

# Hairs for the Complex Exponential Family

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# 1 Introduction

Our goal in this paper is to describe the dynamics and the parameter plane for the family of complex exponential functions

$$E_\lambda(z) = \lambda e^z$$

where  $\lambda \in \mathbb{C}$  is a parameter. This family has been discussed extensively when  $\lambda \in \mathbb{R}$  (see [11]). Our main goal is to prove the existence of “hairs” in both the dynamical and parameter planes for this family.

## 1.1 The Quadratic Family

The hairs in the exponential family are the analogues of the *external rays* for the family of quadratic polynomials

$$Q_c(z) = z^2 + c$$

as introduced by Douady and Hubbard [9]. We first recall the definition of these rays, and then describe their analogues in the exponential family.

For  $Q_c$ , the filled Julia set is defined as

$$K_c = \{z \in \mathbb{C} \mid Q_c^n(z) \text{ is bounded, } n = 1, 2, \dots\}.$$

The boundary of the filled Julia set,  $J_c$ , is called the Julia set of  $Q_c$ . Equivalently,

$$J_c = \{z \in \mathbb{C} \mid \{Q_c^n\} \text{ fails to be a normal family at } z\}.$$

It is known that all of the interesting dynamics of  $Q_c$  occurs on  $J_c$ . In particular,  $J_c$  is the closure of the set of repelling periodic points for  $Q_c$ . See [4], [8], and [9] for general background on the dynamics of the quadratic family.

Near  $\infty$ , the dynamics of  $Q_c$  are equivalent to those of  $z \rightarrow z^2$ . More precisely, we may find a neighborhood  $U_c$  of  $\infty$  and an analytic conjugacy

$$\phi_c : U_c \rightarrow \mathbb{C}$$

such that

$$(\phi_c(z))^2 = \phi_c(Q_c(z)).$$

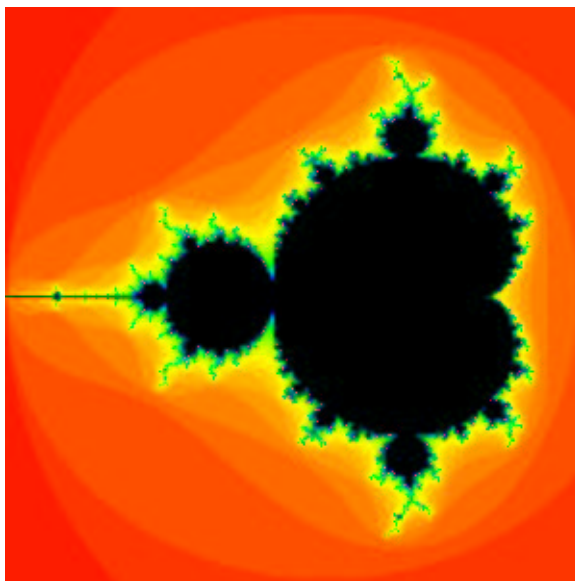


Figure 1: The Mandelbrot set for  $Q_c(z)$ .

The preimage under  $\phi_c$  of a straight ray of the form  $t \rightarrow te^{2\pi i\theta}$  for  $t > 1$  is called an external ray for  $Q_c$  and denoted  $\gamma_\theta(t)$ . By the conjugacy,  $Q_c(\gamma_\theta) = \gamma_{2\theta}$ .

The filled Julia sets for the quadratic family exhibit a fundamental dichotomy as determined by the fate of the orbit of the critical point, 0. Either  $K_c$  is connected, in which case the orbit of 0 under  $Q_c$  is bounded, or  $K_c$  is a Cantor set, in which case  $Q_c^n(0) \rightarrow \infty$ . When  $K_c$  is connected, the conjugacy  $\phi_c$  may be extended to  $\mathbb{C} - K_c$ . In the case that  $\lim_{t \rightarrow 1} \gamma_\theta(t)$  exists, we call this point the *landing point* of the ray  $\gamma_\theta$ .

## 1.2 The Mandelbrot Set

The fundamental dichotomy above is also reflected in the parameter plane for  $Q_c$ . The picture in the parameter plane, the well known Mandelbrot set  $M$ , consists of all  $c$ -values for which  $K_c$  is connected or, equivalently, for which  $Q_c^n(0)$  is bounded. See Figure 1.

According to Douady and Hubbard [9], there is a uniformization of the

exterior of  $M$

$$\Phi: \mathbb{C} - M \rightarrow \mathbb{C} - \{z \mid |z| > 1\}.$$

As in the case of the dynamical plane, we define the external ray

$$R_\theta(t) = \Phi^{-1}(te^{2\pi i\theta}),$$

for  $t > 1$ . Again as in the case of  $K_c$ , certain of these rays are known to land at points of  $M$ . There is a beautiful relationship between the dynamics of the landing points for certain of the external rays in  $M$  and the doubling map  $\theta \rightarrow 2\theta \pmod{1}$ . For example, suppose the binary expansion of  $\theta$  is either periodic or preperiodic under angle doubling, i.e., doubling modulo one. Then it is known that the ray  $R_\theta$  lands at a point in  $M$ . Moreover, if  $\theta$  is periodic,  $R_\theta$  lands at a root point of a hyperbolic component of  $M$  (a bifurcation point). If  $\theta$  is preperiodic, then  $R_\theta$  lands at a Misiurewicz point in  $M$ , i.e., a  $c$ -value for which the orbit of 0 eventually cycles and  $K_c = J_c$  is a dendrite. See [9] for details.

### 1.3 The Exponential Family

We now turn to the exponential family  $E_\lambda(z) = \lambda e^z$  where  $\lambda \in \mathbb{C}$  is a parameter. The Julia set of  $E_\lambda$  is defined as before as

$$J(E_\lambda) = \{z \in \mathbb{C} \mid \{E_\lambda^n\} \text{ fails to be a normal family at } z\}.$$

It is known that  $J(E_\lambda)$  is also the closure of the set of repelling periodic points of  $E_\lambda$  [3]. It is also known that if  $E_\lambda^n(z) \rightarrow \infty$ , then  $z \in J(E_\lambda)$ . This follows immediately from the fact that complex exponentials cannot have wandering domains or domains at  $\infty$ . See [2], [14], and [17]. Recall that, for the quadratic family, points whose orbits escape do not belong to the Julia set.

This is one difference between the exponential and quadratic families, but there are many others. Chief among them is the fact that  $\infty$  is an attracting fixed point for  $Q_c$ , but  $\infty$  is an essential singularity for  $E_\lambda$ . As we show below,  $J(E_\lambda)$  contains  $\infty$  in its closure. As a consequence, there is no analogue of the conjugacies  $\phi_c$  or  $\Phi$  in either the dynamical or parameter planes for  $E_\lambda$ .

However, we may still identify certain curves in the dynamical plane that play the role of the external rays  $\gamma_\theta$ . We will show that for each  $\lambda \in \mathbb{C}' = \mathbb{C} - \{\text{non-positive reals}\}$  and  $K \in \mathbb{Z}^+$  there is an invariant Cantor set contained

in the Julia set of  $E_\lambda$  on which the dynamics of  $E_\lambda$  are conjugate to the one-sided shift map on  $2K$  symbols. Attached to each point in this Cantor set is a curve or “hair” that extends to  $\infty$ . The point in the Cantor set is called the endpoint of the hair. The orbit of any point on a hair (excluding the endpoint) tends to  $\infty$  under iteration of  $E_\lambda$  and, moreover,  $E_\lambda$  permutes these hairs as dictated by the shift map.

Because of this, the hairs are the analogues of the external rays  $\gamma_\theta$ . However, there are several differences. First, the hairs for  $E_\lambda$  lie in the Julia set of  $E_\lambda$  whereas the  $\gamma_\theta$  lie in the complement of  $J_c$  for  $Q_c$ . Secondly, the hairs do not foliate an open neighborhood of  $\infty$  for  $E_\lambda$  as they do for  $Q_c$ .

For the quadratic family, the orbit of the critical point plays a crucial role in determining the dynamics. For  $E_\lambda$ , there is no such critical point. However,  $E_\lambda$  has a unique asymptotic value (or omitted value) 0. This orbit plays a somewhat similar role for  $E_\lambda$ . For example, it is known ([5], [15]) that the Julia set of  $E_\lambda$  is the entire plane if  $E_\lambda^n(0) \rightarrow \infty$ . Also, if  $E_\lambda$  admits an attracting cycle, then the orbit of 0 must tend to this cycle, showing that  $E_\lambda$  may have at most one attracting cycle. In this case it is known that  $J(E_\lambda)$  is a nowhere dense subset of the plane.

## 1.4 The Parameter Plane for the Exponential Family

We do not have a fundamental dichotomy for  $E_\lambda$  based on the fate of the orbit of 0 as we do for  $Q_c$ , for there are many values of  $\lambda$  for which the orbit of 0 is bounded yet the Julia set of  $E_\lambda$  is the entire plane. Nevertheless, we may begin to paint the picture of the parameter plane for  $E_\lambda$  based on the fate of the orbit of 0. In Figure 2, we show a picture of this plane. The visible black regions are “hyperbolic components”,  $C_k$ , i.e.,  $\lambda$ -values for which  $E_\lambda$  has an attracting cycle of period  $k$ . The cardioid-shaped region in the center of the picture is the attracting fixed point region. The large black region to the left is  $C_2$ . The large horizontal strips extending to the right comprise  $C_3$ . It is known that for each  $k > 2$ ,  $C_k$  consists of infinitely many distinct components and each component extends to  $\infty$  in the direction of the right half plane [5].

Between these components sit the analogues of the external rays in the parameter plane. See Figures 2-4. In these images, these rays appear to occupy open sets. However, we will show that there are actually infinitely

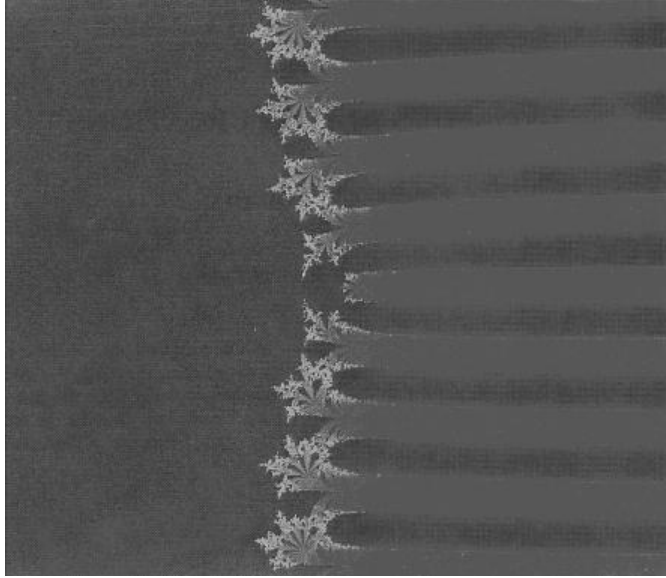


Figure 2: The parameter plane for  $E_\lambda$ .

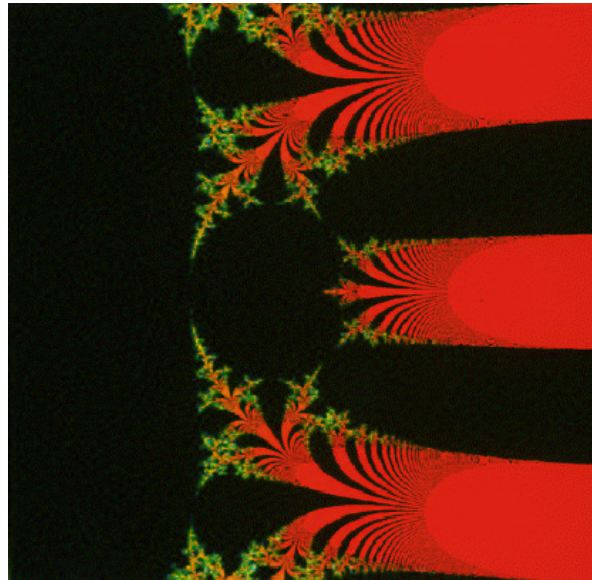


Figure 3: Detail of the parameter plane near  $\lambda = 0$ .

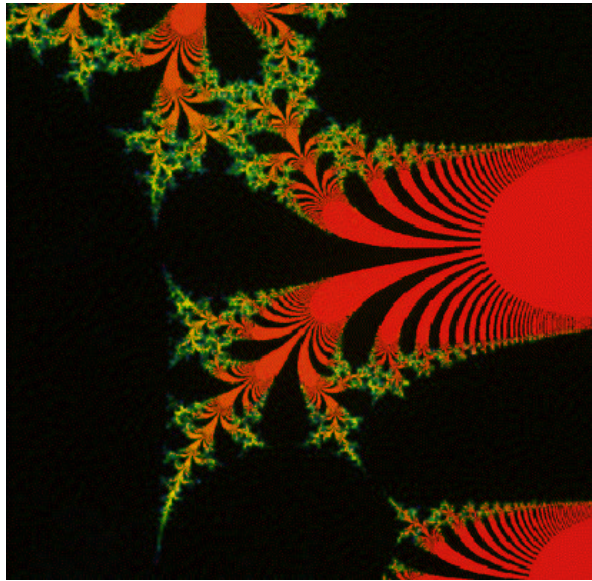


Figure 4: More detail of the hairs of the parameter plane.

many distinct curves (or hairs) in the complement of the hyperbolic components. These curves consist of  $\lambda$  values for which  $E_\lambda^n(0) \rightarrow \infty$  and so the Julia set of  $E_\lambda$  is  $\mathbb{C}$ . As in the case of the dynamical plane, we will study the arrangement of these hairs using symbolic dynamics.

## 2 The dynamical plane for $E_\lambda(z)$ .

In this section we discuss the dynamics of  $E_\lambda(z) = \lambda e^z$ . We assume throughout that  $\lambda \in \mathbb{C}$ . The dynamics of  $E_\lambda$  when  $\lambda \in \mathbb{R}^-$  are well understood; we exclude this case merely to gain continuity of the Markov partitions discussed below.

We will prove the existence of an invariant Cantor set in  $J(E_\lambda)$  for all  $\lambda \in \mathbb{C}$ . A similar construction was obtained for real  $\lambda$  in [1], [11], and [12].

## 2.1 A Markov partition for $E_\lambda$

For each integer  $k$ , we define horizontal strips  $R(k) = R_\lambda(k)$  by

$$R(k) = \{z \in \mathbb{C} \mid (2k-1)\pi - \arg \lambda < \operatorname{Im} z < (2k+1)\pi - \arg \lambda\}.$$

Here  $\arg \lambda$  is the principal branch of the argument, taking values between  $\pm\pi$ . Note that  $E_\lambda$  maps the boundary of  $R(k)$  onto the negative real axis. Consequently,  $E_\lambda$  maps  $R(k)$  onto  $\mathbb{C}'$  for each  $k$ , and  $\mathbb{R}^- \cup \{0\} \subset R(0)$  for each  $\lambda$ . In particular,  $E_\lambda(R(k)) \supset R(j)$  provided  $j \neq 0$ . Hence the  $R(k)$  provide a type of Markov partition for  $E_\lambda$ , at least when  $k \neq 0$ .

**Definition 2.1** For  $z \in \mathbb{C}$ , the itinerary of  $z$  under  $E_\lambda$  is the sequence of integers  $s(z) = s_0 s_1 s_2 \dots$  where  $s_n = k$  iff  $E_\lambda^n(z) \in R(k)$ . We do not define the itinerary of  $z$  if  $E_\lambda^n(z) \in \mathbb{R}^-$  for some  $n$ .

**Definition 2.2** An itinerary  $s_0 s_1 s_2 \dots$  is called regular if  $s_j \neq 0$  for all  $j$ .

Let  $\Sigma_K$  consist of all one-sided sequences  $s_0 s_1 s_2 \dots$  where each  $s_j \in \mathbb{Z}$ ,  $s_j \neq 0$ , and  $|s_j| \leq K$ , i.e., all regular sequences. The one-sided shift  $\sigma$  on  $\Sigma_K$  (also called the shift on  $2K$  symbols) is defined by

$$\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 s_3 \dots).$$

It is well known that  $\sigma$  has dense periodic points in  $\Sigma_K$ , has dense orbits, and exhibits sensitive dependence on initial conditions.

## 2.2 Invariant Cantor sets for $E_\lambda$

In this section we will construct invariant Cantor sets for  $E_\lambda$  on which  $E_\lambda$  is conjugate to the one-sided shift on  $\Sigma_K$  for each integer  $K > 0$ .

For each  $\lambda \in \mathbb{C}'$ , we will construct an invariant Cantor set homeomorphic to  $\Sigma_K$ . Toward that end, we define  $L_{\lambda, s_j}$  to be the inverse of  $E_\lambda$  defined on  $\mathbb{C}'$  and taking values in the strip  $R_\lambda(s_j)$ . Note that

$$L_{\lambda, s_j}(z) = -\log \lambda + \log z + 2\pi i s_j$$



where  $\log$  is the principal branch of the logarithm.

Given  $\lambda$ , define

$$\begin{aligned} M_K &= \{\sup |\operatorname{Im} z| \mid z \in R_\lambda(\pm K)\} \\ m_K &= \{\inf |\operatorname{Im} z| \mid z \in R_\lambda(\pm 1)\} \end{aligned}$$

$M_K$  and  $m_K$  give the maximal and minimal vertical distances to the real line in the union of the strips  $R_\lambda(s_i)$  when  $0 < |s_i| \leq K$ . Note that  $M_K > 2\pi$ .

Define

$$V_\alpha^\beta(s_i) = \{z \in \overline{R(s_i)} \mid \alpha \leq \operatorname{Re} z \leq \beta\}.$$

Each  $V_\alpha^\beta(s_i)$  is a closed rectangle whose interior lies in  $R_\lambda(s_i)$ . Throughout this section we fix  $\alpha$  and  $\beta$  so that

$$\begin{aligned} 0 < |\lambda|e^\alpha &< m_K \\ |\lambda|e^\beta &> M_K + |\beta| + |\alpha|. \end{aligned}$$

With this choice of  $\alpha$  and  $\beta$ , the image of each  $V_\alpha^\beta(s_i)$  under  $E_\lambda$  is an annular region given by

$$|\lambda|e^\alpha \leq |z| \leq |\lambda|e^\beta.$$

Our choice of  $\alpha$  and  $\beta$  guarantees that  $E_\lambda(V_\alpha^\beta(s_i))$  covers  $V_\alpha^\beta(s_j)$  for each  $i$  and  $j$  (assuming  $0 < |s_i|, |s_j| \leq K$ ). See Figure 5.

Let

$$V_\alpha^\beta = \bigcup_{0 < |j| \leq K} V_\alpha^\beta(j).$$

$V_\alpha^\beta$  depends on  $K$  and  $\lambda$ . Let  $\Lambda_{K,\lambda}$  be the set of points whose orbits remain for all time in  $V_\alpha^\beta$ .

**Theorem 2.3** *Suppose  $K > 0$ . For each  $\lambda \in \mathbb{C}'$ ,  $\Lambda_{K,\lambda}$  is homeomorphic to  $\Sigma_K$  and  $E_\lambda|_{\Lambda_{K,\lambda}}$  is conjugate to the shift map on  $\Sigma_K$ .*

**Proof.** Let  $s = (s_0 s_1 s_2 \dots) \in \Sigma_K$ . Let  $z \in V_\alpha^\beta$ . Define

$$L_{\lambda,s}^n(z) = L_{\lambda,s_0} \circ \dots \circ L_{\lambda,s_{n-1}}(z)$$

We claim that

$$\lim_{n \rightarrow \infty} L_{\lambda,s}^n(z)$$

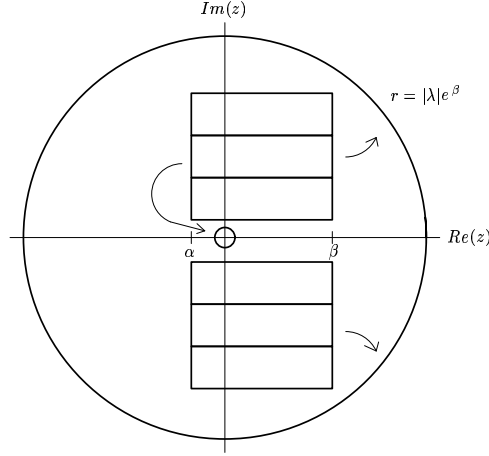


Figure 5:  $E_\lambda(V_\alpha^\beta(s_j)) \supset V_\alpha^\beta(s_k)$  for all  $k$ .

exists and is independent of  $z$ .

To see this, note that  $E_\lambda(\text{Int } V_\alpha^\beta(s_j))$  is a simply connected region in  $\mathbb{C}$  for each  $j$ . Also,  $L_{\lambda,s_j}$  maps this region strictly inside itself and so each  $L_{\lambda,s_j}$  is a strict contraction in the Poincaré metric on  $E_\lambda(\text{Int } V_\alpha^\beta(s_j))$ . In particular, each  $L_{\lambda,s_j}$  is a strict contraction in the Poincaré metric on the subset  $V_\alpha^\beta$ . As a consequence, the sets  $L_{\lambda,s}^n(V_\alpha^\beta)$  are nested and decrease in diameter to 0 as  $n \rightarrow \infty$ . Hence  $\lim_{n \rightarrow \infty} L_{\lambda,s}^n(z)$  exists and is independent of  $z$ .

We therefore define  $\Phi(s) = \lim_{n \rightarrow \infty} L_{\lambda,s}^n(z)$  for any  $z \in V_\alpha^\beta$ . Standard arguments (see [4], Theorem 9.9) then show that  $\Phi$  is a homeomorphism which gives the conjugacy between  $E_\lambda$  and the shift map.  $\square$

Given  $\lambda, K$ , we define  $z_\lambda(s)$  to be the unique point in  $\Lambda_{K,\lambda}$  whose itinerary under  $E_\lambda$  is  $s$ .

**Corollary 2.4** *Suppose  $s = \overline{s_0 \dots s_{n-1}}$  is a repeating sequence in  $\Sigma_K$ . Then  $z_\lambda(s)$  is a repelling periodic point of period  $n$  for  $E_\lambda$ .*

**Proof.** Since  $L_{\lambda,s}^n$  is a composition of analytic maps, it follows that  $L_{\lambda,s}^n$  is analytic. Also,  $L_{\lambda,s}^n(V_\alpha^\beta(s_0))$  is contained in the interior of  $V_\alpha^\beta(s_0)$ . Since  $L_{\lambda,s}^n$  is a strict contraction in the Poincaré metric on  $V_\alpha^\beta(s_0)$ , it follows that

$L_{\lambda,s}^n$  has a unique fixed point in this rectangle and that this fixed point is attracting for  $L_{\lambda,s}^n$ , hence repelling for  $E_\lambda$ . Since this point has itinerary  $s$  for  $E_\lambda$ , it must be  $z_\lambda(s)$  and the result follows.

**Corollary 2.5** *Let  $s \in \Sigma_K$ . Then  $z_\lambda(s)$  lies in the Julia set of  $E_\lambda$ .*

**Proof.** By the previous corollary,  $z_\lambda(s)$  is a limit of repelling periodic points given by the conjugacy with the shift map. By a result of Baker [3],  $J(E_\lambda)$  is the closure of the set of repelling periodic points. Hence  $z_\lambda(s) \in J(E_\lambda)$ .

**Remarks.**

1. The hypothesis that the  $s_j \neq 0$  cannot be dropped in the theorem or its corollaries. For example, if  $\lambda \in \mathbb{R}$  and  $\lambda > 1/e$ , then  $E_\lambda$  has two repelling fixed points in the strip  $R_\lambda(0)$ . See [11]. However, when  $0 < \lambda < 1/e$ , we may allow sequences with 0's, as we show in the next section.
2. It is important to note that  $z_\lambda(s)$  is not the only point in the strip  $R_\lambda(s_0)$  that has itinerary  $s$ . In fact, there are infinitely many points in this strip that share the itinerary  $s$ . These are the hairs that we deal with in the next chapter.
3. It is possible to give a more complete symbolic description of the dynamics of  $E_\lambda$  using both unbounded and non-regular itineraries. See [11], where this is shown for  $e^z$ . However, we will not make use of this fact in this paper.

### 2.3 The case $0 < \lambda < 1/e$

The case where  $0 < \lambda < 1/e$  will play a special role in the remainder of this paper. In this case we do not need to restrict to regular sequences. All of the results below appear elsewhere ([1], [12]), but we include them here for completeness.

Let  $\Sigma'_K$  denote the set of all one-sided sequences  $(s_0s_1s_2\dots)$  where each  $s_j \in \mathbb{Z}$  and  $|s_j| \leq K$ . We specifically allow 0 as an entry for sequences in  $\Sigma'_K$ .

When  $\lambda = 1/e$ ,  $E_\lambda$  has a fixed point at 1 and  $E'_\lambda(1) = 1$ . If  $\lambda \in (0, 1/e)$  then  $E_\lambda$  has two fixed points in  $\mathbb{R}^+$ . We denote them by  $a_\lambda$  and  $r_\lambda$  where

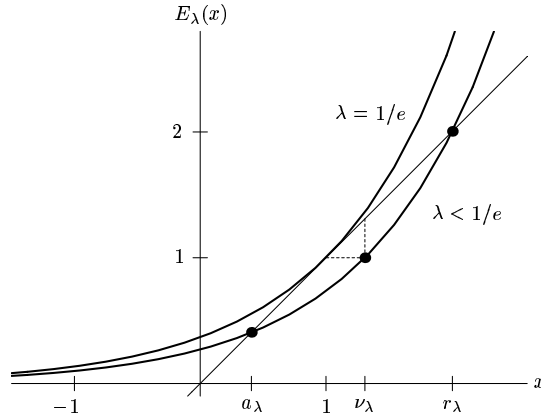


Figure 6: The graphs of  $E_\lambda$  for  $\lambda = 1/e$  and  $\lambda < 1/e$ .

$0 < a_\lambda < 1 < r_\lambda$ . One checks easily that  $a_\lambda$  is attracting while  $r_\lambda$  is repelling. Also,  $E_\lambda(1) < 1$ . See Figure 6.

Since  $\lambda$  is real, the horizontal strips  $R(k)$  are bounded above and below by  $\text{Im } z = (2k \pm 1)\pi$  for each  $k$ . The quantity  $M_K$  is therefore given by  $M_K = (2K + 1)\pi$ .

For any  $\tau > 1$ , we define the collection of rectangles  $\tilde{V}_1^\tau$  by

$$\tilde{V}_1^\tau = \{z \mid 1 \leq \text{Re } z \leq \tau, |\text{Im } z| \leq M_K\}.$$

The only difference between  $\tilde{V}_1^\tau$  and  $V_1^\tau$  is that we now include  $V_1^\tau(0)$ , the rectangle in the 0-strip  $R_\lambda(0)$ , in  $\tilde{V}_1^\tau$ .

Note also that if  $\tau = \tau(\lambda)$  is chosen so that  $\lambda e^\tau > \tau + M_K$ , then  $E_\lambda$  maps the vertical line  $\text{Re } z = \tau$  to a circle of radius  $\lambda e^\tau$  centered at 0. This circle contains all of  $\tilde{V}_1^\tau$  in its interior. Also,  $E_\lambda$  maps the vertical line  $\text{Re } z = 1$  to a circle of radius smaller than 1. It follows that  $L_{\lambda, s_i}(\tilde{V}_1^\tau)$  is contained in the interior of  $\tilde{V}_1^\tau$  for each  $i$ . See Figure 7.

Let  $\nu_\lambda = -\log \lambda$  so that  $E_\lambda(\nu_\lambda) = 1$ . Note that  $1 < \nu_\lambda < r_\lambda$ . Hence  $E_\lambda$  maps the strip  $1 \leq \text{Re } z < \nu_\lambda$  to the annular region  $\lambda e \leq |z| < 1$  which lies in the exterior of  $\tilde{V}_1^\tau$ . Therefore, if  $z \in \tilde{V}_1^\tau$ , then  $\text{Re } L_{\lambda, s_i}(z) \geq \nu_\lambda$  for each  $i$ . Also, if  $\text{Re } z \geq \nu_\lambda$ , then  $0 < |L'_{\lambda, s_i}(z)| < \omega_\lambda$  for some constant  $\omega_\lambda < 1$ . In fact,  $\omega_\lambda = 1/\nu_\lambda$ . It follows that for any sequence  $s_0 s_1 s_2 \dots$  with  $|s_j| \leq K$  and

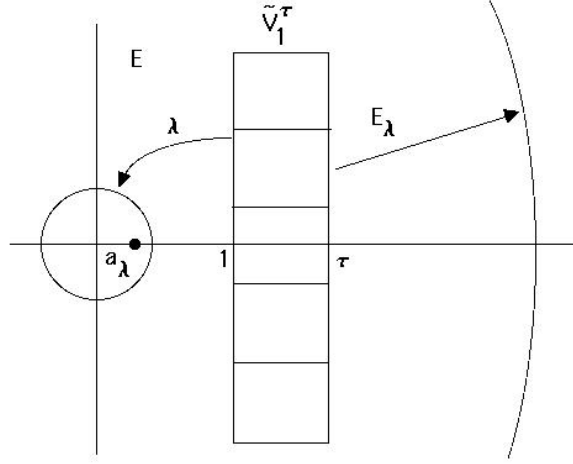


Figure 7: Image of  $\tilde{V}_1^\tau$  under  $E_\lambda$ .

any  $z \in \tilde{V}_1^\tau$ , we have

$$\left| \frac{d}{dz}(L_{\lambda,s}^n)(z) \right| < \omega_\lambda^{n-1}$$

since it takes at most one iteration of  $L_{\lambda,s_i}$  to guarantee that  $\operatorname{Re} L_{\lambda,s_i}(z) \geq \nu_\lambda$ .

Let  $\Lambda'_{K,\lambda}$  denote the set of points whose orbit remains for all time in  $\tilde{V}_1^\tau$ . Using the above facts, we may mimic the proof of Theorem 2.3 to prove:

**Theorem 2.6** *Suppose  $\tau$  is chosen so that  $\lambda e^\tau > \tau + M_K$ . Then  $\Lambda'_{K,\lambda}$  is homeomorphic to  $\Sigma'_K$  and  $E_\lambda|_{\Lambda'_{K,\lambda}}$  is conjugate to the shift map on  $\Sigma'_K$ .*

### 3 Hairs in the Dynamical Plane

Our goal for this section is to show that each point in  $\Lambda_{K,\lambda}$  (or  $\Lambda'_{K,\lambda}$ , if  $0 < \lambda < 1/e$ ) actually lies at the endpoint of a continuous curve, all points of which share the same itinerary.

**Definition 3.1** *A continuous curve  $h_{\lambda,s}: [1, \infty) \rightarrow R_\lambda(s_0)$  is called a hair attached to  $z_\lambda(s)$  if*

1.  $h_{\lambda,s}(1) = z_\lambda(s)$ .

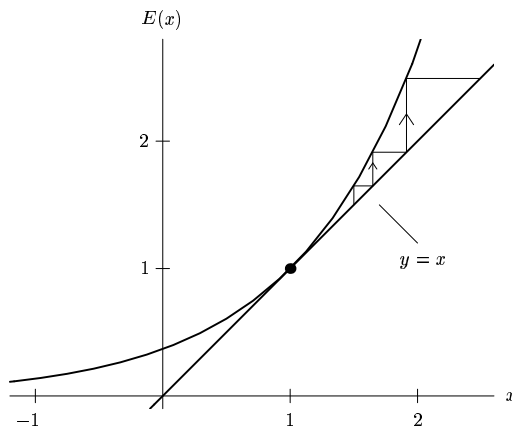


Figure 8: The graph of  $E(x) = (1/e)e^x$ .

2. For each  $t \geq 1$ , the itinerary of  $h_{\lambda,s}(t)$  under  $E_\lambda$  is  $s$ .
3. If  $t > 1$ , then  $\lim_{n \rightarrow \infty} \operatorname{Re} E_\lambda^n(h_{\lambda,s}(t)) = \infty$ .
4.  $\lim_{t \rightarrow \infty} \operatorname{Re} h_{\lambda,s}(t) = \infty$ .

Roughly speaking, a hair attached to  $z_\lambda(s)$  is a continuous curve stretching from  $z_\lambda(s)$  to infinity in the right half plane. Any point  $z$  on this hair with  $z \neq z_\lambda(s)$  shares the same itinerary as  $z_\lambda(s)$  and has orbit which tends to infinity in the right half plane. Thus each hair lies in the Julia set of  $E_\lambda$ . On the other hand, the orbit of the endpoint  $z_\lambda(s)$  remains bounded.

We will show that each point  $z_\lambda(s) \in \Lambda_{K,\lambda}$  has a hair attached to it, and that, up to parametrization, this hair is unique.

We will often encounter the case where  $\lambda = 1/e$  in the sequel, so we choose a simplified notation in this case. Define  $E(z) = (1/e)e^z$ . Note that  $E$  has a fixed point at 1 and  $E'(1) = 1$ . Also,  $E^n(t) \rightarrow \infty$  as  $n \rightarrow \infty$  if  $t > 1$ . See Figure 8.

For any sequence  $s = s_0 s_1 s_2 \dots$  we define the functions  $G_s^n: \mathbb{C}' \times [1, \infty) \rightarrow \mathbb{C}$  by

$$G_s^n(\lambda, t) = L_{\lambda,s}^n \circ E^n(t).$$

Note that  $G_s^n$  is well defined provided that  $L_{\lambda,s}^n \circ E^n(t)$  does not lie in  $\mathbb{R}^- \cup \{0\}$  for each  $j$ . This is one of the reasons for our assumption in Section 3.2 that each  $s_j \neq 0$ , as this implies that  $L_{\lambda,s_j}(z)$  never meets  $\mathbb{R}^- \cup \{0\}$ . Our goal is to show that, for each  $t \geq 1$ , the  $G_s^n$  converge uniformly as a function of  $\lambda$  to an analytic function. For each fixed  $\lambda$ , this limit function will give a parametrization of the hair  $h_{\lambda,s}$  as a function of  $t$ .

### 3.1 The case $0 < \lambda < 1/e$ .

In this section, we will consider hairs for the special case where  $\lambda \in [\alpha, 1/e)$  for some  $0 < \alpha < 1/e$ . In the next section we will consider the general case. Fix a sequence  $s = (s_0 s_1 s_2 \dots) \in \Sigma'_K$ .

**Proposition 3.2** *Suppose  $\lambda \in [\alpha, 1/e)$  with  $\alpha > 0$ . Let  $s = s_0 s_1 s_2 \dots \in \Sigma'_K$ . There is a constant  $M = M(\lambda)$  such that, for all  $n \geq 0$  and  $t \geq 1$ ,*

$$t \leq \operatorname{Re} G_s^n(\lambda, t) \leq t + M.$$

**Proof.** Recall that the constant  $M_K = (2K + 1)\pi$  satisfies

$$|\operatorname{Im} G_{\sigma^i(s)}^n(\lambda, t)| \leq M_K$$

for all  $i$  and  $n$  since the itinerary  $s$  is bounded. Here,  $\sigma^i(s) = (s_i s_{i+1} s_{i+2} \dots)$ . Choose  $M = M(\lambda) \geq M_K$  so that, in addition,

$$M > \log(2M) - \log \lambda + \log 2.$$

Now we use induction. We have

$$L_{\lambda,s_0}(z) = -\log \lambda + \log |z| + 2\pi i s_0 + i \arg z.$$

Therefore

$$\begin{aligned} \operatorname{Re} L_{\lambda,s_0}((1/e)e^t) &= -\log \lambda + \log((1/e)e^t) \\ &= -\log \lambda - 1 + t \\ &\geq t \end{aligned}$$

since  $\lambda \leq 1/e$ . Also,

$$\begin{aligned} \operatorname{Re} L_{\lambda, s_0}((1/e)e^t) &= -\log \lambda - 1 + t \\ &< -\log \lambda + \log 2 + t \\ &< t + M - \log(2M) \\ &\leq t + M \end{aligned}$$

by our assumption on  $M$ . This gives the result when  $n = 1$ . Now let us assume that

$$t \leq \operatorname{Re} G_u^{n-1}(\lambda, t) \leq t + M$$

for any sequence  $u$  with  $|u_j| \leq K$  for all  $j$ . We write  $G_{\sigma(s)}^{n-1}(\lambda, E(t)) = \beta + i\nu$ .

By induction we have

$$E(t) \leq \beta \leq E(t) + M.$$

Hence  $1 + \log \beta \geq t$ . Also  $|\nu| \leq M_K < M$  by hypothesis.

Thus

$$\begin{aligned} \operatorname{Re} G_s^n(\lambda, t) &= \operatorname{Re} L_{\lambda, s_0} G_{\sigma(s)}^{n-1}(\lambda, E(t)) \\ &= -\log \lambda + \log |\beta + i\nu| \\ &\geq -\log \lambda + \log \beta \\ &\geq 1 + \log \beta \\ &\geq t. \end{aligned}$$

For the other inequality, we have

$$\operatorname{Re} G_{\sigma(s)}^{n-1}(\lambda, E(t)) = \beta \leq E(t) + M.$$

Then

$$\begin{aligned} \operatorname{Re} G_s^n(\lambda, t) &= \operatorname{Re} L_{\lambda, s_0} G_{\sigma(s)}^{n-1}(\lambda, E(t)) \\ &= -\log \lambda + \log |\beta + i\nu| \\ &\leq -\log \lambda + \log(E(t) + M + M). \end{aligned}$$

This follows since  $\beta \leq E(t) + M$  and  $|\nu| < M_K < M$ . Recall that  $\log(a+b) < \log a + \log b$  if both  $a, b > 2$ . Using this we have, provided  $E(t) > 2$ ,

$$\begin{aligned} \operatorname{Re} G_s^n(\lambda, t) &\leq -\log \lambda - 1 + t + \log(2M) \\ &\leq -\log \lambda + \log 2 + \log(2M) + t \\ &\leq t + M. \end{aligned}$$



If  $E(t) \leq 2$ , then we have

$$\begin{aligned}
\operatorname{Re} G_s^n(\lambda, t) &\leq -\log \lambda + \log(E(t) + M + M) \\
&\leq -\log \lambda + \log(2(M + 1)) \\
&\leq -\log \lambda + \log 2 + \log(M + 1) \\
&\leq -\log \lambda + \log 2 + \log(2M) \\
&\leq t + M.
\end{aligned}$$

□

This proposition shows that  $G_s^n(\lambda, t)$  lies in a bounded region in the plane, since  $|\operatorname{Im} G_s^n(\lambda, t)| \leq M_K$  by our assumption that  $|s_j| \leq K$  for each  $j$ .

**Proposition 3.3** *Let  $\lambda \in [\alpha, 1/e)$  with  $\alpha > 0$ . For any sequence  $s \in \Sigma'_K$  and any  $t \geq 1$ , the family of functions*

$$\lambda \rightarrow G_s^n(\lambda, t)$$

*converges uniformly in  $\lambda$  on compact subintervals of  $[\alpha, 1/e)$  for fixed  $t \in [\alpha, 1/e)$  as  $n \rightarrow \infty$ .*

**Proof.** Choose  $\beta$  such that  $\alpha < \beta < 1/e$  and let  $\epsilon > 0$ . We will demonstrate the uniform convergence of the  $G_s^n$  on the interval  $[\alpha, \beta]$  for each  $t \geq 1$ . Let  $M^* = \max M(\lambda)$  for  $\lambda \in [\alpha, \beta]$ , where  $M(\lambda)$  is as in Proposition 3.2.

Since we have  $|s_j| \leq K$ , it follows that  $|\operatorname{Im} z| \leq M_K$  for any  $z$  lying in one of the strips  $R_\lambda(s_j)$ . By Proposition 3.2, we have, for each  $\lambda \in [\alpha, \beta]$ ,

$$t \leq \operatorname{Re} G_s^m(\lambda, t) \leq t + M^*$$

for any  $s$  and all  $t \geq 1$ . Thus we have, for any  $n, m \geq 0$

$$|G_u^n(\lambda, t) - G_u^m(\lambda, t)| \leq M^* + 2M_K$$

for any sequence  $u = u_0 u_1 u_2 \dots$  with  $|u_j| \leq K$ .

Now recall that we may choose  $\tau'$  large enough so that for any  $\tau \geq \tau'$

- (1.)  $L_{\lambda, s_i}(\tilde{V}_1^\tau) \subset \operatorname{interior} \tilde{V}_1^\tau$  for each  $\lambda \in [\alpha, \beta]$ , and

(2.) There exists  $\omega = \max_{\lambda \in [\alpha, \beta]} \omega(\lambda) < 1$  so that

$$|(L_{\lambda, s}^N)'(z)| < \omega^{N-1}$$

for all  $z \in \tilde{V}_1^\tau$ .

Suppose  $t > 1$ . We first choose  $N$  so that  $(M^* + 2M_K)\omega^{N-1} < \epsilon$ . We then choose  $\tau > \tau'$  so that  $E^N(t) + M^* < \tau$ . Then for any  $n, m \geq 0$ , we have

$$\begin{aligned} |G_s^{N+n}(\lambda, t) - G_s^{N+m}(\lambda, t)| &= |L_\lambda^N G_{\sigma^N(s)}^n(\lambda, E^N(t)) - L_\lambda^N G_{\sigma^N(s)}^m(\lambda, E^N(t))| \\ &\leq \omega^{N-1} |G_{\sigma^N(s)}^n(\lambda, E^N(t)) - G_{\sigma^N(s)}^m(\lambda, E^N(t))| \\ &\leq \omega^{N-1} (M^* + 2M_K) < \epsilon. \end{aligned}$$

For the case  $t = 1$ , we set

$$\tau^* = \max_{\lambda \in [\alpha, \beta]} \tau(\lambda)$$

(in fact, choosing  $\tau^*$  such that  $\log(\tau^*/\alpha) < \tau^*$  will suffice).

Since  $L_{\lambda, s_i}(V_1^\tau) \subset V_1^\tau$ , by induction we have  $L_{\lambda, s}^n(V_1^\tau) \subset V_1^\tau$ . This allows us to infer that

$$|L_{\lambda, s}^n(1) - L_{\lambda, s}^m(1)| \leq 2(\tau^* + M_K)$$

for each  $n, m \geq 0$ . Now choose  $N > 0$  to satisfy

$$2\omega^{N-1}(\tau^* + M_K) < \epsilon.$$

Then the proof follows as above. □

Since the functions  $G_s^n(\lambda, t)$  converge uniformly on compact subintervals of  $[\alpha, 1/e)$  for each fixed  $t$ , we define  $h_{\lambda, s}(t) = \lim_{n \rightarrow \infty} G_s^n(\lambda, t)$ . In the next section we will prove that  $h_{\lambda, s}$  has the properties of a hair as listed in Definition 3.1. In the next section we will also use this result to prove uniform convergence for all  $\lambda \in \mathbb{C}'$ .

## 3.2 Hairs in the dynamical plane: the general case

Our goal in this section is to prove that if  $s = s_0 s_1 s_2 \dots$  is a regular, bounded sequence, then there is a unique hair attached to  $z_\lambda(s)$  for each  $\lambda \in \mathbb{C}'$ . We define  $G_s^n(\lambda, t)$  exactly as in the previous section. Since we can no longer guarantee that  $L_{\lambda,0}(z)$  does not meet 0 or the negative real axis, we suppose that  $s_j \neq 0$  for each  $j$ , i.e., that our sequence is regular.

We first prove

**Proposition 3.4** *Suppose  $s = s_0 s_1 s_2 \dots \in \Sigma_K$ . For each fixed  $t \geq 1$  the family of functions  $\{\lambda \rightarrow G_s^n(\lambda, t)\}$  is a normal family of functions on  $\mathbb{C}'$ . Moreover, this family converges uniformly as  $n \rightarrow \infty$  to an analytic function  $\lambda \rightarrow h_{\lambda,s}(t)$  for each  $t$ .*

**Proof.** Since each  $s_j \neq 0$ , we have that  $G_s^n(\lambda, t)$  is well defined. For each  $n$ ,  $G_s^n(\lambda, t)$  takes values in the  $s_0$ -strip for  $E_\lambda$ . The union of these strips as  $\lambda$  varies over  $\mathbb{C}'$  forms an open horizontal strip of width  $4\pi$ . Hence  $\lambda \rightarrow G_s^n(\lambda, t)$  omits many more than 3 values in  $\mathbb{C}$ . By Montel's theorem, it follows that  $\{G_s^n(\lambda, t)\}$  is a normal family of functions.

Thus, any sequence of the  $G_s^n$  contains a subsequence that converges uniformly to an analytic function or to  $\infty$ . However, by the results in the previous section, the sequence  $G_s^n(\lambda, t)$  converges uniformly for  $\lambda$  in compact subintervals of the form  $[\alpha, 1/e)$  with  $0 < \alpha < 1/e$ . Since an analytic function is completely determined by its values along any arc in an open region, it follows that the  $G_s^n(\lambda, t)$  for general  $\lambda$  converge uniformly to an analytic function on all of  $\mathbb{C}'$ . □

For each  $s \in \Sigma_K$  and  $t \geq 1$ , we now have defined a function

$$h_{\lambda,s}(t) = \lim_{n \rightarrow \infty} G_s^n(\lambda, t)$$

which is analytic as a function of  $\lambda$  for  $\lambda \in \mathbb{C}'$ . Note that, when  $t = 1$ , we have  $G_s^n(\lambda, 1) = L_{\lambda,s}^n(1)$ . It follows from the results in Section 3.1 that  $h_{\lambda,s}(1) = z_\lambda(s)$ .

We claim now that  $t \rightarrow h_{\lambda,s}(t)$  is continuous on  $[1, \infty)$  and parametrizes the hair attached to  $z_\lambda(s)$ . To prove this we need a preliminary result, which generalizes Proposition 3.2 in the last section.

**Proposition 3.5** *Suppose  $s = s_0 s_1 s_2 \dots \in \Sigma_K$ . Given  $\lambda \in \mathbb{C}$ , there exists  $\tau, M > 0$  such that, for all  $t \geq \tau$  and  $n \geq 1$ , we have*

$$t - M \leq \operatorname{Re} G_s^n(\lambda, t) \leq t + M$$

**Proof.** Since  $|s_i| \leq K$ , there exists  $M_K > 2\pi$  such that  $|\operatorname{Im} L_{\lambda, s_i}(z)| < M_K$  for each  $s_i$  and all  $z \in \mathbb{C}$ .

Given  $\lambda$ , let us choose  $M > 2$  so that the following inequalities hold:

$$M \geq \operatorname{Re}(\log \lambda) + 1 + \log M \quad (*)$$

$$M \geq \operatorname{Re}(-\log \lambda) - 1 + \log M + \log M_K. \quad (**)$$

Suppose  $\tau$  satisfies

$$E(\tau) = \frac{1}{e} e^\tau > \frac{M^2}{M-1} + 1 > M + 2.$$

If  $t \geq \tau$ , we have  $E(t) \geq E(\tau)$ , so

$$E(t) > \frac{M^2}{M-1} + 1 > M + 2.$$

Now let us use induction to prove that, for any sequence  $s$  with  $|s_i| \leq K$ ,

$$t - M \leq \operatorname{Re} G_s^n(\lambda, t)$$

for all  $n$  and all  $t \geq \tau$ .

We have

$$\begin{aligned} \operatorname{Re} G_s^1(\lambda, t) &= \operatorname{Re} L_{\lambda, s_0}(E(t)) \\ &= \operatorname{Re}(-\log \lambda) + \operatorname{Re}(\log(E(t))) \\ &= \operatorname{Re}(-\log \lambda) - 1 + t \\ &\geq t - M + \log M \end{aligned}$$

by equation (\*). Hence  $\operatorname{Re} G_s^1(\lambda, t) \geq t - M$ , which gives the result when  $n = 1$ .

Now let us assume that

$$\operatorname{Re} G_u^n(\lambda, t) \geq t - M$$

for any bounded sequence  $u = u_0 u_1 u_2 \dots$  with  $|u_j| \leq K$  and all  $t \geq \tau$ . We thus have

$$\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t)) \geq E(t) - M$$

so

$$\begin{aligned} \operatorname{Re} G_s^{n+1}(\lambda, t) &= \operatorname{Re} L_{\lambda, s_0}(G_{\sigma(s)}^n(\lambda, E(t))) \\ &= \operatorname{Re}(-\log \lambda) + \operatorname{Re}(\log G_{\sigma(s)}^n(\lambda, E(t))) \\ &= \operatorname{Re}(-\log \lambda) + \log |G_{\sigma(s)}^n(\lambda, E(t))| \\ &\geq \operatorname{Re}(-\log \lambda) + \log |\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t))| \\ &= \operatorname{Re}(-\log \lambda) + \log (\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t))). \end{aligned}$$

This last equality follows since  $\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t)) \geq E(t) - M > 2$ .

Therefore

$$\begin{aligned} \operatorname{Re} G_s^{n+1}(\lambda, t) &\geq \operatorname{Re}(-\log \lambda) + \log (E(t) - M) \\ &\geq \operatorname{Re}(-\log \lambda) + \log E(t) - \log M. \end{aligned}$$

This inequality follows from the fact that

$$\log(a - b) \geq \log a - \log b$$

provided  $a > b > 1$  and  $a > b^2/(b - 1)$ . In our case, we have  $E(t) > E(\tau) > M > 1$  and  $E(t) > M^2/(M - 1)$  by assumption.

Thus

$$\operatorname{Re} G_s^{n+1}(\lambda, t) \geq \operatorname{Re}(-\log \lambda) - 1 + t - \log M \geq t - M$$

because of assumption (\*).

Now we turn to the proof that

$$\operatorname{Re} G_s^n(\lambda, t) \leq t + M$$

for all  $n \geq 1$  and all  $t \geq \tau$ .

We have

$$\begin{aligned} \operatorname{Re} G_s^1(\lambda, t) &= \operatorname{Re} L_{\lambda, s_0}(E(t)) \\ &= \operatorname{Re}(-\log \lambda) + \log(E(t)) \\ &= \operatorname{Re}(-\log \lambda) - 1 + t \\ &\leq t + M - \log M - \log M_K \\ &\leq t + M \end{aligned}$$

by (\*\*). (Recall both  $M, M_K > 1$ .) This proves the case  $n = 1$ .

Now suppose that

$$\operatorname{Re} G_u^n(\lambda, t) \leq t + M$$

for any bounded sequence  $u = u_0 u_1 u_2 \dots$  with  $|u_j| \leq K$ .

Then we have

$$\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t)) \leq E(t) + M.$$

Now

$$\begin{aligned} \operatorname{Re} G_s^{n+1}(\lambda, t) &= \operatorname{Re} \left( L_{\lambda, s_0}(G_{\sigma(s)}^n(\lambda, E(t))) \right) \\ &= \operatorname{Re} (-\log \lambda) + \operatorname{Re} (\log G_{\sigma(s)}^n(\lambda, E(t))) \\ &= \operatorname{Re} (-\log \lambda) + \log |G_{\sigma(s)}^n(\lambda, E(t))| \\ &\leq \operatorname{Re} (-\log \lambda) + \log (|\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t))| \\ &\quad + |\operatorname{Im} G_{\sigma(s)}^n(\lambda, E(t))|) \\ &\leq \operatorname{Re} (-\log \lambda) + \log (\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t))) + \log M_K. \end{aligned}$$

This last inequality follows from the fact that  $\log(a + b) \leq \log a + \log b$  provided both  $a, b > 2$  and the fact that

$$\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t)) \geq E(t) - M \geq E(\tau) - M \geq 2.$$

This uses the previous induction.

We thus have

$$\operatorname{Re} G_s^{n+1}(\lambda, t) \leq \operatorname{Re} (-\log \lambda) + \log (E(t) + M) + \log M_K.$$

Now, as above,  $E(t) > M > 2$ , so

$$\log (E(t) + M) \leq \log E(t) + \log M.$$

Thus

$$\operatorname{Re} G_s^{n+1}(\lambda, t) \leq \operatorname{Re} (-\log \lambda) - 1 + t + \log M + \log M_K \leq t + M$$

by (\*\*).

□

**Remark.** For later use, note that the previous proposition holds for all sequences in  $\Sigma'_K$ , not just regular sequences. Indeed, for  $t$  large enough, the  $L_{\lambda, s_j}$  in the proof are bounded away from the half plane  $\operatorname{Re} z \leq 0$ , and hence these inverse maps are well-defined. With this result in hand, we can now prove continuity with respect to  $t$ .

**Proposition 3.6** *Let  $s \in \Sigma_K$ . Then for each  $\lambda \in \mathbb{C}'$ ,  $h_{\lambda,s}(t)$  is continuous as a function of  $t \in (1, \infty)$ .*

**Proof.** Let  $\epsilon > 0$  and choose  $\omega$  with  $0 < \omega < 1$ . Let  $\tau$  and  $M$  be as given in Proposition 3.5. Choose  $T > \tau + 2M$  so that, if  $\operatorname{Re} z > T$  and  $|\operatorname{Im} z| < M_K$ , then

$$|L'_{\lambda,s_i}(z)| < \omega.$$

By the previous proposition, if  $t > T$ , then

$$E^k(t) - M \leq \operatorname{Re} G_s^n(\lambda, E^k(t)) \leq E^k(t) + M$$

for all  $n, k \geq 0$ . Now choose  $k$  so that  $\omega^k(3M + 2\pi) < \epsilon$ .

**Case 1:** Given  $t_0 > T$ , choose  $\delta$  such that, if  $|t - t_0| < \delta$ , then  $|E^k(t) - E^k(t_0)| < M$ . We claim that, if  $|t - t_0| < \delta$ , then  $|h_{\lambda,s}(t) - h_{\lambda,s}(t_0)| < \epsilon$ . To see this, we note that for each  $n \geq 0$  we have

$$|G_{\sigma^k(s)}^n(\lambda, E^k(t)) - G_{\sigma^k(s)}^n(\lambda, E^k(t_0))| < 3M + 2\pi.$$

This follows since

$$|\operatorname{Re} G_{\sigma^k(s)}^n(\lambda, E^k(t)) - \operatorname{Re} G_{\sigma^k(s)}^n(\lambda, E^k(t_0))| < |E^k(t) - E^k(t_0)| + 2M < 3M$$

and

$$|\operatorname{Im} G_{\sigma^k(s)}^n(\lambda, E^k(t)) - \operatorname{Im} G_{\sigma^k(s)}^n(\lambda, E^k(t_0))| < 2\pi.$$

Therefore

$$\begin{aligned} & |G_s^{n+k}(\lambda, t) - G_s^{n+k}(\lambda, t_0)| = \\ & = |L_{\lambda,s}^k \circ G_{\sigma^k(s)}^n(\lambda, E^k(t)) - L_{\lambda,s}^k \circ G_{\sigma^k(s)}^n(\lambda, E^k(t_0))| \\ & \leq \omega^k |G_{\sigma^k(s)}^n(\lambda, E^k(t)) - G_{\sigma^k(s)}^n(\lambda, E^k(t_0))| \leq \omega^k(3M + 2\pi) < \epsilon. \end{aligned}$$

It follows that the function  $t \rightarrow h_{\lambda,s}(t)$  is continuous for any  $s \in \Sigma_K$  and  $t > T$ .

Note that we did not use the assumption that  $s_i \neq 0$  in this part of the proof.

**Case 2:** To prove continuity for  $1 < t \leq T$ , we will assume from now on that  $|s_i| > 0$  for all  $i$ . Observe that, if  $1 < t < T$ , then there exists  $k$  such that  $E^k(t) > T$ . Then we have

$$t \rightarrow L_{\lambda,s}^k \circ h_{\lambda,\sigma^k(s)}(E^k(t))$$

is continuous provided that each inverse is well defined and continuous. This is true by the assumption that  $s_i \neq 0$  so that

$$L_{\lambda,s}^k \circ h_{\lambda,\sigma^k(s)} \circ E^k(t)$$

never meets 0 or  $\mathbb{R}^-$  for each  $i$ . But this map is given by

$$t \rightarrow \lim_{n \rightarrow \infty} L_{\lambda,s}^k \circ G_{\sigma^k(s)}^n \circ E^k(t) = h_{\lambda,s}(t).$$

Hence we have continuity for all  $t > 1$ . □

**Corollary 3.7** *Let  $s$  be any sequence in  $\Sigma'_K$ . For each  $\lambda \in \mathbb{C}'$ ,  $h_{\lambda,s}(t)$  exists and is continuous as a function of  $t \in (T, \infty)$  for  $T$  sufficiently large.*

**Proof.** Choose  $\tau$  as in Proposition 3.5 so that in addition,  $\tau > M$ . This assures that the family of functions  $G_s^n(\lambda, t)$  is well defined for any  $t > \tau$ . Then Proposition 3.4 goes through exactly as before and the value

$$h_{\lambda,s}(t) = \lim_{n \rightarrow \infty} G_s^n(\lambda, t)$$

exists for each  $t > \tau$ . For continuity, we proceed exactly as in Proposition 3.6, choosing  $T$  as indicated and then following the proof in case 1 to obtain continuity when  $t > T$ . □

The case of continuity at  $t = 1$  is more delicate.

**Proposition 3.8** *Let  $s = (s_0 s_1 s_2 \dots) \in \Sigma_K$ . Then for each  $\lambda \in \mathbb{C}'$ ,  $h_{\lambda,s}$  is continuous at  $t = 1$ .*

**Proof.** Recall from Section 2.2 that for any  $\beta$  sufficiently large,  $L_{\lambda,s_i}(z)$  maps the set  $V_\alpha^\beta$  strictly inside itself. As in the proof of Theorem 2.3, each  $L_{\lambda,s_i}$  is a strict contraction on  $V_\alpha^\beta$ . For a given  $t$  value sufficiently close to one, there exists a first integer  $N$  such that  $E^N(t)$  is larger than the  $\tau$  given to us in Proposition 3.5. For any  $n > N$ , we can invoke the inequality of Proposition 3.5 so that  $G_{\sigma^N(s)}^n(\lambda, E^N(t)) \in V_\alpha^\beta$  (provided  $\beta$  is sufficiently large). Then we can use the Poincaré metric to show that the distance between  $h_{\lambda,s}(1) = z_s(\lambda)$  and

$$h_{\lambda,s}(t) = \lim_{N \rightarrow \infty} L_{\lambda,s}^N \circ G_{\sigma^N(s)}^n(\lambda, E^N(t))$$



can be made arbitrarily small. □

Propositions 3.6 and 3.8 combine to show that the function  $h_{\lambda,s}(t)$  is continuous for  $t \in [1, \infty)$ . We now prove that this function actually parametrizes a hair.

**Theorem 3.9** *Suppose  $s \in \Sigma_K$  is a regular sequence. Then there is a unique hair attached to  $z_\lambda(s)$  and, moreover,  $t \rightarrow h_{\lambda,s}(t)$  is a parametrization of this hair.*

**Proof.** We first verify that  $h_{\lambda,s}$  is in fact a hair. We claim that  $h_{\lambda,s}(t)$  has itinerary  $s$ . Note that

$$\begin{aligned} E_\lambda \circ h_{\lambda,s}(t) &= \lim_{n \rightarrow \infty} E_\lambda \circ G_s^n(\lambda, t) \\ &= \lim_{n \rightarrow \infty} G_{\sigma(s)}^{n-1}(\lambda, E(t)) \\ &= h_{\lambda,\sigma(s)}(E(t)). \end{aligned}$$

It follows that

$$E_\lambda^n \circ h_{\lambda,s}(t) = h_{\lambda,\sigma^n(s)} \circ E^n(t).$$

Hence  $E_\lambda^n(h_{\lambda,s}(t)) \in R_\lambda(s_n)$  as required. Also,

$$E^n(t) - M \leq \operatorname{Re} E_\lambda^n \circ h_{\lambda,s}(t) \leq E^n(t) + M$$

for  $n$  sufficiently large, where  $M$  is as given in Proposition 3.5. Therefore,  $\operatorname{Re} E_\lambda^n h_{\lambda,s}(t) \rightarrow \infty$  as  $n \rightarrow \infty$  as long as  $t > 1$ . Finally, since  $t - M \leq \operatorname{Re} h_{\lambda,s}(t) \leq t + M$  for  $t > \tau$ , it follows that  $\operatorname{Re} h_{\lambda,s}(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . This proves that  $h_{\lambda,s}$  parametrizes a hair.

We finally show that this hair is unique. Suppose this is not true. Then there are two hairs attached to  $z_\lambda(s)$ . Let  $U$  be the open set contained in  $R_\lambda(s_0)$  bounded by these two hairs. The images of  $U$  under  $E_\lambda^n$  are contained within the images of the hairs attached to  $E_\lambda^n(z_\lambda(s))$ , which themselves are contained strictly between the lines  $\operatorname{Im} z = (2s_i \pm 1)\pi - \arg \lambda$  for each  $s_i$ . Moreover, if  $z \in U \cap \{\operatorname{Re} z > \nu\}$  for large  $\nu$ , then  $\operatorname{Re} E_\lambda(z) \gg \operatorname{Re} z$ , and so the orbit of  $z$  tends to  $\infty$ . This yields a wandering domain or domain at  $\infty$  for  $E_\lambda$ , neither of which can occur (see [2], [13], and [17]).

### Remarks.

1. The hypothesis that  $s$  is regular cannot be removed, as it is known that there may be more than one hair attached to a point with bounded orbit if not all of the  $s_i$  are non-zero. A classification of which hairs attach to which points in the case where  $\lambda$  is a Misiurewicz point may be found in [10]. A similar program has been carried out for  $\lambda$  values for which  $E_\lambda$  has an attracting cycle in [6].
2. In certain cases (e.g.,  $0 < \lambda \leq 1/e$ ,  $\lambda = 1$ ) the existence of hairs with regular, unbounded itineraries has been demonstrated using other methods (see [1], [11]). The proof above does not extend to itineraries that grow too quickly. However, the parametrizations given above are necessary for our proof of existence of hairs in the parameter plane, to which we turn in the next section.

## 4 Hairs in the Parameter Plane

Our goal in this section is to show that there also exist hairs in the parameter plane for the exponential family. These hairs consist of  $\lambda$ -values for which the orbit of 0 under  $E_\lambda$  has a specified itinerary in  $\Sigma_K$ . The hairs have an endpoint which determines a  $\lambda$ -value for which the orbit of 0 is bounded, since  $E_\lambda(0) = \lambda$ . All other  $\lambda$ -values on the hair have the property that the orbit of 0 tends to  $\infty$  with the specified itinerary. As a consequence,  $J(E_\lambda) = \mathbb{C}$  for these  $\lambda$ -values.

**Definition 4.1** *Let  $s = s_0s_1s_2\dots$ . A continuous curve  $H_s: [1, \infty) \rightarrow \mathbb{C}$  is called a hair with itinerary  $s$  if  $H_s$  satisfies:*

1. *If  $\lambda = H_s(t)$  and  $t > 1$ , then  $\operatorname{Re} E_\lambda^n(0) \rightarrow \infty$  and the itinerary of  $\lambda$  under  $E_\lambda$  is  $s$ .*
2. *If  $\lambda = H_s(1)$ , then  $E_\lambda(0) = \lambda = z_\lambda(s)$ , and so the orbit of  $\lambda$  under  $E_\lambda$  is bounded and has itinerary  $s$ .*
3.  $\lim_{t \rightarrow \infty} \operatorname{Re} H_s(t) = \infty$ .

**Remark.** We use the term “hair” for curves in both the dynamical plane and parameter plane. When necessary, we use the terms dynamical hair and parameter hair to distinguish between them.

Our goal in this section is to prove the following result.

**Theorem 4.2** *Suppose  $s$  is a bounded, regular sequence. Then there exists a hair in parameter space with itinerary  $s$ . Moreover, if  $s$  is periodic or preperiodic, then  $0$  is preperiodic under  $E_\lambda$  for  $\lambda = H_s(1)$ .*

The proof of the theorem depends upon several lemmas in which we will jump back and forth from dynamical to parameter plane, so we first sketch the main idea.

Given the regular itinerary  $s$ , we will construct a simply connected region  $Q_s$  in parameter space. Consider the map  $F_t(\lambda) = h_{\lambda,s}(t)$ . Note that  $F_t$  is a function of  $\lambda$  and assigns to  $\lambda$  the point on the dynamical hair with itinerary  $s$  and time parameter  $t$ . According to Proposition 3.4,  $F_t$  is an analytic function of  $\lambda$ . We will show that  $F_t$  maps the closure of  $Q_s$  strictly into its interior so therefore  $F_t$  has a unique fixed point in  $Q_s$ . This fixed point is a  $\lambda$ -value that satisfies  $\lambda = h_{\lambda,s}(t)$ , so  $\lambda = E_\lambda(0)$  lies on the hair in dynamical plane that is attached to  $z_\lambda(s)$ . We therefore define the point  $H_s(t)$  on the hair in the parameter plane as the unique fixed point of  $F_t$  for each  $t \geq 1$ . If  $t > 1$  it follows that  $E_\lambda^n(0) \rightarrow \infty$ , whereas, if  $t = 1$ ,  $0$  maps after one iteration of  $E_\lambda$  onto  $z_\lambda(s)$  and so this orbit is bounded. As we vary  $t$ , the fixed point of  $F_t$  varies, and this curve of fixed points produces the hair in parameter plane.

We define  $Q_s$  to be the union of the horizontal strips  $R_\lambda(s_0)$  for  $\lambda \in \mathbb{C}'$ .  $Q_s$  is an open horizontal strip in  $\mathbb{C}$  with height  $4\pi$  bounded by horizontal lines  $\text{Im } z = (2s_0 - 2)\pi$  and  $\text{Im } z = (2s_0 + 2)\pi$ . For each  $t$ , the dynamical hair  $h_{\lambda,s}(t)$  lies in  $R_\lambda(s_0) \subset Q_s$ .

We now think of  $Q_s$  as a subset of the parameter plane. Given  $\lambda \in Q_s$ ,  $F_t(\lambda) = h_{\lambda,s}(t)$  is a point in  $Q_s$ , and so  $F_t$  maps  $Q_s$  into itself. Now this map either has a unique fixed point in the interior of  $Q_s$  or else the  $F_t$ -orbits of all points tend to a fixed point on the boundary or to  $\infty$ . We will show that the latter cases are impossible.

**Lemma 4.3** *The map  $F_t$  has no fixed points on the horizontal boundaries of  $Q_s$ .*

**Proof.** Suppose  $\lambda_*$  is a fixed point for  $F_t$  on a horizontal boundary of  $Q_s$ . Then  $\text{Im } \lambda_* = (2s_0 \pm 2)\pi$ . However,  $R_{\lambda_*}(s_0)$  is given by  $\{z \mid (2s_0 - 1)\pi - \arg \lambda_* < \text{Im } z < (2s_0 + 1)\pi - \arg \lambda_*\}$ . If  $s_0 \neq \pm 1$ , we have  $|\arg \lambda_*| < \pi$ . It follows that  $\lambda_*$  lies outside  $R_{\lambda_*}(s_0)$ , and so  $\lambda_*$  cannot lie on a hair lying in this strip. If  $s_0 = \pm 1$ , we have  $\arg \lambda_* = \pm\pi$  along one of the horizontal boundaries of  $Q_s$ . But this boundary is the negative real axis, and it is easy to check that all such points lie in the basin of attraction of a real attracting fixed point or two-cycle. Consequently, none of these points lie on a hair.  $\square$

For each  $\nu < 0, \eta > 0$ , define

$$Z_{\nu, \eta} = \{\lambda \in \mathbb{C} \mid \text{Re } \lambda \leq \nu, |\text{Im } \lambda| \leq \eta\}.$$

$Z_{\nu, \eta}$  is a closed half strip in the parameter plane.

**Lemma 4.4** *For each  $\eta \in \mathbb{R}^+$ , there is  $\nu = \nu(\eta) < 0$  such that if  $\lambda \in Z_{\nu, \eta}$ , then*

1.  $E_\lambda$  has an attracting two-cycle, and
2.  $W_\lambda = \{z \mid \text{Re } z \leq \text{Re } (\lambda/2)\}$  is contained in the basin of attraction of the two-cycle.

**Proof.** Fix  $\eta > 0$  and suppose that  $|\text{Im } \lambda| \leq \eta$ . Define

$$c = c(\lambda) = |\lambda| \exp(\text{Re } (\lambda/2)).$$

Note that  $|\lambda|c(\lambda) \rightarrow 0$  and  $|\text{Re } \lambda|c(\lambda) \rightarrow 0$  as  $\text{Re } \lambda \rightarrow -\infty$  since  $|\text{Im } \lambda|$  is bounded.

We may choose  $\nu = \nu(\eta)$  such that, if  $\lambda \in Z_{\nu, \eta}$ , then

$$|\lambda|ce^c \leq (|\text{Re } \lambda| + \eta)ce^c \leq |\text{Re } \lambda|/4$$

since  $(|\text{Re } \lambda| + \eta)ce^c$  tends to 0 as  $\text{Re } \lambda \rightarrow -\infty$ .

We claim that  $E_\lambda^2$  maps  $W_\lambda$  inside itself. To see this, first note that  $E_\lambda(W_\lambda)$  is a punctured disk of radius  $c$  centered at 0. On this disk

$$|E'_\lambda(z)| = |E_\lambda(z)| = |\lambda| \exp(\text{Re } z)$$

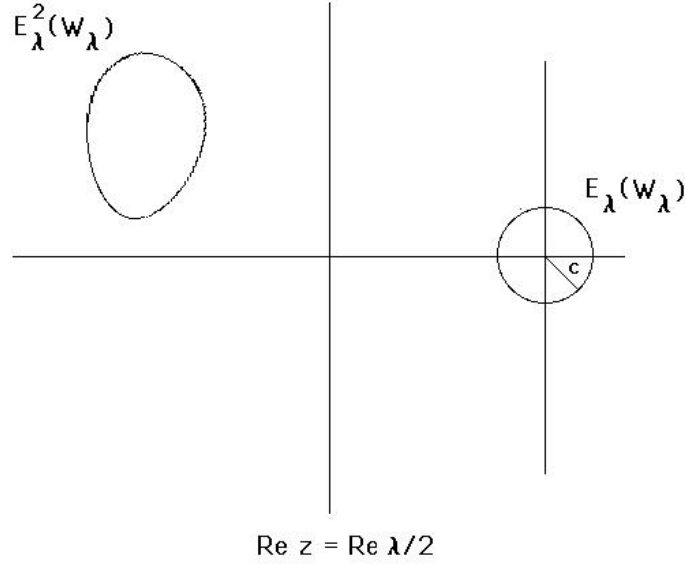


Figure 9:  $E_\lambda^2(W_\lambda) \subset W_\lambda$ .

which is at most  $|\lambda|e^c$ . Clearly,  $E_\lambda(0) = \lambda$ , so points  $z \in E_\lambda(W_\lambda)$  are mapped a maximum distance of

$$\begin{aligned} |E_\lambda(z) - \lambda| &\leq \max_{z \in E_\lambda(W_\lambda)} |E'_\lambda(z)| \cdot |z - 0| \\ &\leq |\lambda|e^c \cdot c \\ &\leq |\operatorname{Re} \lambda|/4 \end{aligned}$$

away from  $\lambda$  under  $E_\lambda$ , and thus well to the left of the line  $\operatorname{Re} z = \operatorname{Re} \lambda/2$  (see Figure 9).

As a consequence,  $E_\lambda^2(W_\lambda)$  is contained in the interior of  $W_\lambda$  provided  $\lambda \in Z_{\nu, \lambda}$ . By the Schwarz lemma,  $E_\lambda^2$  has an attracting fixed point in  $W_\lambda$  and, moreover, each point in  $W_\lambda$  tends to this point under iteration of  $E_\lambda^2$ . This fixed point gives an attracting 2-cycle for  $E_\lambda$ . □

**Corollary 4.5** *Let  $s \in \Sigma_K$ . Let  $\eta = \sup \{|\operatorname{Im} z| \mid z \in Q_s\}$ . Then, if  $\lambda \in Z_{\nu(\eta), \eta}$ , we have*

$$\operatorname{Re} F_t(\lambda) = \operatorname{Re} h_{\lambda, s}(t) > \operatorname{Re} \lambda$$

for all  $t$ .

**Proof.** By the previous lemma, any point in the Julia set lies in the half plane  $\operatorname{Re} z > \operatorname{Re} \lambda/2$ . Since  $\lambda \in Z_{\nu(\eta), \eta}$ , we have  $\operatorname{Re} \lambda < \operatorname{Re} \lambda/2 < 0$ . It follows that  $\operatorname{Re} z > \operatorname{Re} \lambda$  for all points in the Julia set of  $E_\lambda$ . Since  $h_{\lambda, s}(t)$  lies on a hair in the Julia set, it follows that this point lies to the right of the line  $\operatorname{Re} z = \operatorname{Re} \lambda$  as well. □

Recall from Section 2.2 that

$$V_\alpha^x(s_i) = \{z \in \overline{R(s_i)} \mid \alpha \leq \operatorname{Re} z \leq x\}$$

and that  $V_\alpha^x = \cup_{0 < |j| \leq K} V_\alpha^x(j)$ . Recall also from Section 2.2 that  $m_K$  is the smaller of the vertical distances from 0 to  $R(\pm 1)$  and that  $M_K$  is the largest. Note that  $(2K + 2)\pi > M_K$ . Let  $\alpha = \alpha(\lambda) = \log(m_K/(2|\lambda|))$ . Finally, we write  $E(z) = (1/e)e^z$  as before.

For a given  $t \geq 1$ , we may choose  $\lambda_1 = \lambda_1(t)$  so that the following three properties hold:

1.  $\lambda_1(t) > t \geq 1$ ,
2. for each  $x > t$ , we have  $\lambda_1 e^x > E(x) + (2K + 2)\pi$ , and
3. if  $\operatorname{Re} \lambda > \lambda_1$ , then  $|\lambda| > \log 2 + \log |\lambda| + (2K + 2)\pi$ .

Then, if  $\lambda \in Q_s$ ,  $\operatorname{Re} \lambda > \lambda_1$ , and  $\operatorname{Re} z \geq t$ , it follows that

$$\begin{aligned} |E_\lambda(z)| &= |\lambda| \exp(\operatorname{Re} z) \geq (\operatorname{Re} \lambda) \exp(\operatorname{Re} z) > \lambda_1 \exp(\operatorname{Re} z) \\ &> E(\operatorname{Re} z) + (2K + 2)\pi > E(\operatorname{Re} z) + M_K. \end{aligned}$$

**Lemma 4.6** *Given  $t \geq 1$ , let  $\lambda \in Q_s$  satisfy  $\operatorname{Re} \lambda > \lambda_1(t)$ . Then for any  $n$  we have*

$$\alpha \leq \operatorname{Re} G_s^n(\lambda, t) < t.$$

**Proof.** Consider  $V_\alpha^x$  for any  $x \geq t$ . We claim that

$$E_\lambda(V_\alpha^x) \supset V_\alpha^{E(x)}.$$

To see this, we first note that the horizontal boundaries of  $V_\alpha^x$  are mapped to  $\mathbb{R}^-$ , hence into the exterior of  $V_\alpha^{E(x)}$ . The left hand boundary of  $V_\alpha^x$  is mapped to the circle of radius  $m_K/2$  centered at 0, which also lies in the exterior of  $V_\alpha^{E(x)}$ . The right hand boundary of  $V_\alpha^x$  is mapped to the circle of radius  $|\lambda|e^x$  centered at 0. Since

$$|\lambda|e^x > (\operatorname{Re} \lambda)e^x \geq \lambda_1 e^x \geq E(x) + M_K,$$

this circle is outside the furthestmost right corner of  $V_\alpha^{E(x)}$ . We need to check that the furthestmost left corners of  $V_\alpha^{E(x)}$  are also within the circle of radius  $|\lambda|e^x$ . If  $\alpha > 0$ , then clearly  $|\lambda|e^x > E(x) + M_K > \alpha + M_K$  since  $x \geq 1 > \alpha$ . If  $\alpha < 0$ , then since  $\operatorname{Re} \lambda > \lambda_1 > 1$ , we have  $|\arg \lambda| < \pi/2$ , so  $m_K > \pi/2$  and  $\log m_K > 0$ . Hence by condition 3 above,

$$\begin{aligned} |\lambda|e^x > |\lambda| &> \log 2 + \log |\lambda| + M_K \\ &> \log 2 + \log |\lambda| - \log(m_K) + M_K \\ &= -\alpha + M_K. \end{aligned}$$

This implies that the circle of radius  $|\lambda|e^x$  contains the furthestmost left corner of  $V_\alpha^{E(x)}$ . Combining these results, we see that  $V_\alpha^x$  is mapped to an annular region that contains  $V_\alpha^{E(x)}$  in its interior.

Recall that  $L_{\lambda, s_i}$  is the inverse of  $E_\lambda$  that takes values in  $R(s_i)$ . Using the above fact, we have  $L_{\lambda, s_i} V_\alpha^{E(x)} \subset V_\alpha^x$  for any  $x \geq t$  and any  $i$ , provided  $\operatorname{Re} \lambda > \lambda_1$ . Also

$$L_{\lambda, s}^n(V_\alpha^{E^n(x)}) \subset V_\alpha^x.$$

Now recall that

$$G_s^m(\lambda, t) = L_{\lambda, s}^n(E^n(t)).$$

Therefore we have

$$\alpha \leq \operatorname{Re} G_s^n(\lambda, t) < t$$

for each  $n$ . □

Now we complete the proof of Theorem 4.2. It follows from Lemma 4.6 that  $\operatorname{Re} F_t(\lambda) \leq t$  for all  $\lambda \in Q_s$  with  $\operatorname{Re} \lambda > \lambda_1 > t$ . Thus we see that  $F_t$

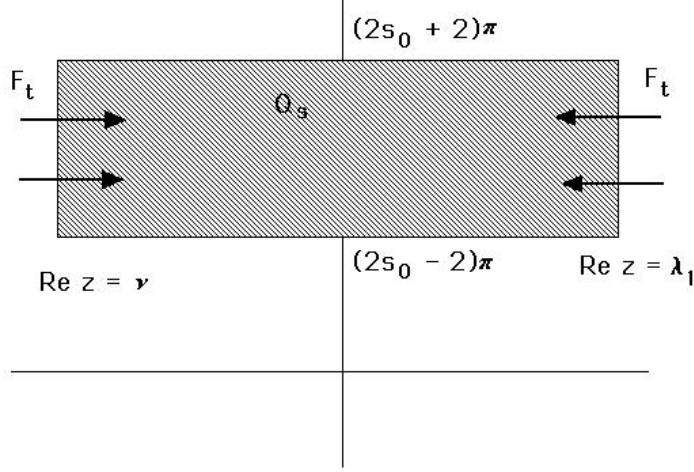


Figure 10: The rectangle  $Q_s$ .

moves  $\lambda$ -values in the far right portion of  $Q_s$  to the left (see Figure 10). From Corollary 4.5,  $F_t$  maps  $\lambda$ -values in the far left region of  $Q_s$  to the right. And from Lemma 4.3,  $F_t$  has no fixed points on the horizontal boundaries of  $Q_s$ . Therefore it follows that  $F_t$  has a unique attracting fixed point in  $Q_s$ . This fixed point is the  $\lambda$ -value for which  $\lambda = E_\lambda(0) = h_{\lambda,s}(t)$ , and so  $\lambda$  lies on the appropriate hair. We therefore define  $H_s(t)$  to be this  $\lambda$ -value. Clearly, the itinerary of this  $\lambda$ -value is  $s$ . Now  $F_t$  depends continuously on  $t$ . Since the fixed point for  $F_t$  is attracting, it follows that these fixed points move continuously with  $t$ . Hence  $H_s$  is continuous in  $t$ .

When  $t > 1$ ,  $\operatorname{Re} E_\lambda^n(h_{\lambda,s}(t)) \rightarrow \infty$  as  $n \rightarrow \infty$ . When  $t = 1$ ,  $h_{\lambda,s}(1) = z_\lambda(s)$  lies at the endpoint of a hair in the dynamical plane. This verifies conditions 1 and 2 in Definition 4.1.

Thus all that remains to prove is condition 3:  $H_s(t)$  extends to  $\infty$  in the right half plane as  $t \rightarrow \infty$ . To show this, suppose  $\lim_{t \rightarrow \infty} \operatorname{Re} H_s(t) \neq \infty$ . This means that there exist sequences  $t_i \rightarrow \infty$  and  $\lambda_i \rightarrow \lambda^*$  in  $Q_s$  with  $|\operatorname{Re} \lambda^*| < \infty$ , where  $\lambda_i = H_s(t_i)$  for each  $i$ . From Proposition 3.5, given  $\lambda$ , there exists  $\tau, M$  such that, for  $t \geq \tau$ , we have

$$(\dagger) \quad t - M \leq \operatorname{Re} G_s^n(\lambda, t) \leq t + M.$$

where  $M = M(\lambda)$  was chosen to satisfy  $(*)$  and  $(**)$  as given in the proof of the Proposition and  $\tau(\lambda)$  was chosen so that  $E(\tau) > M + 2$ . In our case,



since the  $\lambda_i$  are bounded, so are  $\operatorname{Re}(\log \lambda_i)$  and  $\operatorname{Re}(-\log \lambda_i)$ , which implies that  $M(\lambda_i)$  and  $\tau(\lambda_i)$  are bounded too. Letting  $t_i \rightarrow \infty$ , we see from (†) that  $\operatorname{Re} G_s^n(\lambda_i, t_i) \rightarrow \infty$  for any  $n$ . This implies that  $h_{\lambda_i, s}(t_i) = \lambda_i \rightarrow \infty$ , contradicting our assumption that the  $\lambda_i$  are bounded. This completes the proof of Theorem 4.2. □

**Remark.** As before, it is possible to relax the assumption that all the  $s_i$  be non-zero. In this case we get the existence of hairs in the parameter plane, but they are only defined in the far right half plane, that is, for  $t$  sufficiently large. In particular, these hairs do not terminate at  $\lambda$ -values for which 0 is preperiodic.

## 5 Conclusion

In this paper we have shown the existence of hairs in both the dynamical and parameter planes for the exponential family  $E_\lambda(z) = \lambda e^z$ . We emphasize that we have not obtained the complete picture for these hairs. First of all, we have proved the existence of hairs only for bounded, regular itineraries. For these itineraries we showed that each hair was attached to a unique point in either the dynamical or parameter plane.

When the itinerary is not regular, we have only shown that there is a tail for each hair. The existence of the full hairs for these types of itineraries remains a problem. In the dynamical plane, this question has been partially addressed in the case where  $\lambda$  is a Misiurewicz point ([10]) or  $E_\lambda$  admits an attracting cycle ([6]). However, the situation in the parameter plane is completely open.

It is also known that there are hairs in the dynamical plane corresponding to unbounded itineraries for certain  $\lambda$ -values ([1], [11]). We do not have a general proof of the existence of hairs in either plane in this case.

As is well known, the family of polynomials

$$P_{d,\lambda}(z) = \lambda \left(1 + \frac{z}{d}\right)^d$$

converges uniformly on compact subsets to  $E_\lambda$ . Moreover, this family has unique critical point at  $-d$  and critical value at 0. Using techniques developed by Douady and Hubbard, one can uniformize the exterior of (connected) filled Julia sets for these polynomials, just as in the case of the quadratic maps

discussed in the Introduction. Hence one gets a theory of external rays for these maps. In a subsequent paper [7], we plan to use the results in this paper to discuss the convergence of these external rays as  $d \rightarrow \infty$  to the hairs for the exponential family. These results depend heavily on the parametrizations of the hairs discussed herein.

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