Discrete Dynamical Systems:

A Pathway for Students to Become Enchanted with Mathematics

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Abstract. Discrete dynamical systems and fractal geometry are two of the most interesting fields of research in contemporary mathematics. One reason for this is the absolutely beautiful images that often arise in these fields. A second reason is that many topics in these fields are completely accessible to all, including high school students. One of the aims of this paper is to describe one such topic, namely, the chaos game. Not only do students get quite excited when they first encounter this topic, but they also see how the fractal geometry they use to understand the chaos game relates directly to what they are currently studying in their geometry classes.

Keywords. Chaos game, fractal, Sierpinski triangle, geometry of transformations.

1. Introduction

One of the things that we in the US do not do well is to expose our K-12 students to what is new, exciting, and beautiful in contemporary mathematics. We have these students in our math classes for twelve years, during which we show them fourth century BC geometry, eleventh century algebra, and, if they really work hard and do well, some seventeenth century calculus. No wonder many students think that there is nothing interesting or important going on in mathematics. Just imagine physicists restricting attention to eleventh century physics or biologists to fourth century BC biology! No way that would happen!

In an effort to change this in our area of the country, my University has organized Math Field Days for the past twenty years or so. These are held two or three times a year, and, each time, around five hundred high school students and their teachers show up for a day where we expose them to some exciting areas of contemporary mathematics. There are no competitions; students who participate in math contests are already ``hooked'' on mathematics and will very likely end up in STEM careers. But there are plenty of other students out there who are very talented and quite creative, but who have no idea of what is going on in today's mathematics.

In an effort to reach out to these students, at the Field Days we focus primarily on discrete dynamical systems. This is a topic that is incredibly accessible to younger students. Indeed, one of the major areas of interest in the field is what happens when you iterate the simple quadratic expression $x^2 + c$. This iterative process often leads to extremely chaotic behavior, and viewing the corresponding chaotic regime in the complex plane produces incredibly beautiful fractal objects like the Julia and Mandelbrot sets. When students hear that we finally understood what happens when the real expression $x^2 + c$ is iterated in the 1990's, and that we still don't understand what happens when the complex expression $z^2 + c$ is iterated, they become quite intrigued. I cannot count the number of letters and emails I have received from teachers and students over the years raving about how great it was to see how beautiful and exciting mathematics can be.

As an illustration of how these topics can be used to excite students, we shall restrict attention in this paper to just one of the many topics in discrete dynamics that we delve into at the Field Days, namely, the ``chaos game," or, as dynamicists call it, an iterated function system. One of the beauties of this topic, besides the exquisite and quite surprising fractal images that arise, is the fact that it brings together many of the topics that high school students are currently studying, like the geometry of transformations, geometric measurement, and probability, in a very different and appealing way.

2. The ``Classical" Chaos Game

The easiest ``chaos game" to explain is played as follows. Start with three points at the vertices of an equilateral triangle. Color one vertex red, one green, and one blue. Take a die and color two sides red, two sides green, and two sides blue. Then pick any point whatsoever in the triangle; this is the *seed*. Now roll the die. Depending upon which color comes up, move the seed half the distance to the similarly colored vertex. Then repeat this procedure, each time moving the previous point half the distance to the vertex whose color turns up when the die is rolled. After a dozen rolls, start marking where these points land.

When you repeat this process many thousands of times, the pattern that emerges is a surprise: it is not a ``random mess," as most first-time players would expect. Rather, the image that unfolds is one of the most famous fractals of all, the Sierpinski triangle shown in Figure 1. Notice that there are no points in the ``missing'' triangles in this set. This is why we did not plot the first few points when we rolled the die.

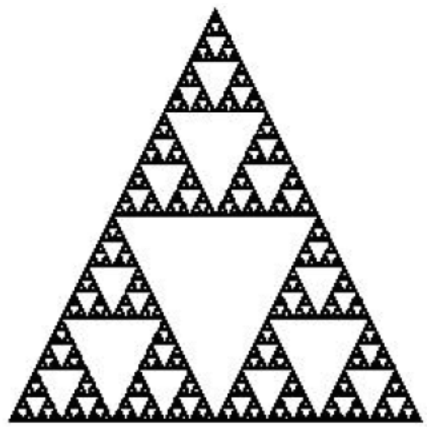


Figure 1. The Sierpinski triangle. The original red, green and blue vertices are located at the vertices of this image.

To enable students to understand what is going on here, it is helpful to provide them with a copy of the Sierpinski triangle. Then, given a particular point in the original triangle, have them plot the three possible points to which this point is moved when the die is rolled. Then, have them plot the nine points at the next level.

And the 27 points at the next level, and so on. It is probably easiest to start this process with a point in the middle of the largest empty triangle. This explains why, after just a very few rolls of the die, the corresponding point is in an empty triangle that is too hard to see because the size of this triangle has become miniscule. So this shows students why the Sierpinski triangle emerges when this game is played, and it also helps their geometric visualization as well as their measurement skills. Starting this process at a point that lies on the Sierpinski triangle leads to a more complicated process, and also helps the student to understand the algorithm for the chaos ``game'' described in Section 5.

It is nice at this point to show students an interesting connection between the Sierpinski triangle and Pascal's triangle. Have them list the numbers in Pascal's triangle down to some level. Then have them erase all of the even numbers and block out each odd number with a black disk. As this process continues down Pascal's triangle, they should begin to see the Sierpinski triangle emerging.

Now here is an observation that fosters other geometric skills: the Sierpinski triangle consists of three self-similar pieces, each of which is exactly one half the size of the original triangle in terms of the lengths of the sides. And these are precisely the numbers that we used to play the game: three vertices and move half the distance to the vertex after each roll. So we can go backwards: just by looking at the Sierpinski triangle, and with a keen eye for its geometry, we can read off the rules of the game we played to produce it.

3. Other Chaos Games

For a different example of a chaos game, put six points at the vertices of a regular hexagon. Number them one through six and erase the colors on the die. We change the rules a bit here: instead of moving the point half the distance to the appropriate vertex after each roll, we now “compress the distance by a factor of three." By this we mean we move the point so that the resulting distance from the moved point to the chosen vertex is one-third the original distance. We say that the *compression ratio* for this game is three.

Again we get a surprise: after rolling the die thousands of times the resulting image is a “Sierpinski hexagon” depicted in Figure 2. And again we can go backwards: this image consists of six self-similar pieces, each of which is exactly one-third the size of the full Sierpinski hexagon --- the same numbers we used to design the game. By the way, there is much more to this picture than meets the eye at first: notice that the interior white regions of this figure are all bounded by the well known Koch snowflake fractal!

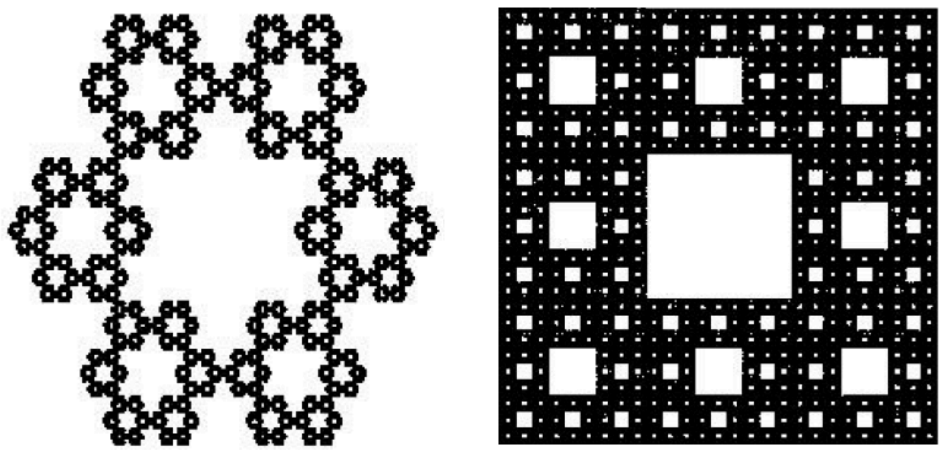


Figure 2. The Sierpinski hexagon and carpet.

This is where the geometry of transformations arises: given a fractal that results when a certain chaos game is played, can you determine the rules that were used to produce this image? In Figure 2, the second fractal is the Sierpinski carpet. How many vertices were used to produce this image, and what was the compression ratio? You need to determine a collection of different geometric transformations that take the entire carpet onto a certain number of distinct, self-similar pieces. An applet called Fractalina that can be used to create a variety of chaos game images is available at the Boston University Dynamical Systems and Technology website (**http://math.bu.edu/DYSYS/applets**).

One quick question: which object emerges when we play the chaos game with four vertices at the corners of a square and a compression ratio of two? As you see, fractals do not always emerge when the chaos game is played.

4. Rotations

Now let's add rotations to the mix. This is where the geometry of transformations becomes even more important. Start with the vertices of a triangle as in the case of the Sierpinski triangle. For the bottom two vertices, the rules are as before: just move half the distance to that vertex when that vertex is called. For the top vertex, the rule is: first move the point half the distance to that vertex, and then rotate the point 90 degrees about the vertex in the clockwise direction. The result of this chaos game is shown in Figure 3a: note that there are basically three self-similar pieces in this fractal, each of which is half the size of the original, but the top one is rotated by 90 degrees in the clockwise direction. Again, as before, we can use geometric transformations to go backwards and determine the rules of the chaos game that produced the image. In addition, plotting the possible images of a given point in the fractal now involves both contractions and rotations and hence more and different geometric skills.

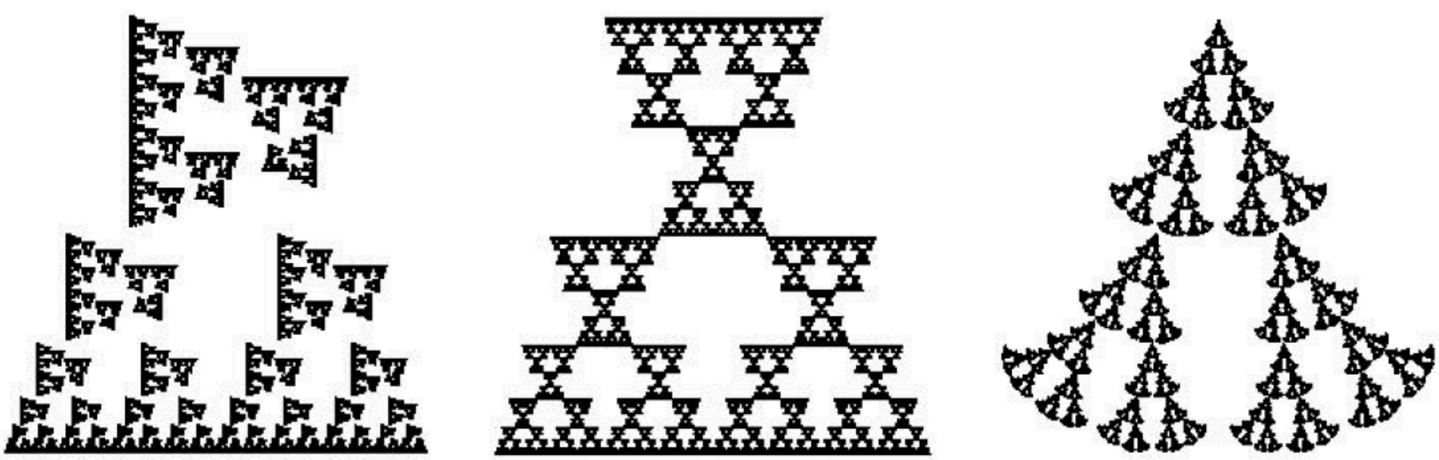


Figure 3. Sierpinski with rotations.

Changing the rotation at this top vertex to 180 degrees yields the image in Figure 3b. This time, the top self-similar piece is rotated 180 degrees. For the fractal in Figure 3c, we rotated twenty degrees in the clockwise direction around the lower left vertex, twenty degrees in the counterclockwise direction around the lower right vertex, and there was no rotation around the top vertex.

In the math classes that most students take, usually the geometry of transformations involves rotations, expansions, or contractions of simple geometric objects, like squares or circles. Here the objects are much more interesting to look at, and determining these transformations can be difficult at times. We often challenge students at the Field Days to figure out the rules of a chaos game that produced a certain image. For example, in Figure 4, we give you the opportunity to try your hand at this. You must determine the number of vertices, the compression ratio, and the rotations involved in each case. Not so easy!

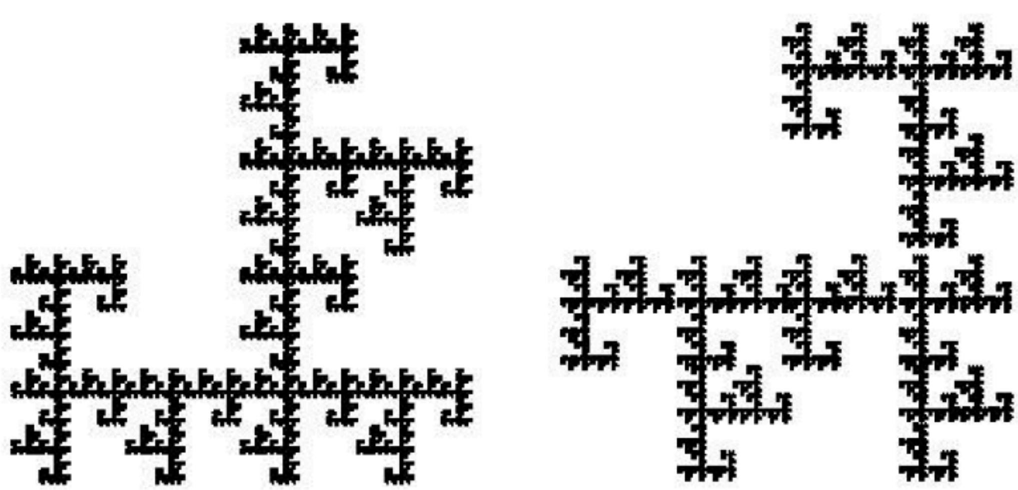
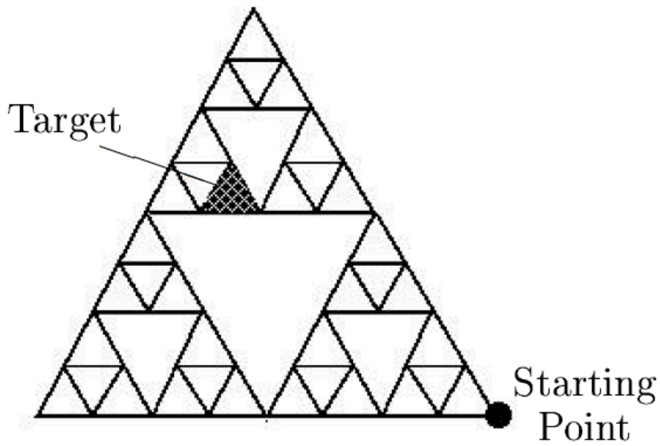


Figure 4. Challenging chaos game images.

Another activity that greatly motivates students is fractal movie-making. Once you know how to create a single fractal pattern via the chaos game, you can slowly vary some of the rotations, compression ratios, or locations of the vertices to create a fractal movie. I often challenge students to make a movie that is "beautiful"" and that I cannot figure out how they made it. The students often work for hours to make these animations. Of course, beautiful here means "with a lot of symmetry," so there really is a lot of geometry in this activity. Another applet called Fractanimate is available to make these movies at the Boston University DS & T website. A number of fractal movies created by students are also posted at this site.

5. The Chaos "Game"

One final topic that is always a big hit at the Field Days arises when I challenge the students present to beat me at the Chaos "Game." To play this game, we begin with the outline of the Sierpinski triangle down to some level. That is, we begin with the original triangle and successively remove groups of sub-triangles at each level. The first level is defined to be the case where the "middle" triangle has been removed from the original triangle, leaving behind three equal-sized triangles. At the second level, the three smaller middle triangles are removed from these three, leaving behind nine equal-sized triangles. At the level n stage, there are then 3^n triangles. Then highlight one of the remaining small triangles at this given level. This triangle is the *target*. Now place a point at the lower right vertex of the original triangle. This is the *starting point*. The goal of the game is to move the starting point into the *interior* of the target. The moves are just the moves of our original chaos game: At each stage the point is moved half the distance to one of the three original vertices. The chaos game setup for a level three game is displayed in Figure 5.



Level three of the chaos game.

At a given level, it is always possible to move the starting point into the interior of the target in the same number of moves, no matter where the target is located. For example, for the three targets available at level one, it is possible to hit any target in exactly three moves. (Recall that you must end up in the interior of the target, not the boundary.) At level two, four moves suffice, and at level n, exactly n+2 moves can be found to hit any target. The challenge to students is to figure out the algorithm for hitting any possible target. Students can usually come up fairly quickly with a way to hit a specific target, but the algorithm necessary to hit any target is much more difficult both to formulate and to explain. But that, as I always tell the students, is what mathematics is all about --- being able to figure out a solution, and then being able to explain it in a coherent fashion.

For example, in Figure 5, the moves to hit the prescribed target are, in order: top, left, right, left, and top. There is only one other way to hit this target in five moves: left, top, right, left, top. This in general is the case: there are exactly two sequences of moves that allow you to hit the target in the minimum number of moves. An interactive version of this game is also available at the DS & T website. At this website, there are also several other variants of this game that include rotations in the mix. These are even more challenging!

6. Some solutions

In this section we briefly describe the answers to some of the questions posed earlier in this paper. First, what happens when you play the chaos game with four vertices at the corners of a square and a compression ratio of two? Well, the points generated by this process eventually fill up the entire square densely. Indeed, the square is a self-similar object: it can be broken into four equal-sized sub-squares, each with sides exactly half the length of the original square.

In Section 4, the two fractals displayed in Figure 4 were obtained as follows. Each was generated using three vertices, a compression ratio of two, and a rotation of 90 degrees. In the first figure, the rotation was in the clockwise direction, and, in the second, in the counterclockwise direction. The three vertices were placed at the corners of an isosceles right triangle. In particular, using the applet Fractalina, in the first case the vertices were placed at (100, 50), (-50, 0), and (50, -50) and, in the second, the vertices were placed at (0, 50), (-50, -100), and (50, -50).

Finally, winning the chaos game in the previous section is a two-step process. The first step involves moving the starting point into the interior of original triangle. This can be accomplished by either moving left then top, or by moving top then left, since the starting point is located at the lower right vertex of the triangle. The second step involves determining the "address" of the target triangle. For example, at stage one, there are three possible target triangles, one on the top, one at the left, and one at the right. We denote these targets by T, L, and R respectively. At stage two, each of these level one triangles is sub-divided into three smaller triangles. For example, the upper triangle T now contains three smaller target triangles, which we denote by TT, TL, and TR. Then each of these targets contains three smaller targets. So, for example, TL can be divided into TLT, TLL, and TLR. Continuing, each possible target at level n has a unique address consisting of a sequence of n letters T, L, and R. Then, to reach the given target at phase two of the process, we just reverse the letters in the address and follow that pattern to move the point into the interior of the target. That is why, in the example in Section 5, the winning strategy to reach the given target was either LTRLT or TLRLT.

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