Baby Mandelbrot Sets Adorned with Halos in Families of Rational Maps

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ABSTRACT. In this paper we prove the existence of a number of different small copies of Mandelbrot sets for the families of rational maps given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}.$$

We also show that certain of the "antennas" of these sets are attached to the boundaries of Sierpinski holes. These holes are open subsets of the parameter plane for which the corresponding members of the family have Julia sets that are homeomorphic to the Sierpinski carpet fractal.

In this paper we consider families of rational maps $F_\lambda : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

where $\lambda \in \mathbb{C} - \{0\}$ and $n \geq 3$. These families contain a rich variety of both topological structures and dynamical behavior. For example, it is known [**6**] that there are infinitely many disjoint, simply connected, open sets in the parameter plane (the λ-plane) for this family having the property that if λ lies in one of these sets, then the Julia set for F_λ is a *Sierpinski curve*. A Sierpinski curve is an interesting topological space that is homeomorphic to the well known Sierpinski carpet fractal. Hence the Julia sets drawn from each of these open subsets of the parameter plane are homeomorphic. However, it is also known [**6**] that the dynamics of the maps drawn from two of these open sets that are disjoint are quite different in the sense that the maps are not topologically conjugate on their respective Julia sets. Regions in the parameter plane for which the Julia set of F_λ is a Sierpinski curve are called Sierpinski holes. It is known ([**14**]) that each Sierpinski hole is a simply connected open subset of the parameter plane.

There are many other types of Julia sets that arise in these families. For example, in this paper we shall show that there exist small copies of Mandelbrot sets in the parameter plane for each $n \geq 3$. We call these sets *baby Mandelbrot sets.* This implies that the Julia sets for parameters drawn from one of these Mandelbrot sets contain forward invariant subsets that are homeomorphic to the Julia sets of quadratic polynomials.

1991 *Mathematics Subject Classification.* Primary 37F45; Secondary 37F10.

Key words and phrases. Mandelbrot set, rational map, singular perturbation, halo.

Partially supported by NSF grant 0205779.

We shall also show that these Mandelbrot sets in the parameter plane come with an additional structure: each of these sets has infinitely many "halos" attached. These halos are (usually) Sierpinski holes whose boundaries touch the Mandelbrot sets at a unique point at the tip of one of the antennas of the set. More precisely, to use the terminology of external rays in parameter space, the halos are attached to the baby Mandelbrot set at parameter values where, in the case of a quadratic polynomial, the external rays with angle $p/2^q$ land, where $p, q \in \mathbb{Z}$.

In Figures 1 and 2, we display the parameter plane for the families where $n = 3$ and $n = 4$, including a magnification of one of the baby Mandelbrot sets in each case. Note how white disks seem to hang off these Mandelbrot sets at certain extreme points on the antennas of the Mandelbrot set. These are the halos. We shall prove that there are infinitely many such halos for each of the Mandelbrot sets described below.

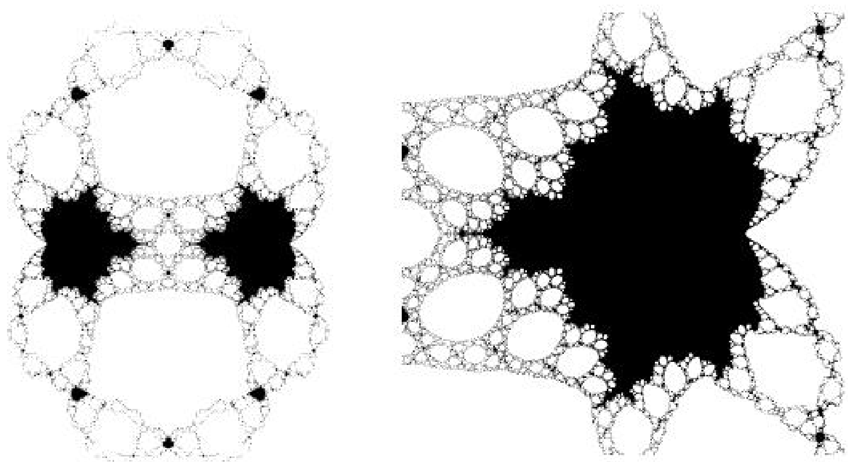


FIGURE 1. The parameter plane for the family $z^3 + \lambda/z^3$ and a magnification showing one of the baby Mandelbrot sets and its halos.

We remark that the central hole in each of these figures is not a Sierpinski hole. Rather, if λ lies in this region, the Julia set of F_λ is a Cantor set of circles. McMullen ([**9**]) was the first to observe this phenomenon, so we call this region the McMullen domain. See [**6**] for a dynamical description of this region.

1. Preliminaries

We consider the family of rational maps of degree $2n$ given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

for a fixed $n \geq 3$ and $\lambda \in \mathbb{C}$. The cases $n = 1$ and $n = 2$ are excluded for several reasons which are discussed below. Each of these maps has a superattracting fixed point at ∞ and hence there is an immediate basin of attraction at ∞ which we denote by B_λ. Note that ∞ is not superattracting when $n = 1$; this is one of the reasons we exclude this case. Since 0 is a pole of order n, there is a neighborhood

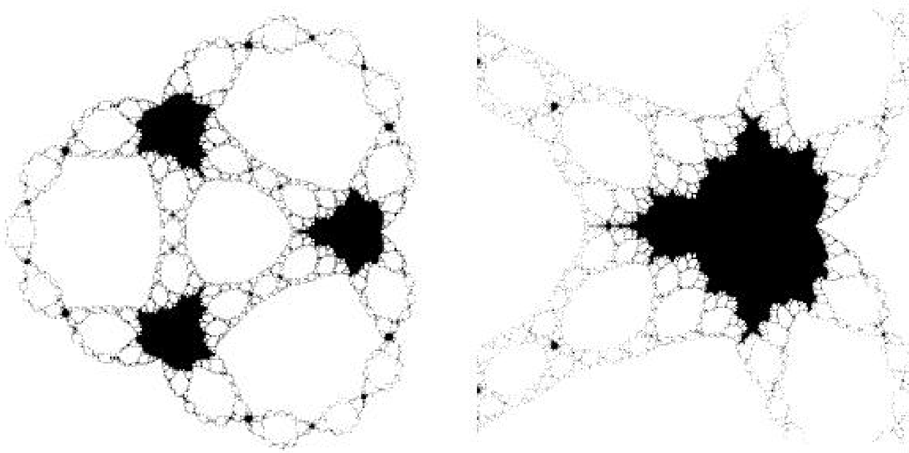


FIGURE 2. The parameter plane for the family $z^4 + \lambda/z^4$ and a similar magnification.

of 0 that is mapped into B_λ. Either this neighborhood lies in B_λ or else it forms a disjoint preimage of B_λ. In the latter case, we denote the preimage of B_λ containing 0 by T_λ. In this case T_λ is the only other preimage of B_λ since F_λ is n to 1 on both B_λ and T_λ. So any point not in B_λ whose orbit eventually enters B_λ must pass through T_λ. For this reason, we call T_λ the *trap door.*

One checks easily that, besides 0 and ∞, this map has $2n$ additional critical points given by $c_\lambda = \lambda^{1/2n}$. However, there are only two critical values given by $v_\lambda = \pm 2\sqrt{\lambda}$. There are also $2n$ prepoles for this map at the points $p_\lambda = (-\lambda)^{1/2n}$.

Note the symmetric arrangement of these critical points and prepoles. This is no accident, for if ω is a primitive $2n$-th root of unity, then we have

$$F_\lambda(\omega z) = \omega^n F_\lambda(z) = -F_\lambda(z).$$

Therefore, if n is even, we have $F_\lambda^j(\omega z) = F_\lambda^j(z)$ for all $j \geq 2$, so the orbit of all symmetrically arranged points $\omega^k z$ behave in the same manner. If n is odd, then we have $F_\lambda^j(\omega z) = -F_\lambda^j(z)$ for all $j \geq 1$, so there are two distinct orbits for the points $\omega^k z$, but each orbit is symmetric under $z \mapsto -z$, so again, these orbits all behave in the same manner. In particular, the orbits of all of the critical points behave symmetrically. Hence we essentially have only one critical orbit, which we call the *free critical orbit.* Note also that if $z \in B_\lambda$, then $\omega^k z \in B_\lambda$ for all k. The same is true for T_λ. We say that these sets have $2n$-fold symmetry.

There is another symmetry for these families. Let H_λ be one of the n maps given by

$$H_\lambda(z) = \frac{\lambda^{1/n}}{z}.$$

Then H_λ is an involution and one checks easily that $F_\lambda(H_\lambda(z)) = F_\lambda(z)$. The involution maps B_λ to T_λ and vice versa.

Consider the circle C_λ given by $r = |\lambda|^{1/2n}$. This circle contains all of the free critical points as well as the prepoles of F_λ. A straightforward computation shows that this circle is mapped onto the straight line segment that connects the two critical values $\pm 2\sqrt{\lambda}$ and passes through the origin. We call C_λ the *critical circle* and its image the *critical segment*. The map F_λ takes the critical circle onto the critical segment in $2n$ to 1 fashion except at the endpoints $\pm 2\sqrt{\lambda}$, for which there are only n preimages, each of which is a critical point.

Consider the straight line passing through the origin and any one of the prepoles. This line is given by $t(-\lambda)^{1/2n}$ with $t \in \mathbb{R}$. We have

$$F_\lambda(t(-\lambda)^{1/2n}) = i\sqrt{\lambda}\left(t^n - \frac{1}{t^n}\right),$$

so this line mapped in 2 to 1 fashion onto the entire straight line passing through the origin and perpendicular to the critical segment.

2. The Connectedness Locus

The set of points whose orbits under F_λ are bounded is called the filled Julia set of F_λ and denoted by $K(F_\lambda)$. The boundary of $K(F_\lambda)$ is the Julia set of F_λ and is denoted by $J(F_\lambda)$. It is well known that both $J(F_\lambda)$ and $K(F_\lambda)$ are compact subsets of the plane that do not contain B_λ or any of the preimages of B_λ. Indeed, $K(F_\lambda)$ is the complement of the union of all of the preimages of B_λ.

The following theorem, proved in [**6**], is known as the escape trichotomy; it shows that the Julia set of F_λ assumes three different forms depending on how the critical orbit escapes to ∞.

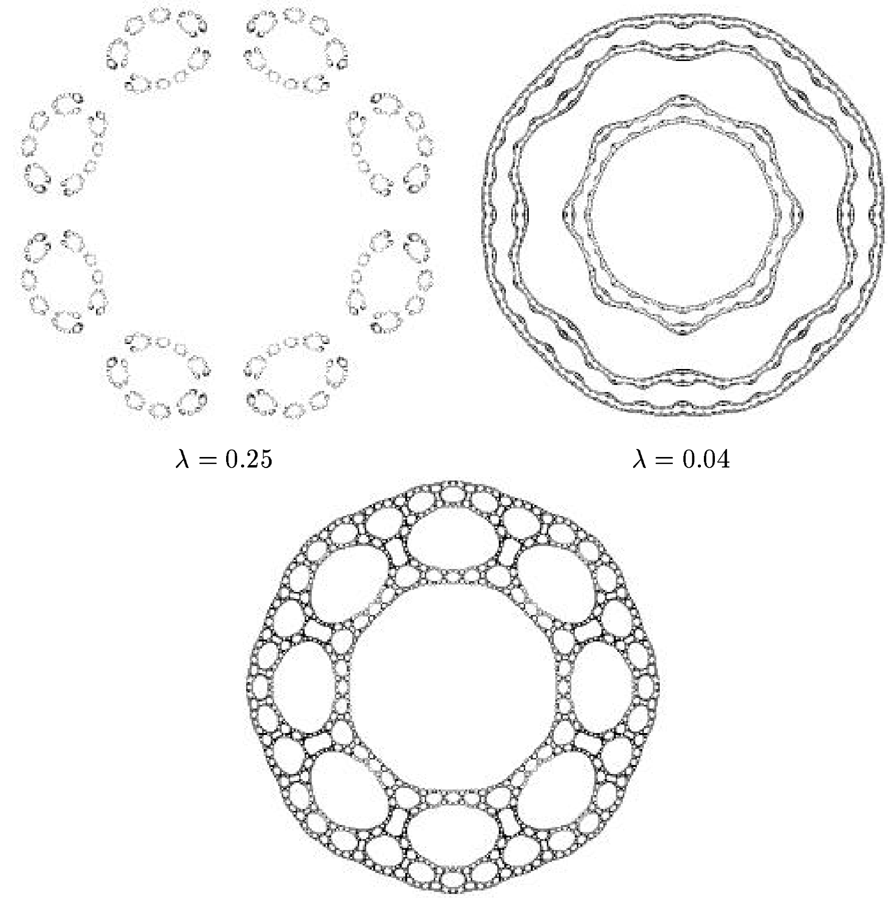
Theorem (The Escape Trichotomy). *Suppose that the free critical orbit tends to* ∞. *Then*

1. *If the critical values lie in* B_λ, *then* $J(F_\lambda)$ *is a Cantor set;*
2. *If the critical values lie in* T_λ, *then* $J(F_\lambda)$ *is a Cantor set of simple closed curves;*
3. *If the critical values do not lie in* B_λ *or* T_λ, *then* $J(F_\lambda)$ *is a connected set. In particular, if the critical values lie in some other preimage of* B_λ, *then* $J(F_\lambda)$ *is a Sierpinski curve.*

A *Sierpinski curve* is a planar set that is characterized by the following five properties: it is a compact, connected, locally connected and nowhere dense set whose complementary domains are bounded by simple closed curves that are pairwise disjoint. It is known [**16**] that any two Sierpinski curves are homeomorphic. In fact, they are homeomorphic to the well-known Sierpinski carpet fractal. From the point of view of topology, a Sierpinski curve is a universal set in the sense that it contains a homeomorphic copy of any planar, compact, connected, one-dimensional set. The first example of a Sierpinski curve Julia set was given by Milnor and Tan Lei [**12**]. See also Ushikii [**15**].

From a dynamical systems point of view, however, these types of Julia sets can be quite different from one another. In [**6**] it is also shown that, for each $n \geq 2$, there are infinitely many disjoint open sets $\mathcal{O}_j$ in the λ-plane such that, if $\lambda_j \in \mathcal{O}_j$, then $J(F_{\lambda_j})$ is a Sierpinski curve. However, F_{λ_j} is not topologically conjugate to F_{λ_k} if $j \neq k$. So the Julia sets are the same but the dynamics on them are different. In [**1**] it is shown that, in the special case where $n = 2$, there are infinitely many such $\mathcal{O}_j$ in every neighborhood of 0 in the parameter plane.

We remark that case 2 of the Escape Trichotomy where the Julia set is a Cantor set of simple closed curves does not occur if $n = 1, 2$. Hence there is no McMullen



$\lambda = 0.25$ $\lambda = 0.04$

$\lambda = -0.07$

FIGURE 3. Some Julia sets for $z^4 + \lambda/z^4$: if $\lambda = 0.25$, $J(F_\lambda)$ is a Cantor set; if $\lambda = 0.04$, $J(F_\lambda)$ is a Cantor set of simple closed curves; and if $\lambda = -0.07$, $J(F_\lambda)$ is a Sierpinski curve.

domain when $n = 1$ or $n = 2$. In Figure 3 we display examples of each of the three types of Julia sets that arise in the trichotomy.

Let $\mathcal{C}$ denote the set of λ-values for which $J(F_\lambda)$ is a connected set. $\mathcal{C}$ is called the *connectedness locus*. So $\mathcal{C}$ contains all λ-values except those for which v_λ lies in either B_λ or T_λ. There are countably many open sets in $\mathcal{C}$ for which the critical orbit lies in a preimage of B_λ that is not T_λ. These are the Sierpinski holes in $\mathcal{C}$. The region for which v_λ lies in T_λ is the McMullen domain. It is known that that there is a unique McMullen domain for each $n \geq 3$ [**3**].

The next few results show that $\mathcal{C}$ lies in an annular region contained inside the unit disk in the plane.

Proposition. *Suppose that* $|\lambda| \geq 1$. *Then* $v_\lambda \in B_\lambda$ *so that* λ *does not belong to the connectedness locus.*

Proof: Recall that the critical values of F_λ are of the form $\pm 2\sqrt{\lambda}$. Suppose that $|z| \geq |2\sqrt{\lambda}|$. We claim that z lies in B_λ. We have

$$\begin{aligned} |F_\lambda(z)| &\geq |z|^n - \frac{|\lambda|}{|z|^n} \\ &\geq |z|^n - \frac{1}{4|z|^{n-2}} \\ &\geq |z|^n - \frac{1}{2^n} \\ &\geq |z|^{n-1}. \end{aligned}$$

Inductively we have

$$|F_\lambda^k(z)| \geq |z|^{(n-1)^k}$$

and, since $n \geq 3$, it follows that $z \in B_\lambda$. In particular, $v_\lambda \in B_\lambda$.

□

The following result gives a criterion to determine if a point lies in either B_λ or T_λ when $|\lambda| \leq 1$.

Proposition (The Escape Criterion). *Suppose that $|\lambda| \leq 1$. If $|z| \geq 2$, then $|F_\lambda(z)| \geq |z|^{n-1}$ so that $z \in B_\lambda$. If $|z| \leq |\lambda|^{1/n}/2$, then $|F_\lambda(z)| \geq 2$ so that $F_\lambda(z) \in B_\lambda$.*

Proof: First suppose that $|z| \geq 2$. We have

$$\begin{aligned} |F_\lambda(z)| &\geq |z|^n - \frac{|\lambda|}{|z|^n} \\ &\geq |z|^n - \frac{1}{2^n} \\ &\geq |z|^{n-1}. \end{aligned}$$

So by induction we have

$$|F_\lambda^k(z)| \geq |z|^{(n-1)^k}$$

and it again follows that $z \in B_\lambda$.

Now recall that the involution $H(z) = \lambda^{1/n}/z$ has the property that $F_\lambda(H(z)) = F_\lambda(z)$. This involution takes the region $|z| \geq 2$ to the closed disk of radius $|\lambda|^{1/n}/2$ about the origin. Hence the image of any point in this disk lies in B_λ.

□

Using this result, we can show that there is a McMullen domain about the origin in parameter plane.

Proposition. *Suppose that*

$$|\lambda| < \left(\frac{1}{4}\right)^{2n/(n-2)}.$$

Then v_λ lies in T_λ so $J(F_\lambda)$ is a Cantor set of circles.

Proof: By hypothesis, we have

$$|\lambda|^{\frac{1}{2}-\frac{1}{n}} = |\lambda|^{(n-2)/(2n)} < \frac{1}{4}$$

so that $|\lambda|^{1/2} < |\lambda|^{1/n}/4$. Therefore

$$|v_\lambda| = 2|\lambda|^{1/2} < \frac{|\lambda|^{1/n}}{2}.$$

By the previous result, since $|\lambda| < 1$, it follows that the image of the critical value lies in B_λ.

We therefore have to show that T_λ is disjoint from B_λ and that v_λ lies in T_λ. Recall that the critical circle whose radius is $|c_\lambda| = |\lambda|^{1/(2n)}$ is mapped to the critical segment whose endpoints are the two critical values. By the above inequality we have

$$|v_\lambda| < \frac{|\lambda|^{1/n}}{2} < |\lambda|^{1/n} < |\lambda|^{1/(2n)}$$

since $|\lambda| < 1$. Therefore the image of the critical circle lies strictly inside itself. Hence we may choose δ slightly greater than $|\lambda|^{1/n}$ so that the circle of radius δ about the origin is also mapped strictly inside itself.

Now consider the annular region A given by $\delta \leq |z| \leq 2$. The boundaries of A are mapped strictly outside of A and there are no critical points in A. Hence F_λ is a covering map of A onto its image. By the Riemann-Hurwitz Theorem, it follows that $F_\lambda^{-1}(A) \cap A$ is a subannulus of A that is mapped onto A. It follows that A contains a closed invariant set that surrounds the origin. Therefore B_λ cannot meet the inner boundary of A and, in particular, B_λ cannot meet the disk of radius $|\lambda|^{1/n}/2$. Thus v_λ must lie in T_λ.

□

3. Baby Mandelbrot Sets

In this section we prove the existence of $n-1$ baby Mandelbrot sets in the connectedness locus of the family $F_\lambda(z) = z^n + \lambda/z^n$ when $n > 2$.

We first recall the Douady-Hubbard theory of polynomial-like maps. See [**8**] for more details. Suppose $U' \subset U$ are a pair of bounded, open, simply connected subsets of $\mathbb{C}$ with U' relatively compact in U. A map $G : U' \to U$ is called a *polynomial-like* map of degree two if G is analytic and proper of degree two. Hence such a map has a unique critical point $c \in U'$. The filled Julia set of G is defined in the natural manner as the set of points whose orbits never leave the subset U' under iteration of G. By the results in [**8**], it is known that G is topologically conjugate to some quadratic polynomial on a neighborhood of the polynomial's filled Julia set in $\mathbb{C}$, hence the name polynomial-like.

Now suppose that we have a family of polynomial-like maps $G_\lambda : U'_\lambda \to U_\lambda$ depending on a parameter λ and satisfying:

(1) The parameter λ lies in an open set in $\mathbb{C}$ that contains a closed disk W, and the boundaries of U'_λ and U_λ vary analytically as λ varies;
(2) The map $(\lambda, z) \to G_\lambda(z)$ depends holomorphically on both λ and z;
(3) Each $G_\lambda : U'_\lambda \to U_\lambda$ is polynomial-like of degree two.

Then we may consider the set of parameters in W for which the orbit of the critical point, c_λ, does not escape from U'_λ and so the corresponding filled Julia set is connected. Suppose that for each λ in the boundary of W we have that $G_\lambda(c_\lambda)$ lies in $U_\lambda - U'_\lambda$ and that, moreover, $G_\lambda(c_\lambda) - c_\lambda$ winds once around 0 as λ winds once around the boundary of W. Then, in this case, Douady and Hubbard also prove [**8**] that the set of λ-values for which the orbit of c_λ does not escape from

U'_λ is homeomorphic to the Mandelbrot set and that the polynomial to which G_λ corresponds under this homeomorphism is conjugate to G_λ on some neighborhood of its Julia set. This result thus gives a criterion for proving the existence of small copies of a Mandelbrot set inside $\mathcal{C}$.

So to prove the existence of a copy of the Mandelbrot set in $\mathcal{C}$, we first need to define the sets W, U'_λ, and U_λ. Each of these sets will be portions of a sector bounded on the inside and outside by arcs of a circle centered at the origin, and also by two straight rays emanating from the origin whose arguments differ by a constant depending on n.

We first define W to be the set of λ-values in the right half plane enclosed by arcs of the circles given by

$$|\lambda| = \left(\frac{1}{4}\right)^{2n/(n-2)} \quad \text{and} \quad |\lambda| = 1$$

and by portions the rays

$$\operatorname{Arg} \lambda = \pm \frac{\pi}{n-1}.$$

Later we will use a symmetry in the system to consider parameters drawn from rotationally symmetric sectors. Note that it is here that we need $n > 2$. For each $\lambda \in W$, let $\psi = \operatorname{Arg} \lambda$. We then define the sector U'_λ to be points in the open region bounded by arcs of the circles

$$|z| = \frac{|\lambda|^{1/n}}{2} \quad \text{and} \quad |z| = 2$$

and portions the rays

$$\operatorname{Arg} z = \frac{\psi \pm \pi}{2n}.$$

A straightforward computation shows that there is a unique critcal point lying in U'_λ and given by

$$c_\lambda = |\lambda|^{1/(2n)} e^{i\psi/2n}$$

and that the straight boundaries of U'_λ each contain a prepole given by

$$|\lambda|^{1/(2n)} \exp\left(i\,\frac{\psi \pm \pi}{2n}\right).$$

Proposition. *The family of maps F_λ defined on U'_λ with $\lambda \in W$ is a polynomial like family.*

Proof: Let $U_\lambda = F_\lambda(U'_\lambda)$. For each $\lambda \in W$, the two circular boundaries of U'_λ are mapped to the same curve in $\mathbb{C}$ since they are inversions of each other under the involution H. Moreover, by the escape criterion of the previous section, this image curve lies strictly outside of the circle $|z| > 2$ in B_λ since $|\lambda| \leq 1$.

As we saw in Section 1, since the straight line boundaries of each U'_λ contain a prepole, each of these boundaries is mapped onto a closed segment that is perpendicular to the critical segment. Since the critical segment contains the points $\pm 2\sqrt{\lambda}$, the argument of the critical segment is $\pm\psi/2$. Thus the line that forms the image of the straight line boundaries has argument $(\psi \pm \pi)/2$. But since

$$-\frac{\pi}{n-1} < \psi < \frac{\pi}{n-1},$$

it follows easily that

$$\frac{\psi - \pi}{2} < \frac{\psi \pm \pi}{2n} < \frac{\psi + \pi}{2}.$$

Therefore the image of the straight line boundaries of U'_λ also lies outside the sector U'_λ and F_λ maps these boundaries to the image in a 2 to 1 fashion. It follows that $F_\lambda(U'_\lambda)$ contains U'_λ in its interior, and that F_λ maps the boundary of U'_λ around the boundary of U_λ with degree two. Therefore $F_\lambda : U'_\lambda \to U_\lambda$ is a polynomial-like family of degree 2.

□

In Figure 4 we display the sets U'_λ and U_λ.

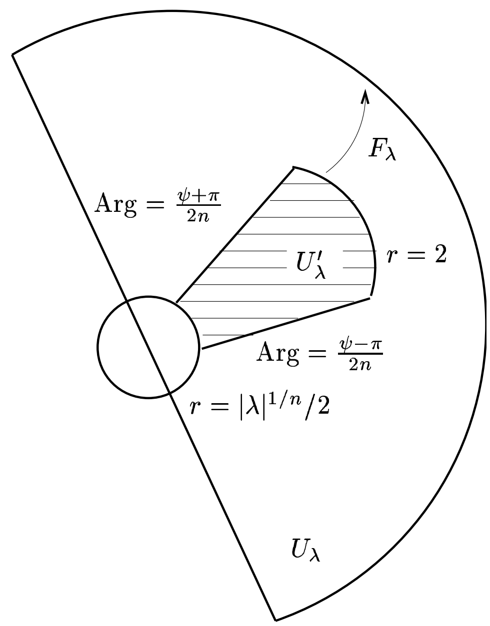


FIGURE 4. The set U'_λ and its image U_λ.

Theorem. *There exists a small copy of the Mandelbrot set in the parameter plane for F_λ in each of the $n-1$ sectors in parameter plane of the form*

$$\frac{(2j-1)\pi}{n-1} < \operatorname{Arg} \lambda < \frac{(2j+1)\pi}{n-1}.$$

Proof: We first deal with λ-values in the region W defined above, so that $j = 0$ and

$$\frac{-\pi}{n-1} \leq \operatorname{Arg} \lambda \leq \frac{\pi}{n-1}.$$

We consider the location of the critical value and the critical points for λ in each of the four different boundary curves of W. We must show that the critical values wind once around the exterior of U'_λ as λ winds once around the boundary of W. In fact, we will show that the critical values actually lie in the boundary of U'_λ for each such λ (recall that U'_λ is open, so this is permissible). Moreover, we will show that whenever λ lies in one of the four distinct boundary components of W, then v_λ also lies in the corresponding boundary component of U'_λ. This will prove the result.

Suppose first that λ lies on the outer circular boundary of W, so that $|\lambda| = 1$. Then

$$|v_\lambda| = |2\sqrt{\lambda}| = 2$$

so v_λ lies on the outer circular boundary of U'_λ. If λ lies on the inner circular boundary of W, then we have

$$|\lambda| = \left(\frac{1}{4}\right)^{2n/(n-2)}.$$

Therefore

$$|v_\lambda| = 2\left(\frac{1}{4}\right)^{n/(n-2)} = \frac{1}{2}\left(\frac{1}{4}\right)^{2/(n-2)} = \frac{1}{2}|\lambda|^{1/n}$$

so that v_λ lies on the inner circular boundary of U'_λ.

Now suppose that λ lies on the upper straight line boundary of W so that $\operatorname{Arg}\lambda = \pi/(n-1)$. Therefore $\operatorname{Arg} v_\lambda = \pi/(2(n-1))$ while the upper boundary of U'_λ is given by

$$\operatorname{Arg} z = \frac{\psi+\pi}{2n} = \frac{1}{2n}\left(\frac{\pi}{n-1}+\pi\right) = \frac{1}{2}\frac{\pi}{n-1} = \operatorname{Arg} v_\lambda$$

so again the result holds. The lower boundary of W is handled analogously. Therefore we see that $F_\lambda(v_\lambda) - c_\lambda$ winds once around the outside of U'_λ as λ winds around the boundary of W. Therefore there is a small copy of the Mandelbrot set inside the sector

$$\frac{\pi}{n-1} < \operatorname{Arg}\lambda < \frac{\pi}{n-1}.$$

To find Mandelbrot sets in the other $n-2$ symmetrically arranged sectors in the parameter plane, we invoke a symmetry in the λ plane. Given n, we let ν be a primitive $2n-2$ root of unity. Then for each j we have

$$\begin{aligned}
F_{\lambda\nu^{2j}}(\nu^j z) &= (\nu^j z)^n + \frac{\lambda\nu^{2j}}{\nu^{jn} z^n} \\
&= \nu^{jn} z^n + \frac{\lambda\nu^{2j-nj}}{z^n} \\
&= \nu^{j(n-1)}\nu^j z^n + \frac{\lambda\nu^{(1-n)j}\nu^j}{z^n} \\
&= (-1)^j \nu^j (F_\lambda(z)).
\end{aligned}$$

In particular, it follows that, if j is even, then the dynamics of F_λ and $F_{\lambda\nu^{2j}}$ are conjugate via the map $z \mapsto \nu^j z$. If, on the other hand, j is odd, the orbits of $F_{\lambda\nu^{2j}}$ are obtained from those of F_λ by first applying the symmetry $z \mapsto \nu^j z$ and then reflecting through the origin. In any event, the dynamics of each map are essentially equivalent (except that, when j is odd, $F_{\lambda\nu^{2j}}$ may have cycles that are twice as long as the corresponding cycles for F_λ). Nonetheless, the picture in parameter plane is preserved by the rotation $z \mapsto \nu^{2j} z$. This proves the existence of $n-2$ additional symmetrically located baby Mandelbrot sets in $\mathcal{C}$ and concludes the proof of the theorem.

Remark. The case $n = 2$ is slightly different. First of all, there is no McMullen domain surrounding the origin. Secondly, there does appear to be a Mandelbrot set in the parameter plane, but this Mandelbrot set has the tip of its tail "chopped off." That is, it appears that the extreme left hand parameter value (corresponding in the quadratic case to the parameter $c = -2$) actually lies at the origin. Hence the method of polynomial like maps fails in this case. Nonetheless, we conjecture that, when $n = 2$, there exists a homeomorphic copy of a baby Mandelbrot set in parameter space, minus only the tip of the tail. See Figure 5.

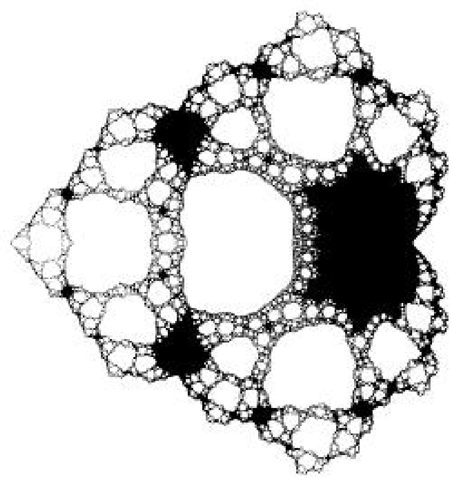


FIGURE 5. The parameter plane for the family $z^2 + \lambda/z^2$.

4. Halos

In this section we prove the main result that the Mandelbrot sets described in the previous section have infinitely many "halos" attached. That is, there are infinitely many points in these Mandelbrot sets that lie on the boundary of Sierpinski holes in parameter space.

We begin by recalling some well known facts about the Mandelbrot set for quadratic polynomials of the form $z \mapsto z^2 + c$. See [**7**] or [**11**] for details on this subject. Like our families, ∞ is a superattracting fixed point and there is only one free critical point at 0 for $z^2 + c$, so c is the sole critical value. As is well known, the Julia set of $z^2 + c$ is connected if and only if the orbit of the critical point is bounded. Those c-values for which the Julia set is connected make up the Mandelbrot set.

When the Julia set of $z^2 + c$ is connected, there is an analytic uniformizing map ϕ_c that takes the exterior of the unit disk in $\mathbb{C}$ in one-to-one fashion onto the exterior of the connected Julia set. The external ray of of argument θ is the image of the straight line $\{te^{2\pi i\theta} \,|\, t > 1\}$ under ϕ_c. It is known that the external ray with argument 0 always limits on a particular fixed point for the map as $t \to 1$, and this fixed point lies in the boundary of the basin of attraction of ∞.

In the parameter plane, there is a similar analytic uniformizing map Φ that takes the exterior of the unit disk in $\mathbb{C}$ in one-to-one fashion onto the exterior of the Mandelbrot set. As above, the external ray of of argument θ is the image of the straight line $\{te^{2\pi i\theta} \mid t > 1\}$ under Φ. It is known that the special external rays whose argument is given by $p/2^j$ for $p, j \in \mathbb{Z}$ always limit on distinct points in the Mandelbrot set. These points are parameter values for which the orbit of the critical point lands on the particular fixed point that is the limiting point for the external ray of argument 0 in the dynamical plane after exactly $j+1$ iterations. These are particular examples of Misiurewicz parameters for $z^2 + c$.

In the case of the Mandelbrot sets for F_λ, we have a similar phenomenon. We have the analogues of the Misiurewicz parameters for which the critical orbit eventually lands on a repelling fixed point that lies on the boundary of B_λ. As in the quadratic case, there are infinitely many such parameter values. We shall show that each of these parameters lies on the boundary of a Sierpinski hole in the parameter plane.

Because of the symmetry present in the family F_λ, we restrict attention to the Mandelbrot set $\mathcal{M}_0$ contained in the sector given by

$$-\frac{\pi}{n-1} < \operatorname{Arg} \lambda < \frac{\pi}{n-1}.$$

Let λ_0 denote one of the parameter values that is the analogue of the point in the standard Mandelbrot set where an external ray of argument $p/2^j$ lands. Also, let p_{λ_0} denote the fixed point where the critical orbit eventually lands. That is, we have

$$F_{\lambda_0}^{j+1}(c_{\lambda_0}) = p_{\lambda_0}.$$

Let $U' = U'_{\lambda_0}$ be the region in dynamical plane described in the previous section. Recall that the straight line boundaries of U' are given by

$$\operatorname{Arg} z = \frac{\psi_0 \pm \pi}{2n}$$

where $\psi_0 = \operatorname{Arg} \lambda_0$. Note that p_{λ_0} lies in the interior of U'.

Now let V' be the region obtained by extending both of these straight line boundaries of U' to ∞. Then F_{λ_0} maps V' strictly outside itself just as in the case of U'. As a consequence, the external ray of argument 0 for F_{λ_0} lies entirely within the region V'. Denote this ray by $\gamma_t(\lambda_0)$ with $t > 1$. Since F_{λ_0} maps V' strictly outside itself, it follows that there is a neighborhood $\mathcal{W}$ of λ_0 in parameter space such that, if $\lambda \in \mathcal{W}$, then

(1) $F_\lambda(U') \supset U'$, and
(2) $F_\lambda(V') \supset V'$.

Here U' and V' are the regions defined for F_{λ_0}, not the analogues of the region U'_λ introduced in the previous section (i.e., U' and V' do not depend on λ). From 1 it follows that, for each $\lambda \in \mathcal{W}$, the fixed point $p_\lambda \in U' \subset V'$. From 2 it follows that the external ray $\gamma_t(\lambda)$ that lands on p_λ is also entirely contained in V'.

For each $\lambda \in \mathcal{W}$, let $R_t(\lambda) = \gamma_t(\lambda)$. It is known that, for each $t > 1$, R_t is an analytic function of λ. Since $R_t(\lambda) \subset V'$ for all t, it follows that the family of functions $\{R_t \mid t > 1\}$ is a normal family in the sense of Montel. Hence any sequence of the the functions R_t for which t tends to 1 contains a subsequence that converges uniformly to an analytic function. But any such sequence converges pointwise to the function $\lambda \mapsto p_\lambda$. Hence any sequence of these functions converges uniformly to the analytic function $\lambda \mapsto p_\lambda$, or, equivalently, to $\lambda \mapsto \gamma_\lambda(1)$.

Now consider the function $G_1 : \mathcal{W} \to \mathbb{C}$ defined by

$$G_1(\lambda) = F_\lambda^{j+1}(c_\lambda) - \gamma_\lambda(1) = F_\lambda^{j+1}(c_\lambda) - p_\lambda.$$

The function G_1 is analytic on $\mathcal{W}$ and, provided that $\mathcal{W}$ is chosen small enough, has a unique zero at λ_0. Now consider the function G_t given by

$$G_t(\lambda) = F_\lambda^{j+1}(c_\lambda) - \gamma_t(\lambda).$$

By the above, G_t converges uniformly to G_1 as $t \to 1$. Hence G_t also has a unique zero in $\mathcal{W}$. But this zero is a λ-value for which the critical orbit lands on the external ray $\gamma_t(\lambda)$ for F_λ after $j+1$ iterations. Therefore we have found a curve of λ values that lies in a Sierpinski hole and approaches λ_0. This completes the proof of the existence of halos for these Mandelbrot sets.

5. Concluding Remarks

In this paper we have established the existence of $n-1$ "principal" Mandelbrot sets in the families $F_\lambda(z) = z^n + \lambda/z^n$, together with their attached halos. But the computer pictures suggest that there are many more copies of such sets in the parameter plane. Indeed, around the outer boundary of the connectedness locus, there appears to be infinitely many small copies of the Mandelbrot set that meet this boundary only at the cusp of the main cardioid. We conjecture that these Mandelbrot sets not only exist, but also meet the boundary at exactly the landing points of the external rays with angles that are periodic under $\theta \mapsto n\theta$. Such rays are known to exist by work of Petersen and Ryd [**13**].

In addition, there also appear to be infinitely many small copies of the Mandelbrot set that are "buried" in the connectedness locus, i.e., that do not meet the boundary of this set. We conjecture that these Mandelbrot sets give rise to the buried Sierpinski curve Julia sets described in [**5**]. We also conjecture that these sets do not have halos attached. In particular, the parameter values in these sets that correspond to the Misiurewicz parameters described in this paper now appear to feature the critical orbits landing on a periodic point that is not on the boundary of B_λ. Hence the critical orbits for nearby parameter values do not lie in Sierpinski holes, or at least there is no curve of such parameter values landing at the Misiurewicz parameter.

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