_{d,\lambda}$ on its Julia set increasingly resembles that of E_λ (see [7] as d becomes large in the sense that we can take the bound on the s_j larger and thus obtain larger classes of regular itineraries corresponding to repelling periodic points.

3.3 Existence of Hairs

In this section we will prove the existence of invariant hairs for $P_{d,\lambda}(z)$. These invariant hairs are defined in a similar way to those for the exponential family of maps [7], with endpoints $z_{d,\lambda}(s)$.

Definition 3.3 *A continuous curve $h_{d,\lambda,s} : [\frac{d}{d-1}, \infty) \to R_{d,\lambda}(s_0)$ is called a hair attached to $z_{d,\lambda}(s)$ if*

1. $h_{d,\lambda,s}(\frac{d}{d-1}) = z_{d,\lambda}(s)$.
2. *For each $t \geq \frac{d}{d-1}$, the itinerary of $h_{d,\lambda,s}(t)$ under $P_{d,\lambda}$ is s.*
3. *If $t > \frac{d}{d-1}$, then*
$$\lim_{n\to\infty} |P_{d,\lambda}^n(h_{d,\lambda,s}(t))| = \infty.$$
4. $\lim_{t\to\infty} |h_{d,\lambda,s}(t)| = \infty$.

Clearly the endpoint of each hair lies in the filled Julia set of $P_{d,\lambda}(z)$, whereas the other points on the hair do not. Later in this paper the hairs will be shown to be equivalent to the familiar external rays for the family of polynomial maps.

We make use of the "model map" $P_{d,*}(t)$, where $* \equiv (\frac{d-1}{d})^{d-1}$. This parameter value corresponds to a saddle-node bifurcation for the real-valued polynomial.

For any value of $\lambda \in \mathbb{C}'$ and itinerary s, we define the functions $h^n_{d,s}$: $\mathbf{C}' \times [\frac{d}{d-1}, \infty) \to \mathbf{C}$ by

$$h^n_{d,s}(\lambda, t) = Q_{d,\lambda,s_0} \circ \cdots \circ P^n_{d,*}(t).$$

where $Q_{d,\lambda,s_j}(z) = d((z/\lambda)^{1/d} e^{2\pi s_j i/d} - 1)$ is the s_jth branch of the inverse of $P_{d,\lambda,s}$. That is, the function $h^n_{d,s}$ is obtained by iterating the model map forward n times, and then taking n appropriate branches of the inverse of the polynomial.

Theorem 3.4 *For each* $t \in (\frac{d}{d-1}, \infty)$, *the function*

$$h_{d,s}(\lambda, t) \equiv \lim_{n\to\infty} h^n_{d,s}(\lambda, t)$$

exists and is a non-trivial function of t.

To prove this theorem we will need four lemmas.

Lemma 3.5

$$\lim_{n\to\infty} \left(P^n_{d,*}(t)\right)^{\frac{1}{d^n}} \quad \text{existsforall} \quad \mathrm{t} > \frac{\mathrm{d}}{\mathrm{d}-1}.$$

Proof of Lemma 3.5. Since the graph of $t_{n+1} = P_{d,*}(t_n)$ is tangent to the line $t_{n+1} = t_n$ at the point $t_n = \frac{d}{d-1}$, it follows easily that $t^d > P_{d,*}(t) > 1$ for $t > \frac{d}{d-1}$. If we now iterate the functions, we can extend the inequality to:

$$t^{d^n} > P^n_{d,*}(t) > 1.$$

Taking the dth roots of each side, we see that $P^n_{d,*}(t)$ is bounded above by t and below by 1. Next, we notice that $P^n_{d,*}(t)$ is a decreasing sequence:

$$\left(P^{n+1}_{d,*}(t)\right)^{\frac{1}{d^{n+1}}} = \left(* \left(1 + \frac{P^n_{d,*}(t)}{d}\right)^d\right)^{\frac{1}{d^{n+1}}} < *^{\frac{1}{d^{n+1}}} \left(\frac{P^n_{d,*}}{d}\right)^{\frac{1}{d^n}} < (P^n_{d,*}(t))^{\frac{1}{d^n}}.$$

(The last inequality follows from the fact that $*^{1/d} < d$). Since $(P^n_{d,*})^{\frac{1}{d^n}}$ is a decreasing sequence bounded below, it must converge.

Lemma 3.6 $\lim_{n\to\infty}\left(P^n_{d,*}(t)\right)^{\frac{1}{d^n}}$ *is a non-trivial function of* t.

Proof of Lemma 3.6. We first claim that $j_n\ t \leq (P^n_{d,*}(t))^{\frac{1}{d^n}}$ for each $n \geq 1$, where

$$j_n = \frac{*^{\frac{1}{d}+\frac{1}{d^2}+\cdots+\frac{1}{d^n}}}{d^{1+\frac{1}{d}+\cdots+\frac{1}{d^{n-1}}}} = \frac{*^{\frac{1}{d^n}}}{d}\left(\frac{*}{d}\right)^{\sum_{i=1}^{n-1}\frac{1}{d^i}}.$$

This is proved by induction: for $n = 1$, we have $(P_{d,*}(t))^{\frac{1}{d}} = *^{\frac{1}{d}}(1+\frac{t}{d}) > \frac{*^1 d}{d}\ t = j_1\ t$. Assuming the result holds for n,

$$\begin{aligned}
(P^{n+1}_{d,*}(t))^{\frac{1}{d^{n+1}}} &= \left(*\left(1+\frac{P^n_{d,*}(t)}{d}\right)^d\right)^{\frac{1}{d^{n+1}}}\\
&= *^{\frac{1}{d^{n+1}}}\left(1+\frac{P^n_{d,*}(t)}{d}\right)^{\frac{1}{d^n}}\\
&\geq \frac{*^{\frac{1}{d^{n+1}}}}{d^{\frac{1}{d^n}}}\left(P^n_{d,*}(t)\right)^{\frac{1}{d^n}}\\
&\geq \frac{*^{\frac{1}{d^{n+1}}}}{d^{\frac{1}{d^n}}}\ j_n\ t = j_{n+1}\ t.
\end{aligned}$$

Also, from the proof of Lemma 3.5, we know that $(P^n_{d,*}(t))^{\frac{1}{d^n}} < t\ \forall\ n \geq 1$. These two inequalities bound $(P^n_{d,*}(t))^{\frac{1}{d^n}}$; taking the limits of each as $n \to \infty$, we get:

$$\lim_{n\to\infty} j_n\ t \leq \lim_{n\to\infty}(P^n_{d,*}(t))^{\frac{1}{d^n}} \leq t,$$

which means that $\frac{1}{d}\left(\frac{*}{d}\right)^{\frac{1}{d-1}}\ t \leq \lim_{n\to\infty}(P^n_{d,*}(t))^{\frac{1}{d^n}} \leq t$. Thus we have that $\lim_{n\to\infty}(P^n_{d,*}(t))^{\frac{1}{d^n}}$ is a non-trivial function of t.

Lemma 3.7 *There exists a conjugacy,* $\Phi_{d,\lambda}(z)$*, near infinity such that*

$$\Phi_{d,\lambda}\circ P_{d,\lambda}\circ\Phi^{-1}_{d,\lambda}(z) = z^d,$$

where

$$\Phi(z) = c\ z + \mathrm{O}\left(\frac{1}{\mathrm{z}}\right) \qquad \left\{ \begin{array}{l} c \neq 0 \\ c \in \mathbf{C} \end{array} \right. .$$

Proof of Lemma 3.7. This is a well known fact in complex dynamics, see for example [9].

Lemma 3.8 *For fixed t, if $\lim_{n\to\infty} |P_d(t)| = \infty$, where $P_d(t)$ is any polynomial of degree d, and if $\Phi(z)$ is any function of the form*

$$\Phi(z) = c\ z + \mathrm{O}\left(\frac{1}{\mathrm{z}}\right) \left\{ \begin{array}{l} c \neq 0 \\ c \in \mathbf{C} \end{array} \right. ,$$

then

$$\lim_{n\to\infty} (\Phi \circ P_d^n(t))^{\frac{1}{d^n}} = \lim_{n\to\infty} (P_d^n(t))^{\frac{1}{d^n}}.$$

Proof of Lemma 3.8. Since $\lim_{n\to\infty} |P_d(t)| = \infty$, we can write

$$\Phi(P_d^n(t)) = c\ P_d^n(t) + \Phi(P_d^n(t)) - c\ P_d^n(t).$$

But $|\Phi(P_d^n(t)) - c\ P_d^n(t)| = \mathrm{O}\left(\frac{1}{\mathrm{P_d^n(t)}}\right)$, which is certainly bounded as $n \to \infty$. Suppose for all sufficiently large n this quantity is bounded by K. Then

$$\begin{aligned} \lim_{n\to\infty} (\Phi(P_d^n(t)))^{\frac{1}{d^n}} &= \lim_{n\to\infty} (c\ P_d^n(t) + K)^{\frac{1}{d^n}} \\ &= \lim_{n\to\infty} c^{\frac{1}{d^n}} (P_d^n(t) + K)^{\frac{1}{d^n}} \\ &= \lim_{n\to\infty} c^{\frac{1}{d^n}} \lim_{n\to\infty} (P_d^n(t))^{\frac{1}{d^n}} \\ &= \lim_{n\to\infty} (P_d^n(t))^{\frac{1}{d^n}}, \end{aligned}$$

since $\lim_{n\to\infty} c^{\frac{1}{d^n}} = 1$.

With the help of these lemmas we now proceed to prove the existence of hairs for the family of polynomial maps.

Proof of Theorem 3.4. We know from Lemma 3.5 that $\lim_{n\to\infty}(P^n_{d,*}(t))^{\frac{1}{d^n}}$ exists for all $t > \frac{d}{d-1}$. Also, from Lemma 3.7, we know there exists a conjugacy $\Phi_{d,\lambda}(z)$ near infinity such that $\Phi_{d,\lambda} \circ P_{d,\lambda} \circ \Phi^{-1}_{d,\lambda}(z) = z^d$. Then we have

$$Q_{d,\lambda,s_j}(z) = \Phi^{-1}_{d,\lambda}(e^{\frac{2\pi s_j i}{d}}(\Phi_{d,\lambda}(z))^{\frac{1}{d}}).$$

By extension, we have

$$Q_{d,\lambda,s_0} \circ Q_{d,\lambda,s_1} \circ \cdots \circ Q_{d,\lambda,s_{n-1}}(z) = \Phi^{-1}_{d,\lambda}(e^{\frac{2\pi m_n i}{d}}(\Phi_{d,\lambda}(z))^{\frac{1}{d^n}}),$$

where $m_n = \sum_{i=0}^{n-1} \frac{s_i}{d^i}$.

Then

$$\begin{aligned}
\lim_{n\to\infty} H^n_{d,s}(\lambda, t) &= \lim_{n\to\infty} Q_{d,\lambda,s_0} \circ Q_{d,\lambda,s_1} \circ \cdots \circ Q_{d,\lambda,s_{n-1}} \circ P^n_{d,*}(t) \\
&= \lim_{n\to\infty} \Phi^{-1}_{d,\lambda}(e^{\frac{2\pi m_n i}{d}}(\Phi_{d,\lambda} \circ P^n_{d,*}(t))^{\frac{1}{d^n}}) \\
&= \Phi^{-1}_{d,\lambda}(\lim_{n\to\infty} e^{\frac{2\pi m_n i}{d}} \lim_{n\to\infty}(\Phi_{d,\lambda} \circ P^n_{d,*}(t))^{\frac{1}{d^n}}) \\
&= \Phi{-1}_{d,\lambda}(e^{\frac{2\pi m_s i}{d}} \lim_{n\to\infty}(P^n_{d,*}(t))^{\frac{1}{d^n}}),
\end{aligned}$$

using Lemma 3.8, and where $m_s = \lim_{n\to\infty} m_n$. Each hair $h_{d,s}(\lambda, t)$ is a non-trivial function of t by an easy application of Lemma 3.6. This completes the proof of the theorem.

Theorem 3.9 *The hairs $h_{d,s}(\lambda, t)$ are continuous in t and analytic in λ.*

Proof of Theorem 3.9. To prove continuity in t, we first show that $\lim_{n\to\infty}(P^n_{d,*}(t))^{\frac{1}{d^n}}$ is continuous in t. We define

$$R_n = \frac{*^{\frac{1}{d}+\frac{1}{d^2}+\cdots+\frac{1}{d^n}}}{d^{1+\frac{1}{d}+\cdots+\frac{1}{d^{n-1}}}}.$$

For $n = 1$ and t, t_0 on the hair, we have:

$$\begin{aligned}
|(P_{d,*}(t))^{\frac{1}{d}} - (P_{d,*}(t_0))^{\frac{1}{d}}| &= *^{\frac{1}{d}}\left|\left(1+\frac{t}{d}\right) - \left(1+\frac{t_0}{d}\right)\right| \\
&= \frac{*^{\frac{1}{d}}}{d}|t - t_0| \\
&= R_1|t - t_0|.
\end{aligned}$$

Under the induction hypothesis that $|(P^n_{d,*}(t))^{\frac{1}{d^n}} - (P^n_{d,*}(t_0))^{\frac{1}{d^n}}| < R_n|t - t_0|$, we find that:

$$\begin{aligned}
|(P^{n+1}_{d,*}(t))^{\frac{1}{d^{n+1}}} &- (P^{n+1}_{d,*}(t_0))^{\frac{1}{d^{n+1}}}| \leq \\
&\leq *^{\frac{1}{d^{n+1}}} \left| \left(1 + \frac{P^n_{d,*}(t)}{d}\right)^{\frac{1}{d^n}} - \left(1 + \frac{P^n_{d,*}(t_0)}{d}\right)^{\frac{1}{d^n}} \right| \\
&\leq *^{\frac{1}{d^{n+1}}} \left| \left(1 + \frac{P^n_{d,*}(t)}{d}\right) - \left(1 + \frac{P^n_{d,*}(t_0)}{d}\right) \right|^{\frac{1}{d^n}} \\
&= \frac{*^{\frac{1}{d^{n+1}}}}{d^{\frac{1}{d^n}}} |P^n_{d,*}(t) - P^n_{d,*}(t_0)|^{\frac{1}{d^n}} \\
&\leq \frac{*^{\frac{1}{d^{n+1}}}}{d^{\frac{1}{d^n}}} R_n|t - t_0| \\
&= R_{n+1}|t - t_0|.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\lim_{n\to\infty} |(P^n_{d,*}(t))^{\frac{1}{d^n}} - (P^n_{d,*}(t_0))^{\frac{1}{d^n}}| &\leq \lim_{n\to\infty} R_n|t - t_0| \\
&= R|t - t_0|,
\end{aligned}$$

where $R = \lim_{n\to\infty} R_n = (\frac{*}{d})^{\frac{1}{d-1}}$. Clearly $\lim_{n\to\infty}(P^n_{d,*}(t))^{\frac{1}{d^n}}$ is continuous in t. Finally, from the proof of Theorem 3.4 we see that $h_{d,s}(\lambda, t)$ is the composition of functions which are continuous in t, hence itself continuous in t.

To show that $h_{d,s}(\lambda, t)$ is analytic in λ, we use a normal families argument: for each n and all valid t , $H^n_{d,s}(\lambda, t)$ is an analytic function in λ and takes values in $\bigcup_{\lambda\in\mathbf{C}'} R(s_0)$, thus missing at least three values in $\mathbf{C}$. By Montel's Theorem, the $H^n_{d,s}(\lambda, t)$ must then be a normal family on $\mathbf{C}'$. Hence $\lim_{n\to\infty} H^n_{d,s}(\lambda, t) = h_{d,s}(\lambda, t)$ is an analytic function in λ.

This completes the proof of the theorem.

3.4 Convergence of Polynomial Hairs to Exponential Hairs

In this section we will prove that the hairs defined for the polynomial maps $P_{d,\lambda}(z)$ in the dynamical plane converge pointwise to the corresponding hairs defined for the exponential map $E_\lambda(z)$ as d tends to infinity.

Recall that for the exponential family there is defined a Markov family of rectangles,

$$R(k) = \{z \in \mathbf{C} \,|\, (2k-1)\pi - \text{Arg } \lambda < \text{Im}\, z < (2k+1)\pi - \text{Arg } \lambda\}$$

where the hairs live. See [7]. For the polynomial family, we have defined a similar Markov family of wedges $R_d(k)$ where the hairs for P_d live. Note that the wedges converge to the rectangles as $d \to \infty$.

We now prove that the hairs converge pointwise.

Theorem 3.10 *For each fixed* $0 < \lambda < 1/e$, $t \in [1, \infty)$, $s \in \Sigma_K$,

$$\lim_{d\to\infty} \lim_{n\to\infty} H^n_{d,\lambda,s}(t) = \lim_{n\to\infty} \lim_{d\to\infty} H^n_{d,\lambda,s}(t).$$

This theorem is proven by mimicking estimates produced for the exponential hairs in [7] for the polynomial hairs. However, these estimates cannot be used in general to prove existence of the polynomial hairs. Geometrically, this is because the polynomial hairs are contained in wedges around $z = -d$, with unbounded imaginary part, whereas the exponential hairs which are contained in strips of width 2π parallel to the real axis.

Lemma 3.11 *Suppose* $0 < \lambda < *$, $t \in [1, \infty)$, *and* $s \in \Sigma_K$. *if* $d > K$ *is sufficiently large so that* $(2\pi(K+1))/d < \pi/2$ *and*

$$\left(\frac{*}{\lambda}\right)^{1/d} \cos \frac{2\pi(K+1)}{d} > 1, \tag{1}$$

and r *satisfies* $1 \le r \le (\frac{*}{\lambda})^{1/d} \cos \frac{2\pi(K+1)}{d}$, *then*

$$\text{Re } H^n_{d,\lambda,s}(t) \ge rt.$$

Remark. Condition (1) is always satisfied for sufficiently large d as long as $0 < \lambda < *$. This can be seen by a expanding $(\frac{*}{\lambda})^{1/d}$ and $\cos \frac{2\pi(K+1)}{d}$ as power series in $1/d$.

Proof of Lemma 3.11. The proof is by induction on n. When $n = 1$,

$$\begin{aligned} H_{d,\lambda,s}(t) &= Q_{d,\lambda,s_0} \circ P_{d,*}(t) \\ &= d\left(\left(\frac{P_{d,*}(t)}{\lambda}\right)^{1/d} e^{\frac{2\pi s_0 i}{d}} - 1\right) \\ &= d\left(\left(\frac{*}{\lambda}\right)^{1/d} \left(1+\frac{t}{d}\right) e^{\frac{2\pi s_0 i}{d}} - 1\right). \end{aligned}$$

Hence

$$\begin{aligned} \text{Re } H_{d,\lambda,s}(t) &= \left(\frac{*}{\lambda}\right)^{1/d} \cos\frac{2\pi s_0}{d}\ t + d\left(\left(\frac{*}{\lambda}\right)^{1/d} \cos\frac{2\pi s_0}{d} - 1\right) \\ &\geq rt. \end{aligned}$$

For the induction step, we assume the lemma holds in the nth case. Then,

$$\begin{aligned} & \text{Re } H^{n+1}_{d,\lambda,s}(t) = \\ = \ & \text{Re } \mathrm{Q_{d,\lambda,s_0} \circ H^n_{d,\lambda,\sigma(s)} \circ P_{d,*}(t)} \\ = \ & \text{Re } d\left(\left(\frac{H^n_{d,\lambda,\sigma(s)} \circ P_{d,*}(t)}{\lambda}\right)^{1/d} e^{\frac{2\pi s_0 i}{d}} - 1\right) \\ = \ & d\left(\left(\frac{|H^n_{d,\lambda,\sigma(s)} \circ P_{d,*}(t)|}{\lambda}\right)^{1/d} \cos\frac{2\pi s_0 + \text{Arg}(H^n_{d,\lambda,\sigma(s)} \circ P_{d,*}(t))}{d} - 1\right) \\ \geq \ & d\left(\left(\frac{r*}{\lambda}\right)^{1/d} \left(1+\frac{t}{d}\right)\ \cos\frac{2\pi|s_0| + \pi/2}{d} - 1\right) \\ \geq \ & d\left(\left(\frac{r*}{\lambda}\right)^{1/d} \left(1+\frac{t}{d}\right)\ \cos\frac{2\pi(K+1)}{d} - 1\right) \\ \geq \ & \left(\frac{r*}{\lambda}\right)^{1/d} t + d\left(\left(\frac{r*}{\lambda}\right)^{1/d}\ \cos\frac{2\pi(K+1)}{d} - 1\right) \\ \geq \ & \left(\frac{*}{\lambda}\right)^{1/d} t + d\left(\left(\frac{*}{\lambda}\right)^{1/d}\ \cos\frac{2\pi(K+1)}{d} - 1\right) \\ \geq \ & rt. \end{aligned}$$

This completes the proof of Lemma 3.11.

Lemma 3.12 *Suppose $0 < \lambda < *$, $t \in [1, \infty)$, and $s \in \Sigma_K$. Suppose also that $d > K$ is sufficiently large so that there exist constants (with respect to t) A, B and R, S which satisfy:*

$$\begin{aligned} A &\geq \frac{2\pi(K+1)}{d}\left(\frac{*(A+R)(B+S)}{\lambda}\right)^{1/d}, \\ B &\geq 2\pi(K+1)\left(\frac{*(A+R)(B+S)}{\lambda}\right)^{1/d}, \\ R &\geq \left(\frac{*(A+R)(B+S)}{\lambda}\right)^{1/d}, \\ S &\geq d\left(\left(\frac{*(A+R)(B+S)}{\lambda}\right)^{1/d} - 1\right). \end{aligned}$$

Then

$$\begin{aligned} \text{Re } H^n_{d,\lambda,s}(t) &\leq Rt + S \\ |\text{Im } H^n_{d,\lambda,s}(t)| &\leq At + B. \end{aligned}$$

Proof of Lemma 3.12. The proof is again by induction on n. When $n = 1$,

$$\begin{aligned} |\text{Im } H_{d,\lambda,s}(t)| &= |d\left(\frac{*}{\lambda}\right)^{1/d}\left(1+\frac{t}{d}\right)\sin\frac{2\pi s_0}{d}| \\ &\leq d\left(\frac{*}{\lambda}\right)^{1/d}\left(1+\frac{t}{d}\right)\left(\frac{2\pi|s_0|}{d}\right) \\ &\leq \frac{2\pi K}{d}\left(\frac{*}{\lambda}\right)^{1/d} t + 2\pi K\left(\frac{*}{\lambda}\right)^{1/d} \leq At + B. \end{aligned}$$

Similarly,

$$\begin{aligned} \text{Re } H_{d,\lambda,s}(t) &= \left(\frac{*}{\lambda}\right)^{1/d}\cos\frac{2\pi s_0}{d}\ t + d\left(\left(\frac{*}{\lambda}\right)^{1/d}\cos\frac{2\pi s_0}{d} - 1\right) \\ &\leq Rt + S. \end{aligned}$$

Hence the lemma holds for $n = 1$. For the inductive step, if we assume the lemma holds in the nth case, we have:

$$
\begin{aligned}
& |\text{Im } H^{n+1}_{d,\lambda,s}(t)| = |\text{Im } d\left(\left(\frac{H^n_{d,\lambda,\sigma(s)} \circ P_{d,*}(t)}{\lambda}\right)^{1/d} e^{\frac{2\pi s_0 i}{d}} - 1\right)| \\
= \;& \frac{d}{\lambda^{1/d}}|H^n_{d,\lambda,\sigma(s)} \circ P_{d,*}(t)|^{1/d}|\sin\frac{2\pi s_0 + \text{Arg}(H^n_{d,\lambda,\sigma(s)} \circ P_{d,*}(t))}{d}| \\
\leq \;& \frac{d}{\lambda^{1/d}}(|\text{Re } H_{d,\lambda,\sigma(s)}(P_{d,*}(t))| + |\text{Im } H_{d,\lambda,\sigma(s)}(P_{d,*}(t))|)^{1/d}\ \frac{2\pi(K+1)}{d} \\
\leq \;& \frac{d}{\lambda^{1/d}}(RP_{d,*}(t) + S + AP_{d,*}(t) + B)^{1/d}\ \frac{2\pi(K+1)}{d} \\
\leq \;& d\left(\frac{*(A+R)(B+S)}{\lambda}\right)^{1/d}\left(1+\frac{t}{d}\right)\ \frac{2\pi(K+1)}{d} \\
\leq \;& \frac{2\pi(K+1)}{d}\left(\frac{*(A+R)(B+S)}{\lambda}\right)^{1/d} t \\
& \quad +2\pi(K+1)\left(\frac{*(A+R)(B+S)}{\lambda}\right)^{1/d} \\
\leq \;& At + B.
\end{aligned}
$$

Similarly, for the real part,

$$
\begin{aligned}
& \text{Re } H^{n+1}_{d,\lambda,s}(t) = \\
= \;& d \text{ Re}\left(\left(\frac{H^n_{d,\lambda,\sigma(s)} \circ P_{d,*}(t)}{\lambda}\right)^{1/d} e^{\frac{2\pi s_0 i}{d}} - 1\right) \\
\leq \;& d\left(\left(\frac{1}{\lambda}\right)^{1/d}(|\text{Re } H_{d,\lambda,\sigma(s)}(P_{d,*}(t))| + |\text{Im } H_{d,\lambda,\sigma(s)}(P_{d,*}(t))|)^{1/d} - 1\right) \\
\leq \;& d\left(\left(\frac{1}{\lambda}\right)^{1/d}(RP_{d,*}(t) + S + AP_{d,*}(t) + B)^{1/d} - 1\right) \\
\leq \;& d\left(\left(\frac{*(A+R)(B+S)}{\lambda}\right)^{1/d}\left(1+\frac{t}{d}\right) - 1\right) \\
\leq \;& \left(\frac{*(A+R)(B+S)}{\lambda}\right)^{1/d} t + d\left(\left(\frac{*(A+R)(B+S)}{\lambda}\right)^{1/d} - 1\right) \\
\leq \;& Rt + S.
\end{aligned}
$$

Hence, by induction on n, the lemma holds for all $n \geq 1$, completing the proof of the lemma.

We need to show that such choices of r, A, B and R, S are possible for each λ and for sufficiently large d.

Lemma 3.13 *For each fixed* $0 < \lambda < 1/e$, $t \in [1, \infty)$ *and* $s \in \Sigma_K$, *the choice of constants*

$$\begin{aligned} r_d &= \left(\frac{*}{\lambda}\right)^{1/d} \cos \frac{2\pi(K+1)}{d}, \\ A_d &= \frac{2\pi(K+1)}{d} \left(\frac{*4\pi(K+1)}{\lambda^2} \left(1 + \frac{2\pi(K+1)}{d}\right)\right)^{1/(d-1)}, \\ B &= \frac{2\pi(K+1)}{\lambda}, \\ R_d &= \left(\frac{*4\pi(K+1)}{\lambda^2} \left(1 + \frac{2\pi(K+1)}{d}\right)\right)^{1/(d-1)}, \\ S &= \frac{2\pi(K+1)}{\lambda} \end{aligned}$$

satisfy the conditions of Lemmas 3.11 and 3.12 for sufficiently large d.

Proof of Lemma 3.13. The choice of r_d easily holds, since $* < 1/e$ for all d. By noticing that $A_d = 2\pi R_d(K+1)/d$, and substituting the choices of B and S, we can check that indeed

$$\begin{aligned} A_d &= \frac{2\pi(K+1)}{d} \left(\frac{*(A_d + R_d)(B+S)}{\lambda} \right)^{1/d}, \\ R_d &= \left(\frac{*(A_d + R_d)(B+S)}{\lambda} \right)^{1/d}, \end{aligned}$$

the maximum possible values for A and R. For S, we see that

$$\lim_{d \to \infty} d \left(\left(\frac{*(A_d + R_d)(B+S)}{\lambda} \right)^{1/d} - 1 \right)$$

$$
\begin{aligned}
&= \lim_{d\to\infty} d(R_d - 1) \\
&= \lim_{d\to\infty} d\left(\left(\frac{*4\pi(K+1)}{\lambda^2}\left(1+\frac{2\pi(K+1)}{d}\right)\right)^{1/(d-1)} - 1\right) \\
&= \log\frac{4\pi(K+1)}{e\lambda^2} \le S,
\end{aligned}
$$

with a tedious application of L'Hôpital's rule required - the details are left to the reader. Finally,

$$
\begin{aligned}
\lim_{d\to\infty} 2\pi(K+1)\left(\frac{*(A_d+R_d)(B+S)}{\lambda}\right)^{1/d} &= \lim_{d\to\infty} 2\pi(K+1)R_d \\
&= 2\pi(K+1) \le B,
\end{aligned}
$$

confirms the validity of the B estimate, via another application of L'Hôpital's rule to show that $\lim_{d\to\infty} R_d = 1$. This completes the proof of the lemma.

It can easily be seen from Lemma 3.13 that $\lim_{d\to\infty} r_d = \lim_{d\to\infty} R_d = 1$, and $\lim_{d\to\infty} A_d = 1$. We now compare the relative positions of the $\tilde{H}_s(\lambda,t) = \lim_{d\to\infty} H_{d,s}(\lambda,t)$ and $H_s(\lambda,t) = \lim_{n\to\infty} H^n_{\lambda,s}(t)$.

Lemma 3.14 *For each fixed $0 < \lambda < 1/e$, $t \in [1,\infty)$ and $s \in \Sigma_K$, there exists a constant $W = W(\lambda)$ such that, for all sufficiently large d,*

$$
|\tilde{H}_s(\lambda,t) - H_s(\lambda,t)| \le W.
$$

Proof of Lemma 3.14. We use the bounds on $H_s(\lambda,t)$ given in [7], namely, for fixed $0 < \lambda < 1/e$, $t \in [1,\infty)$, $s \in \Sigma_K$,

$$
\begin{aligned}
t \le \mathrm{Re} H_s(\lambda,t) &\le t + M, \qquad (2) \\
|\mathrm{Im} H_s(\lambda,t)| &\le (2K+1)\pi, \qquad (3)
\end{aligned}
$$

where $M = M(\lambda)$. From Lemmas 3.11, 3.12 and 3.13, we also have:

$$
\begin{aligned}
r_d t \le \mathrm{Re}\ H^n_{d,\lambda,s}(t) &\le R_d t + S \\
|\mathrm{Im}\ H^n_{d,\lambda,s}(t)| &\le A_d t + B.
\end{aligned}
$$

Taking the limits of the above bounds as $n \to \infty$, we find the following estimates:

$$\begin{aligned} r_d t \leq \text{Re } H_{d,s}(\lambda, t) &\leq R_d t + S & (4) \\ |\text{Im } H_{d,s}(\lambda, t)| &\leq A_d t + B. & (5) \end{aligned}$$

Combining bounds (2) and (4) and bounds (3) and (5), we have:

$$\begin{aligned} -M \leq \text{Re } H_{d,s}(\lambda, t) - r_d t - (\text{Re } H_s(\lambda, t) - t) &\leq (R_d - r_d)t + S \\ |\text{Im } H_{d,s}(\lambda, t) - \text{Im } H_s(\lambda, t)| &\leq |A_d t + B - (2K+1)\pi|. \end{aligned}$$

Taking the limit as $d \to \infty$ in the above bounds, we obtain bounds independent of t:

$$\begin{aligned} -M \leq \text{Re } \tilde{H}_s(\lambda, t) - \text{Re } H_s(\lambda, t) &\leq S \\ |\text{Im } \tilde{H}_s(\lambda, t) - \text{Im } H_s(\lambda, t)| &\leq |B - (2K+1)\pi|. \end{aligned}$$

A simple application of the triangle inequality gives the required bounds. This completes the proof of the lemma.

We are now in a position to prove the main result of this section:

Proof of Theorem 3.10. We need some invariance properties of $\tilde{H}_s(\lambda, t)$ and $H_s(\lambda, t)$. For any $m \geq 0$,

$$H^{n+m}_{d,\lambda,s}(t) = Q^m_{d,\lambda,s} \circ H^n_{d,\lambda,\sigma^m(s)} \circ P^m_{d,*}(t).$$

Taking first the limit as $n \to \infty$ then as $d \to \infty$, we have

$$\tilde{H}_s(\lambda, t) = L^m_{\lambda,s} \circ \tilde{H}_{\sigma^m(s)}(\lambda, E^m(t)).$$

Reversing the order of the limits we find

$$H_s(\lambda, t) = L^m_{\lambda,s} \circ H_{\sigma^m(s)}(\lambda, E^m(t)).$$

For each fixed $0 < \lambda < 1/e$, and fixed $t \geq 1$, take any $\epsilon > 0$. Recall from [7] that $|(L^m_{s,\lambda}(z))'| \leq \omega$, for some $\omega = \omega(\lambda) < 1$. We choose m such that $W\omega^{m-1} < \epsilon$. Then,

$$\begin{aligned} |\tilde{H}_s(\lambda, t) - H_s(\lambda, t)| &= |L^m_{\lambda,s} \circ \tilde{H}_{\sigma^m(s)}(\lambda, E^m(t)) - L^m_{\lambda,s} \circ H_{\sigma^m(s)}(\lambda, E^m(t))| \\ &\leq \omega^{m-1} \, |\tilde{H}_{\sigma^m(s)}(\lambda, E^m(t)) - H_{\sigma^m(s)}(\lambda, E^m(t))| \\ &\leq \omega^{m-1} W \leq \epsilon, \end{aligned}$$

using Lemma 3.14. This completes the proof of the theorem.

Using a normal families argument, we can extend this result to all $\lambda \in \mathbf{C}'$. The fact is crucial to proving the convergence of hairs in the parameter plane.

Corollary 3.15 *For a fixed t and fixed regular sequence s, the family of functions $H_{d,s,t}(\lambda)$ converges uniformly as functions in λ to $H_{s,t}(\lambda)$ as $d \to \infty$ for $\lambda \in \mathbf{C}'$.*

Proof: We use a normal families argument. The $H_{d,s,t}(\lambda)$ are analytic functions in λ, and $\lambda \to H_{d,s,t}(\lambda)$ misses many more than three values in $\mathbf{C}$ (since each $H_{d,s,t}(\lambda)$ only takes values in the s_0th wedge around $-d$). By Montel's Theorem, it follows that $H_{d,s,t}(\lambda)$ is a normal family of functions. Every subsequence converges uniformally on $\mathbf{C}'$ to either an analytic function or to infinity. Since $H_{d,s,t}(\lambda)$ converges for $0 < \lambda < 1/e$ to the analytic function $H_{s,t}(\lambda)$, the family cannot converge to ∞ and so it follows that it must converge uniformally on $\mathbf{C}'$ to the analytic function $H_{s,t}(\lambda)$ – uniquely determined by its values on the arc $0 < \lambda < 1/e$.

4 Hairs in the Parameter Plane

4.1 Existence of Hairs

Our goal in this section is to show that there also exist hairs in the parameter plane for the polynomial family. Except for the endpoints, these hairs consist of λ-values for which the orbit of $-d$ (the critical point) under $P_{d,\lambda}$ tends to infinity with a specified itinerary and hence the Julia set for $P_{d,\lambda}$ (which equals the filled Julia set) is a Cantor set. For the exponential family, the λ-values on the hairs in the parameter plane correspond to Julia sets which are the entire plane. We will show that the polynomial hairs converge as $d \to \infty$ to the exponential hair with the appropriate itinerary.

Definition 4.1 *Let $s = s_0s_1s_2\ldots$. A continuous curve $H_{s,d}\colon [\frac{d}{d-1}, \infty) \to \mathbf{C}$ is called a* **hair** *with itinerary s if $H_{s,d}$ satisfies:*

1. *If $\lambda = H_{s,d}(t)$ and $t > \frac{d}{d-1}$, then $P^n_{d,\lambda}(-d) \to \infty$ and the itinerary of λ under $P_{d,\lambda}$ is s.*

2. *If* $\lambda = H_{s,d}(\frac{d}{d-1})$, *then the orbit of* λ *under* $P_{d,\lambda}$ *is bounded and has itinerary* s.

3. $\lim_{t\to\infty} \text{Re}\, H_{s,d}(t) = \infty$.

Remark: We use the term "hair" for curves in both the dynamical plane and parameter plane. When necessary, we use the terms dynamical hair and parameter hair to distinguish between them.

Theorem 4.2 *Suppose* s *is a bounded, regular sequence. Then there exists a hair in parameter space with itinerary* s. *Moreover, if* s *is periodic or preperiodic, then* $-d$ *is preperiodic under* $P_{d,\lambda}$ *for* $\lambda = H_{s,d}(\frac{d}{d-1})$.

Proof: The last statement of the theorem follows from the fact that if s is periodic or preperiodic, then so is the endpoint of the hair $h_{d,\lambda,s}(\frac{d}{d-1}) = \lambda = P^2_{d,\lambda}(-d)$.

For a given bounded, regular sequence s, a fixed d and t, we will construct a wedge $Q_{s,d}$ in the parameter plane which is mapped strictly inside itself by the map $F_{t,d}(\lambda) = h_{d,\lambda,s}(t)$. We will show that this map has a unique fixed point λ_0. In other words, $\lambda_0 = h_{d,\lambda_0,s}(t)$, so that λ_0 sits on its own hair $h_{d,\lambda_0,s}$ at time t. If $t > \frac{d}{d-1}$, it follows that $P^n_{d,\lambda_0}(-d) \to \infty$, whereas, if $t = \frac{d}{d-1}$, the orbit of $z = -d$ is bounded under P_{d,λ_0}.

Let $s = s_0 s_1 s_2 \ldots$ be a bounded, regular sequence and let Z_s denote the union of the wedges $R_{d,\lambda}(s_0)$ for $\lambda \in \mathbf{C} - \mathbf{R}^- = \mathbf{C}'$. We will assume that $-d$ is contained in Z_s. The map $F_{t,d}(\lambda)$, which is analytic in λ, takes $\mathbf{C}'$ to Z_s since the hair $h_{d,\lambda,s}$ lies in the wedge $R_{d,\lambda}(s_0)$. So in particular, $F_{t,d}$ is an analytic map of Z_s to itself. This map either has a unique fixed point in Z_s or else all points tend to a fixed point on the boundary. However, if $F_{t,d}$ had a fixed point on the boundary, say λ_0, then $h_{d,\lambda_0,s}(t) = \lambda_0$ and the hair $h_{d,\lambda_0,s}$ intersects a ray which is mapped to $\mathbf{R}$ under P_{d,λ_0}. But then $s_1 = 0$, contradicting the assumption that s is a regular sequence. Thus, $F_{t,d}$ has a unique fixed point and it remains to be shown that this fixed point is finite.

We do this by finding a number r_t such that

$$|\lambda + d| \geq r_t \quad \text{implies that} \quad |F_{t,d}(\lambda) + d| \leq r_t. \tag{6}$$

It follows that the fixed point will be contained in $Z_s \cap \{\lambda : |\lambda + d| \leq r_t\}$.

Choose r_t such that

$$\begin{aligned} (1+\frac{t}{d})^d + \frac{d}{*} &< (1+\frac{r_t}{d})^d \\ r_t &> d + \frac{d^{d+1}}{t^d} + *(1+\frac{d}{t})^d \end{aligned} \tag{7}$$

where $* = (\frac{d-1}{d})^{d-1}$ from Section 3. Fix $\lambda \in Z_s$ with $|\lambda + d| \geq r_t$ and denote its corresponding d wedges by $R(k)$. Recall that

$$V_\alpha^\beta(s_i) = \{z \in R(s_i) \mid \alpha \leq |z+d| \leq \beta\}.$$

We will need to alter the definition of $V_\alpha^\beta(0)$ slightly. In Section 3.2, we chose α small enough so that the circle $|z| = |\lambda|(\alpha/d)^d = C_\alpha$ was entirely contained in $R(0)$. Here we choose α so that in addition, $|\lambda|(\alpha/d)^d < \frac{d}{d-1}$. Then set

$$V_\alpha^\beta(0) = \{z \in R(0) \,:\, \alpha \leq |z+d| \leq \beta \text{ and } z \notin C_\alpha\}.$$

Now let

$$V_\alpha^\beta = \bigcup_{0 \leq |s_i| < [(d-1)/2]} V_\alpha^\beta(s_i).$$

We claim that for any $x \geq t$,

$$V_\alpha^{P_{d,*}(x)} \subset P_{d,\lambda}(V_\alpha^x). \tag{8}$$

The only thing which needs to be checked is that the circle $|z| = |\lambda|(x/d)^d$ contains the circle $|z+d| = P_{d,*}(x)$ in its interior.

We have by the triangle inequality that $|\lambda + d| \geq r_t$ implies

$$\begin{aligned} |\lambda| &\geq r_t - d \\ &> \frac{d^{d+1}}{t^d} + *(\frac{d}{t}+1)^d \\ &> \frac{d^{d+1}}{x^d} + *(\frac{d}{x}+1)^d \end{aligned}$$

which yields

$$|\lambda|(\frac{x}{d})^d > d + *(1+\frac{x}{d})^d$$

as desired.

Relation (8) means that $P_{d,\lambda,s_i}^{-1}(V_\alpha^{P_{d,*}(x)}) \subset V_\alpha^x$ for any $x \geq t$ and any i. Moreover, repeated iteration of this relation gives

$$P_{d,\lambda,s_0}^{-1} \circ P_{d,\lambda,s_1}^{-1} \circ \cdots \circ P_{d,\lambda,s_{n-1}}^{-1}((V_\alpha^{P_{d,*}^n(x)}) \subset V_\alpha^x \tag{9}$$

for all $x \geq t$. By our choice of r_t in (8), it is clear that $r_t > t$ so relation (9) holds for $x = r_t$ as well. Finally, (8) also implies that $P_{d,*}^n(t) + d < P_{d,*}^n(r_t)$ which means that $P_{d,*}^n(t) \in V_\alpha^{P_{d,*}^n}(r_t)$. Therefore, $h_{d,\lambda,s}^n(t) \in V_\alpha^{r_t}$ for all n, and in particular

$$|F_{t,d}(\lambda) + d| \leq r_t$$

which verifies (6).

We have shown that $F_{t,d}$ has a unique fixed point λ_0 in the parameter plane which is contained in $Z_s \cap \{\lambda : |\lambda + d| \leq r_t\}$. Consequently, we can define a map to the parameter plane $H_{s,d}(t) : [\frac{d}{d-1}, \infty) \to \mathbf{C}'$ as the unique fixed point λ_0 of $F_{t,d}$. This fixed point satisfies $\lambda_0 = h_{d,\lambda_0,s}(t)$ and thus the itinerary of λ_0 under $P_{d,\lambda}$ is s. To show this yields a hair in the parameter plane, we need to verify that it is continuous and that $\lim_{t\to\infty} \mathrm{Re}\, H_{s,d}(t) = \infty$.

For continuity, choose a $t_0 \in [\frac{d}{d-1}, \infty]$ and fix $\epsilon > 0$. We have shown that $F_{t_0,d}$ has a unique fixed point λ_0 and moreover, that $F_{t_0,d}$ is a contraction near λ_0. Let A be the disc of radius ϵ about λ_0 in the parameter plane and assume that ϵ was chosen small enough so that $F_{t_0,d}$ contracts A. If we fix λ and vary t, $F_{t,d}$ then parameterizes the hair $h_{d,\lambda,s}$ in the dynamical plane. We know that $F_{t,d}$ varies continuously in t, as the hairs in the dynamical plane are continuous. Therefore, we can find a δ such that if $|t - t_0| < \delta$, then $F_{t,d}$ also contracts A. This implies that the fixed point for $F_{t,d}$ is also in A and in particular, less than ϵ away from λ_0.

To complete the proof on the existence of hairs in the parameter plane we need to show the real part of the hairs heads off to infinity. But these hairs are identical to the external rays of Douady and Hubbard [9], since they have the correct itinerary, and these rays are known to extend to infinity.

Remark. As before, it is possible to relax the assumption that all the s_i be non-zero. In this case we get the existence of hairs in the parameter plane, but they are only defined in the far right half plane, that is, for t sufficiently large. In particular, these hairs do not necessarily terminate at λ-values for which $-d$ is preperiodic.

4.2 Convergence of Polynomial Hairs to Exponential Hairs

Fix s to be some regular sequence. Let $F_{d,t}(\lambda)$ be the t-point on the hair in the dynamical plane for $P_{d,\lambda}$ and let $F_t(\lambda)$ be the t-point on the hair in the dynamical plane for E_λ. Both $F_{t,d}$ and F_t are analytic functions in λ on $\mathbf{C}'$. More importantly, Corollary 3.15 that $F_{t,d} \to F_t$ uniformly on all of $\mathbf{C}'$.

We have shown that $F_{t,d}$ has a unique attracting fixed point for each d. Similar results in [7] show that F_t also has a unique attracting fixed point. Since $F_{t,d}$ converges to F_t uniformly, it follows that the fixed points of $F_{t,d}$ converge to the corresponding fixed point for F_t. But these fixed points are exactly the λ values on the associated hairs in the parameter plane. We have proven the following:

Theorem 4.3 *Let s be a regular sequence. For each t,*

$$\lim_{d\to\infty} H_{s,d}(t) = H_s(t).$$

In other words, the hairs in the parameter plane with itinerary s for the polynomial family converge pointwise to the hair in the parameter plane with itinerary s for the exponential family.

References

[1] Aarts, J. and Oversteegen, L. The Geometry of Julia Sets, *Trans. Amer. Math. Soc.* **338** (1993) 897-918.

[2] Baker, I. N. Wandering Domains in the Iteration of Entire Functions, *Proc. London Math. Soc.* **49** (1984), 563-576.

[3] Baker, I. N. Repulsive Fixpoints of Entire Functions, *Math. Zeitschrift* **104** (1968), 252-256.

[4] Blanchard, P. Complex Analytic Dynamics on the Riemann Sphere, *B.A.M.S.* Vol. II, No.1, 1984, 85-141.

[5] Baker, I. N. and Rippon, P. Iteration of Exponential Functions, *Ann. Acad. Sci. Fenn.*, Series 1A Math. Vol. 9, 1984, pp. 49-77.

[6] Bhattarcharjee, R. and Devaney, R. L. Tying Hairs for Structurally Stable Exponentials. Preprint.

[7] Bodelón, C., et.al. Hairs for the Complex Exponential Family. To appear in *Int'l J. Bifurcation Chaos.*

[8] Carleson, L. and Gamelin, T. *Complex Dynamics.* Springer-Verlag, New York, 1993.

[9] Douady, A. and Hubbard, J. Itération des Polynômes quadratiques complexes, *C.R. Acad. Sci. Paris*, t.29, Serie I-1982, pp. 123-126.

[10] Devaney, R. L. and Jarque, X. Misiurewicz Points for Complex Exponentials *Int. J. Bifurcation and Chaos* **7** (1997), 1599-1616.

[11] Devaney, R. L. and Krych, M. Dynamics of Exp(z), *Ergodic Theory and Dynamical Systems* **4** (1984), 35-52.

[12] Devaney, R. L. and Tangerman, F. Dynamics of Entire Functions Near the Essential Singularity, *Ergodic Theory and Dynamical Systems* **6** (1986), 489-503.

[13] Eremenko, A. and Lyubich, M. Yu. Iterates of Entire Functions. *Dokl. Akad. Nauk SSSR* **279** (1984), 25-27. English translation in *Soviet Math. Dokl.* **30** (1984), 592-594.

[14] Eremenko, A. and Lyubich, M. Yu. Structural stability in some families of entire functions. *Funk. Anal. i Prilo.* **19** (1985), 86-87.

[15] Eremenko, A. and Lyubich, M. Yu. Dynamical Properties of Some Classes of Entire Functions. *Ann. Inst. Fourier (Grenoble)* **42** (1992), no. 4, 989–1020.

[16] Fagella, N. Limiting Dynamics for the Complex Standard Family. *Internat. J. Bifur. Chaos* **5** (1995), 673-699.

[17] Fatou, P., Sur l'Itération des fonctions transcendentes Entières, *Acta Math.* **47** (1926), 337-370.

[18] Goldberg, L. R. and Keen, L. A Finiteness Theorem For A Dynamical Class of Entire Functions, *Ergodic Theory and Dynamical Systems* **6** (1986), 183-192.

[19] Julia, G. Iteration des Applications Fonctionelles, *J. Math. Pures Appl.* (1918), 47-245.

[20] Misiurewicz, M., On Iterates of e^z, *Ergodic Theory and Dynam. Syst.* **1** (1981), 103-106.

[21] Sullivan, D., Quasiconformal Maps and Dynamical Systems I, Solutions of the Fatou-Julia Problem on Wandering Domains. *Ann. Math.* **122** (1985), 401-418.