Indecomposable Continua in Exponential Dynamics

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Abstract

In this paper we prove the existence of uncountably many indecomposable continua in the dynamics of complex exponentials of the form $E_\lambda(z) = \lambda e^z$ with $\lambda > 1/e$. These continua contain points that share the same itinerary under iteration of E_λ. These itineraries are bounded but consist of blocks of 0's whose lengths increase, and hence these continua are never periodic.

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1 Introduction

Our goal in this paper is to discuss the set of points that share the same itinerary under iteration of complex exponential functions of the form $E_\lambda(z) = \lambda e^z$, where $\lambda > 0$.

For the moment suppose that $\lambda = 1$ so that we consider the usual exponential function $E(z) = e^z$. It is known [14] that the Julia set of E, $J(E)$, is the entire complex plane. Hence E is chaotic everywhere in $\mathbb{C}$. As in [6] we may use symbolic dynamics to describe the fates of orbits of E.

We partition the plane into horizontal strips R_j of height 2π and centered about the line $\text{Im}\, z = 2j\pi$. We may then assign an infinite sequence of integers $S(z) = s_0 s_1 s_2 \ldots$ to each z via the rule $s_j = k$ iff $E^j(z) \in R_k$. $S(z)$ is called the *itinerary* of z. We make this definition more precise in Section 2 below.

A natural question in dynamics is to determine the set of points whose orbits share the same itinerary. A number of results are known in this context for E. For example, if $S(z)$ is a bounded sequence that consists of at most finitely many zeroes, then the set of points that share this itinerary is a continuous curve homeomorphic to the half line $[0, \infty)$ and extending to ∞ in the right half plane. These curves are called *hairs*. All orbits on this curve (except possibly that of the endpoint) tend to ∞ in the right half plane [6]. This phenomenon occurs for all exponentials [2].

A contrasting case occurs for the itinerary that is identically zero. In this case the set of points having this itinerary is a pair of invariant indecomposable continua [4], [13]. An *indecomposable continuum* is a compact, connected subset of the Riemann sphere that cannot be written as a union of two such compact, connected sets. The prototype for such a set is the Knaster continuum [11], [10]. These objects appear often in real dynamical systems [1]. The dynamics on this continuum are quite simple. There is a unique repelling fixed point which is the α-limit set for all orbits in the continuum. All other orbits either tend to ∞ or else accumulate on both the orbit of 0 and ∞. This phenomenon also occurs for exponentials for which the orbit of 0 tends to ∞ with periodic itinerary. See [15].

Our goal in this paper is to show that there are many other itineraries that yield indecomposable continua in this way. For example we show that any sequence that consists of infinitely many blocks of zeroes whose lengths grow sufficiently quickly yields a continuum of this sort.

We remark that these continua differ somewhat in a dynamical sense from those that arise from itineraries ending in all zeroes. In this case, there is no fixed point in the continuum. However, there is a unique point in the continuum whose orbit is bounded. All orbits either tend to ∞ or accumulate on the orbit of 0 and ∞.

2 Tails and Hairs

For the real exponential family $E_\lambda(z) = \lambda e^z$ with $\lambda > 0$, the following facts are known.

1. If $0 < \lambda \leq 1/e$, then $J(E_\lambda)$ is a Cantor bouquet. This is a nowhere dense subset of the plane consisting of an uncountable collection of hairs. See [6], [12]. There are no invariant indecomposable continua in $J(E_\lambda)$ in this case.

2. If $\lambda > 1/e$, then $J(E_\lambda)$ is the entire plane [14].

To simplify the constructions, we will work here only with $E(z) = e^z$. All of the results below hold for any $\lambda > 1/e$ with only minor modifications.

Define the horizontal strips

$$R_j = \{z \,|\, (2j-1)\pi < \operatorname{Im} z \leq (2j+1)\pi\}.$$

For each $z \in \mathbb{C}$, we define the *itinerary* of z as the sequence of integers $S(z) = s_0 s_1 s_2 \ldots$ where $s_j \in \mathbb{Z}$ and $s_j = k$ iff $E^j(z) \in R_k$.

Fix a positive integer M. Let

$$\Sigma_M = \{s = (s_0 s_1 s_2 \ldots) \,|\, |s_j| \leq M \text{ for each } j\}.$$

We will only consider points whose itineraries lie in Σ_M. Note that the imaginary parts of the orbits of such points satisfy

$$-(2M+1)\pi < \operatorname{Im} z \leq (2M+1)\pi.$$

It is known [6] that there are infinitely many points whose itineraries correspond to each $s \in \Sigma_M$.

Remark. Many points in $\mathbb{C}$ have unbounded itineraries (see [6]). However, not all unbounded sequences are itineraries. See [7] for a complete description of the allowable unbounded sequences for λe^z.

For $s \in \Sigma_M$, let $I(s)$ denote the set of all points whose itinerary is s. Our goal is to describe the structure of $I(s)$ for special sequences that consist of increasingly larger blocks of zeroes.

For $x > 0$, define the half strip

$$H_x = \{z \,|\, \operatorname{Re} z \geq x, \ -(2M+1)\pi < \ \operatorname{Im} z \leq (2M+1)\pi\}.$$

We say that the orbit of z *tends directly to* ∞ *in* H_x if the entire orbit of z lies in H_x and $\operatorname{Re} E^{n+1}(z) > \operatorname{Re} E^n(z)$ for all n. Let $\omega_s(x)$ denote the set of points in H_x whose orbits tend directly to ∞ with itinerary s. It is known (see [2]) that there exists $\zeta \in \mathbb{R}^+$ such that $\omega_s = \omega_s(\zeta)$ is a continuous curve of the form $(t, h_s(t))$ with $\zeta \leq t < \infty$. In cases like this we say that ω_s is a *graph over* $[\zeta, \infty)$. The value ζ depends only on M, not the particular sequence $s \in \Sigma_M$.

We call ω_s the *tail* of $I(s)$. Note that, by definition, if $z \in \omega_s$, then $E^n(z) \in H_\zeta$ for all $n \geq 0$.

Remark. In the strip R_j there are two straight lines whose itineraries are $j000\ldots$, namely the lines with imaginary parts $2j\pi$ and $(2j+1)\pi$. However, since we require that $\zeta > 0$ and the orbit tend directly to ∞ in H_ζ, it follows that the tail corresponding to this itinerary is the single straight line with imaginary part $2j\pi$ and real part $\geq \zeta$.

By choosing ζ larger if necessary we may assume that E maps the vertical line $\operatorname{Re} z = \zeta$ to a circle that crosses both horizontal boundaries of H_ζ at points with real parts strictly larger than ζ. That is, the circle of radius e^ζ intersects H_ζ to the right of the line $\operatorname{Re} z = \zeta$. Let

$$F_\zeta = \{z \in H_\zeta | \; |z| < e^\zeta\}.$$

We call F_ζ the *fundamental domain* for the tails. The portion of $\omega_s(\zeta)$ contained in F_ζ is called the *base* of the tail. We denote the base by $\alpha_s = \alpha_s(\zeta)$. It is known (see [6]) that $\{\alpha_s \,|\, s \in \Sigma_M\}$ is homeomorphic to the product of a Cantor set and the interval $[0, 1)$.

We will assume for the moment that s is a sequence that does not end in all zeroes. Hence the orbit of a point with itinerary s never lands on the real axis. We will now pull back the curves ω_s to produce longer curves, each point of which will have itinerary s. These longer curves will not in general be graphs of the form $(t, h_s(t))$ as is ω_s. To accomplish this, let $\sigma : \Sigma_M \to \Sigma_M$ denote the shift map given by $\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots)$. Note that $\omega_{\sigma(s)}$ is a tail that properly contains the curve $E(\omega_s)$. Indeed, $E(\omega_s)$ lies in $H_\zeta - F_\zeta$. As such $E(\omega_s)$ misses the base $\alpha_{\sigma(s)}$. Consider $L_{s_0}(\omega_{\sigma(s)})$, where L_{s_0} is the branch of the logarithm taking values in R_{s_0}. This is a continuous curve that lies in R_{s_0} and extends ω_s. Clearly, any $z \in L_{s_0}(\omega_{\sigma(s)})$ has itinerary s.

Inductively, consider

$$L_{s_0} \circ \ldots \circ L_{s_{n-1}}(\omega_{\sigma^n(s)}).$$

This is well defined since 0 never lies in any of these preimages because of our assumption that the sequence s does not end in all zeroes. This curve similarly contains points with itinerary s, and each such curve extends its predecessor. Let

$$\gamma(s) = \bigcup_{n=0}^{\infty} L_{s_0} \circ \ldots \circ L_{s_{n-1}}(\omega_{\sigma^n(s)}).$$

As above, $\gamma(s)$ is a continuous curve containing points with itinerary s. We call $\gamma(s)$ the *hair* associated to s.

Remark. If s ends in all zeroes, we may still pull the tail of s back. In this case, however, $\gamma(s)$ "breaks" into infinitely many disjoint pieces, and hence is no longer a continuous curve. This situation is described in [4].

Clearly, $\gamma(s) \subset I(s)$. In many cases, $I(s)$ is the union of $\gamma(s)$ and a single point whose orbit is bounded. This occurs, for example, if s contains only finitely many zeroes. When $s = 000\ldots$ it is known that $I(s)$ is a pair of indecomposable continua with a single accessible boundary "curve," namely the infinitely many pieces of the broken $\gamma(s)$. Our goal is to show that $I(s)$ is also an indecomposable continuum for certain other sequences in Σ_M.

Note that when $s = k000\ldots$, the set $I(s)$ is unbounded in the left half plane. The following proposition shows that these are the only itineraries for which $I(s)$ is unbounded to the left.

Proposition 2.1. *Suppose $s \in \Sigma_M$ and $s \neq k000\ldots$ for some k. Then $I(s)$ lies in $\mathrm{Re}\, z \geq x$ for some $x > -\infty$.*

Proof. Suppose $s_1 \neq 0$. Then the set of points in R_k that map to R_{s_1} is bounded to the left, since the far left half plane is mapped to a small neighborhood about 0, and hence outside R_{s_1}. Since $I(s)$ lies in this region, it follows that this set is bounded in the left half plane. If $s_1 = 0$, let s_j be the first nonzero digit in s with $j > 1$. Since $E^i(0) \in R_0$ for each $i \geq 0$, we may find a neighborhood U of 0 such that $E^i(U)$ belongs to R_0 for $i = 0, \ldots, j$. Thus the set of points whose itinerary begins $s_1 \ldots s_j$ misses U since $s_j \neq 0$. Consequently, the set of points whose itinerary begins $ks_1s_2\ldots$ misses a far left half plane and so this set of points is again bounded on the left. □

We may also push the base α_s forward to obtain an increasing sequence of curves α_s^n that satisfy:

1. $\alpha_s^0 = \alpha_s$
2. $\alpha_s^{n+1} \supset \alpha_s^n$
3. $\cup_n \alpha_s^n = \omega_s$.

To construct α_s^1, consider the bases $\alpha_{\sigma^{-1}(s)}$. This is a finite collection of bases, one in each R_j. The images $E(\alpha_{\sigma^{-1}(s)})$ all lie in ω_s just to the right of α_s. Let

$$\alpha_s^1 = \alpha_s^0 \cup E(\alpha_{\sigma^{-1}(s)}).$$

This is then a curve in ω_s that extends α_s.

Inductively, set

$$\alpha_s^n = \alpha_s^{n-1} \cup E^n(\alpha_{\sigma^{-n}(s)}).$$

It is easy to check that the α_s^n have the above properties. We call the α_s^n *initial portions* of the tail.

3 Targets

In this section we set up targets around the n^{th} images of the bases of each tail ω_s with $s \in \Sigma_M$. In the next section we will produce hairs that curl through these targets multiple times.

Define

$$V(\xi,\eta) = \{z \in H_\zeta \,|\, \xi - 1 \le \operatorname{Re} z \le \eta + 1\}.$$

$V(\xi,\eta)$ is a rectangle bounded above and below by $|\operatorname{Im} z| = (2M+1)\pi$. We choose to extend the boundaries of $V(\xi,\eta)$ to the left of ξ and right of η by 1 unit for later convenience.

Proposition 3.1. *Given any $n \in \mathbb{Z}^+$, there exists $\xi_n, \eta_n \in \mathbb{R}^+$ such that for any $s \in \Sigma_M$, $E^n(\alpha_s) \subset V(\xi_n, \eta_n)$.*

Proof. Since the fundamental domain for the tails F_ζ is contained inside $\{z \,|\, |z| = e^\zeta\}$, we let $\xi_0 = \zeta$ and $\eta_0 = e^\zeta$. Then $\alpha_s \subset V(\xi_0, \eta_0)$ for each $s \in \Sigma_M$. We also have $|E^n(z)| \le E^{n+1}(\zeta)$ for each z in any α_s. So we set $\eta_n = E^{n+1}(\zeta)$. Since the closure of the union of the α_s for all $s \in \Sigma_M$ is compact (here we include the bases whose itinerary ends in all zeroes), and all orbits of points in this union move to the right under E^j, it follows that there is a maximal ξ_n such that Re $E^n(z) \ge \xi_n$ for each z in the union of the bases α_s. □

From now on we choose ξ_n to be the maximal value for which $E^n(\alpha_s)$ lies to the right of $\operatorname{Re} z = \xi_n$ for all α_s. We also set $\eta_n = E^{n+1}(\zeta)$ as above. We call $V(\xi_n, \eta_n)$ the n^{th} *target* for E.

Proposition 3.2. *We may choose ζ large enough so that, for any $n \ge 0$,*

$$E(V(\xi_n, \eta_n)) \supset V(\xi_{n+1}, \eta_{n+1}).$$

Proof. We may choose ζ large enough so that the image of the vertical line $\operatorname{Re} z = \zeta$ meets the strip H_ζ in an arc whose real part is never less than $e^\zeta - 1$. That is, E maps a portion of the line $\operatorname{Re} z = \zeta$ onto a nearly vertical arc in H_ζ, on which all points have real parts that are within 1 of e^ζ. If $z, E(z) \in H_\zeta$ and $\operatorname{Re} z > \zeta$, then we similarly have that $\operatorname{Re} E(z) \ge |E(z)| - 1$. This follows since the image of this line is an "even more vertical" arc in H_ζ. See Figure 1.

Now any point in the $(n+1)^{st}$ target lies to the right of the circle $r = e^{\xi_n}$ in H_ζ. By the above remarks, we have $\xi_{n+1} \ge e^{\xi_n} - 1$, so $\xi_{n+1} - 1 \ge e^{\xi_n} - 2$. Now $e^{\xi_n} >> e^{-1}e^{\xi_n} + 2$ in the far right half plane. It follows that $\xi_{n+1} - 1 >> e^{\xi_n - 1}$, and so the $(n+1)^{st}$ target lies outside the circle $r = e^{\xi_n - 1}$, provided ζ is large enough.

Similarly,

$$E(\eta_n + 1) = ee^{\eta_n} = e\eta_{n+1} >> \eta_{n+1} + (2M+1)\pi$$

again in the far right half plane. It follows that the $(n+1)^{st}$ target lies inside the circle $r = e^{\eta_n + 1}$. □

Corollary 3.3. *We may choose ζ large enough so that if $\ell > 0$, then, for each $n \ge 0$,*

$$E(V(\xi_n, \eta_{n+\ell})) \supset V(\xi_{n+1}, \eta_{n+\ell+1}).$$

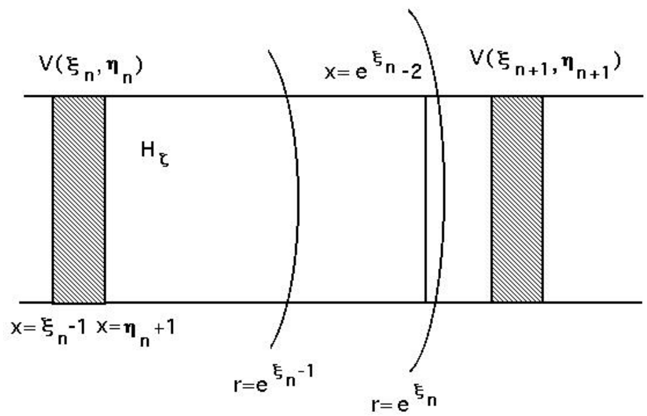

Figure 1: The image of $V(\xi_n, \eta_n)$ covers $V(\xi_{n+1}, \eta_{n+1})$.

Corollary 3.4. *Let $\ell \geq 0$. Given $s = s_0 s_1 s_2 \ldots \in \Sigma_M$, the preimages*

$$L_{s_0} \circ \ldots \circ L_{s_{n-1}}(V(\xi_n, \eta_{n+\ell}))$$

form a nested sequence of subsets of $V(\zeta, \eta_\ell)$. As $n \to \infty$ these neighborhoods tend to a portion of the curve ω_s that contains at least α_s^ℓ.

Proof. The fact that these sets are nested follows from Corollary 3.3. The fact that their intersection lies in ω_s follows from the fact that $|E'| > 1$ in H_ζ and the fact that the only points in H_ζ whose orbits tend directly to ∞ with itinerary s are those in ω_s. Finally, the fact that the intersection contains α_s^ℓ follows from Proposition 3.1 as applied not only to the base α_s but also to all of the bases in $\alpha_{\sigma^{-i}(s)}$. $\square$.

We say that a continuous curve passes twice through $V(\xi_n, \eta_{n+\ell})$ if this curve connects the left and right boundaries of $V(\xi_n, \eta_{n+\ell})$ at least twice. Similarly, a curve passes twice through

$$L_{s_0} \circ \ldots \circ L_{s_{n-1}}(V(\xi_n, \eta_{n+\ell}))$$

if its image under E^n passes twice through $V(\xi_n, \eta_{n+\ell})$.

4 Curly Hairs

In this section we construct hairs corresponding to specific itineraries that pass at least twice through a rectangle $V(\xi_n, \eta_{n+j})$. For fixed $\widetilde{y}$, let I be a segment of the form $x + i\widetilde{y}$ with $x_0 \leq x \leq x_1$. Given $\epsilon > 0$, we say that a continuous curve μ is within ϵ of I if a subset of μ can be parametrized by $(x, h(x))$ with

$x_0 \leq x \leq x_1$ and $|h(x) - \widetilde{y}| < \epsilon$. A sequence $t = t_0 t_1 t_2 \ldots \in \Sigma_M$ is called an *acceptable sequence* if $t_0 \neq 0$ and t does not end in a string of all zeroes.

Now consider a sequence in Σ_M of the form $0_k t$ where 0_k denotes a string of zeroes of length k, and $t = t_0 t_1 t_2 \ldots$ is an acceptable sequence in Σ_M. Without loss of generality, we assume that $t_0 > 0$. In the following construction, we will increase k at times to obtain a new itinerary, but we will always follow the string 0_k with the same sequence t.

Let $\epsilon > 0$. We denote by μ_k the tail corresponding to the sequence $0_k t$. For k large enough, μ_k is a graph over $[\zeta, \infty)$ that lies above (since $t_0 > 0$) and within ϵ of the half line $[\zeta, \infty)$ in the real axis. If we pull this tail back by an application of the logarithm L_0, we obtain a new curve μ_{k+1} whose points have itinerary $0_{k+1} t$. This curve lies within ϵ of the half line $[\log \zeta, \infty)$. Indeed, this curve lies between μ_k and $[\zeta, \infty)$ to the right of ζ. Continuing to pull back by successive applications of L_0, we find a first ℓ so that $\mu_{k+\ell}$ satisfies

1. All points in $\mu_{k+\ell}$ have itinerary $0_{k+\ell} t$
2. There is a subcurve of $\mu_{k+\ell}$ that is a graph over $[0, \infty)$ that lies within ϵ of $[0, \infty)$.

Now pull this subcurve of $\mu_{k+\ell}$ back one more time via L_0. Given any $\tau << 0$, we may choose our initial k so that there is a subcurve of this pullback that lies within ϵ of $[\tau, \infty)$ and that has itinerary $0_{k+\ell+1} t$. Now the next pullback of this curve is a curve that hugs both the interval $[\tau, \infty)$ and a portion of the line $\operatorname{Im} z = \pi$. Assuming τ is sufficiently negative and choosing k even larger, we may assume that the curve corresponding to $0_{k+\ell+2} t$ cuts completely across $V(\xi_n, \eta_{n+j})$ twice. One of these crossings is associated to the tail corresponding to this sequence. Another crossing lies near $\operatorname{Im} z = \pi$. Since t is an acceptable sequence, it follows that the hair corresponding to this sequence is a continuous curve. Note that by choosing k larger we obtain the same result. Hence we obtain infinitely many hairs that pass through $V(\xi_n, \eta_{n+j})$ twice. We call these curves *curly hairs* since they curl around and cross $V(\xi_n, \eta_{n+j})$ at least once in addition to the tail's crossing. We summarize this in the following proposition.

Proposition 4.1. *Let t be an acceptable sequence. Let $n, j \geq 0$. Then there exists $K > 0$ such that, for all $k \geq K$, the hair corresponding to the sequence $0_k t$ passes twice through $V(\xi_n, \eta_{n+j})$.*

5 Indecomposable Continua

In this section we construct indecomposable continua corresponding to certain itineraries. We denote by t_m a block of digits of length m whose first entry is nonzero and that has the property that each digit has absolute value $\leq M$. As above, we also denote by 0_m a block of length m consisting of all zeroes of length m. Our main result is

Theorem 5.1. *Given an infinite sequence of blocks $t_{m_1}, t_{m_2}, \ldots$ of length $m_i < \infty$ and with all digits $\leq M$ in absolute value, we may find a sequence of integers n_j such that the sequence*

$$\hat{s} = t_{m_1}\ 0_{n_1}\ t_{m_2}\ 0_{n_2} \ldots$$

lies in Σ_M and satisfies $I(\hat{s})$ is an indecomposable continuum in the Riemann sphere.

To prove this result, we invoke a result of Curry [3] which states that if a continuous, non-separating curve in the plane accumulates everywhere upon itself, then the closure of this curve is an indecomposable continuum.

Lemma 5.2. *Suppose $s_0, \ldots, s_{n-1}$ satisfy $|s_i| \leq M$ for each i. Let $t \in \Sigma_M$ be any acceptable sequence. Let $s0_kt$ denote the sequence $s_0 s_1 \ldots s_{n-1} 0_k t$. Let $\ell \geq 0$. Then we may choose K such that, if $k \geq K$, then $\gamma(s0_kt)$ passes twice through*

$$L_{s_0} \circ \ldots \circ L_{s_{n-1}}(V(\xi_n, \eta_{n+\ell}))$$

Proof. Choose K as in Proposition 4.1 so that the curve $\gamma(0_kt)$ passes through $V(\xi_n, \eta_{n+\ell})$ twice. Then pull back this curve via $L_{s_0} \circ \ldots \circ L_{s_{n-1}}$. Since we are pulling back by the appropriate branches of the logarithm, we obtain a hair with itinerary $s0_kt$ that passes through the preimage twice. □

We remark that one of the subcurves of $\gamma(s0_kt)$ that passes through the preimage in this lemma contains $\alpha^\ell_{s0_kt}$, an initial portion of the tail lying in $V(\xi_0, \eta_\ell)$. As in Corollary 3.4 the other piece of this curve lies close to this piece of the tail, with the distance between these curves depending on n.

We now turn to the proof of the theorem. Let $\ell > 0$. Define

$$q_j = m_1 + \ldots + m_j + n_1 + \ldots + n_{j-1}.$$

By the lemma, we may inductively construct a hair corresponding to the sequence

$$\hat{s}_j = t_{m_1} 0_{n_1} \ldots t_{m_j}\ 0_{n_j}\ t,$$

where t is any acceptable sequence in Σ_M. (Note that we regard the block $t_{m_1} 0_{n_1} \ldots 0_{n_{j-1}} t_{m_j}$ as the block $s_0 \ldots s_{n-1}$ as in the lemma.) The hair $\gamma(\hat{s}_j)$ has the property that it passes twice through the appropriate preimage of $V(\xi_{q_j}, \eta_{q_j+\ell})$ in $V(\xi_0, \eta_\ell)$. Note that in Σ_M, $\hat{s}_j \to \hat{s}$ as $j \to \infty$.

Now that the lengths of the zero blocks in $\hat{s}$ have been determined, we may regard the sequence $t_{m_{j+1}}\ 0_{n_{j+1}}\ t_{m_{j+2}}\ 0_{n_{j+2}} \ldots$ as the acceptable sequence t in the construction above. Thus we find that the hair $\gamma(\hat{s})$ itself passes twice through the appropriate preimage of $V(\xi_{q_j}, \eta_{q_j+\ell})$ in $V(\xi_0, \eta_\ell)$ for each j. By Corollary 3.4 these preimages are nested and tend to the initial portion of the tail $\alpha^\ell_{\hat{s}}$ as $j \to \infty$. Hence $\gamma(\hat{s})$ accumulates on each point in $\alpha^\ell_{\hat{s}}$.

Since ℓ was arbitrary, we see that $\gamma(\hat{s})$ must accumulate on any point in the full tail $\omega_{\hat{s}}$.

To see that $\gamma(\hat{s})$ accumulates on points in $\gamma(\hat{s})$ that do not lie in the tail, we note that we may perform the same construction for the sequence

$$\hat{\nu}_i = t_{m_i}\ 0_{n_i}\ t_{m_{i+1}}\ 0_{n_{i+1}} \dots .$$

We get the same 0-blocks in this case since the distance needed by $\gamma(0_{n_j}t)$ to curl in the 0-strip is smaller. Then $\gamma(\hat{\nu}_i)$ accumulates on all points in its tail, just as $\gamma(\hat{s})$ does. Now we may pull these hairs and their accumulations back by the appropriate logarithms to find that $\gamma(\hat{s})$ accumulates everywhere on itself.

Next, we claim that $\gamma(\hat{s})$ does not separate the plane. If this were the case, we would have that the complement of the closure of $\gamma(\hat{s})$ contains at least two connected components. One of these sets may be chosen so that it contains the unbounded complement of $\gamma(\hat{s})$ in $\overline{\mathbb{C}}$. Another, say U, must therefore be contained in R_{s_0}, the strip containing $\gamma(\hat{s})$. We claim that $E^n(U) \subset R_{s_n}$. Certainly $E^n(U) \cap R_{s_n} \neq \phi$ since the boundary of U is contained in the closure of $\gamma(\hat{s})$. If $E^n(U)$ also meets a different strip R_k with $k \neq s_n$, then $E^n(U)$ crosses a line with imaginary part an odd multiple of π. But then $E^n(\gamma(\hat{s}))$ also meets this line. But this yields a contradiction, since points on these horizontal lines have itinerary that ends in all zeroes.

Thus $E^n(U) \subset R_{s_n}$ for all n. But this contradicts the fact that $J(E) = \mathbb{C}$, which, by Montel's theorem, implies that $\cup E^n(U) = \mathbb{C}$. This completes the proof that the closure of $\gamma(\hat{s})$ is an indecomposable continuum. It remains to show that this set is equal to $I(\hat{s})$. To show this we need to digress and discuss the dynamics on these sets.

6 Dynamics

Our goal in this section is to prove the following result:

Theorem 6.1. *Let $s \in \Sigma_M$. Then there is a unique point $z_s \in I(s)$ whose orbit is bounded. All other points have ω-limit sets that are either the point at ∞ or the orbit of 0 together with ∞.*

Note that we do not assume in this theorem that s is such that $I(s)$ is an indecomposable continuum. There are many sequences for which $I(s)$ consists of only the hair $\gamma(s)$ together with a single point on which $\gamma(s)$ limits. This is the point z_s in the theorem. The point z_s is called the *endpoint* of $I(s)$. We adopt this terminology for any sequence $s \in \Sigma_M$, even those for which $I(s)$ is an indecomposable continuum. In the case where $I(s) = \gamma(s) \cup \{z_s\}$, the ω-limit set of any point in $I(s)$ (except z_s) is the point at ∞ (see [2]).

In the case where $I(s)$ is an indecomposable continuum, the situation is somewhat different.

Corollary 6.2. *Suppose $s \in \Sigma_M$ is such that $I(s)$ is an indecomposable continuum as in Theorem 5.1. Then the ω-limit set of each point in $\gamma(s)$ is the point at ∞. If $z \in I(s) - (\gamma(s) \cup \{z_s\})$ then the ω-limit set of z is the orbit of 0 together with the point at ∞.*

For simplicity, we will prove this theorem in the special case of a sequence

$$\widehat{s} = 1\ 0_{k_1}\ 1\ 0_{k_2}\ 1\ 0_{k_3}\ \ldots$$

where 1 denotes the block consisting of the single digit 1 and 0_{k_i} is a block of zeroes of length k_i. We assume that the lengths of the blocks k_i grow sufficiently quickly.

Let D_r denote the open disk of radius r centered at 0. Let S_0' denote the closed strip $-100 \leq \mathrm{Re}\, z \leq 100,\ 0 \leq \mathrm{Im}\ z \leq \pi$. Let S_1 denote the closed strip $-100 \leq \mathrm{Re} z \leq 100,\ 2\pi \leq \mathrm{Im}\, z \leq 3\pi$.

Let B_0 denote the open ball of radius $\exp(-100)$ centered at 0, and let $B_j = E^j(B_0)$ for $j = 1, 2, 3$. B_j is a topological disk centered at $1, e$, and e^e for $j = 1, 2, 3$. Let $S_0 = S_0' - \left(\cup_{j=0}^{3} B_j\right)$. S_0 is a closed strip with four open half-disks removed.

Note that the diameters of B_j are much smaller than 1 for $j = 0, 1, 2, 3$. Also, $B_4 = E(B_3)$ is a disk of radius less than 1 about $E^4(0) >> 100$. Indeed, the radius of B_4 is on the order of

$$e^{1+e+e^e-100} << 1.$$

Let $\Gamma(01) = \{z \in S_0 \,|\, E(z) \in S_1\}$.

Lemma 6.3. $\Gamma(01)$ *is a closed subset of* S_0 *which is contained in the interior of* S_0.

Proof. E maps S_0 in one-to-one fashion onto the portion of

$$\overline{D}_{\exp(100)} - \left(\cup_{j=0}^{4} B_j\right)$$

that lies in the upper half plane Im $z \geq 0$. Since each of the B_j lies below the line Im $z = 1$, it follows that the preimage of S_1 in S_0 is strictly contained in the interior of S_0. □

Let $\Gamma(00) = \{z \in S_0 \,|\, E(z) \in S_0\}$ and $\Gamma(10) = \{z \in S_1 \,|\, E(z) \in S_0\}$. As above, we have $\Gamma(00) \subset S_0$, $\Gamma(10) \subset S_1$, although this containment is not strict. Let $\Gamma(k) = \{z \in S_1 \,|\, E^j(z) \in S_0 \text{ for } 1 \leq j \leq k,\ E^{k+1}(z) \in S_1\}$. $\Gamma(k)$ is the set of points in S_1 whose itinerary begins with $1\, 0_k\, 1$. Combining the above observations we have

Proposition 6.4. $\Gamma(k)$ *is a closed subset of* S_1 *which is properly contained in the interior of* S_1.

Note that E^{k+1} maps $\Gamma(k)$ in one-to-one fashion onto S_1. We claim that in fact $E^{k+1}|\Gamma(k)$ is an expansion. This follows from the following lemma, first proved by Misiurewicz [14]. We include the proof for completeness.

Lemma 6.5. $|\mathrm{Im}\ (E_\lambda^n(z))| \leq |(E_\lambda^n)'(z)|$.

Proof. If $z = x + iy$, we have

$$\begin{aligned} |\text{Im } (E_\lambda(z))| &= \lambda e^x |\sin y| \\ &\le \lambda e^x |y| \\ &= |E'_\lambda(z)||\text{Im } (z)| \end{aligned}$$

so that

$$\frac{|\text{Im } (E_\lambda(z))|}{|\text{Im } (z)|} \le |E'_\lambda(z)|$$

if $z \notin \mathbb{R}$. More generally, if $E^n_\lambda(z) \notin \mathbb{R}$, we may apply this inequality repeatedly to find

$$\begin{aligned} \frac{|\text{Im } (E^n_\lambda(z))|}{|\text{Im } (E_\lambda(z))|} &= \prod_{i=1}^{n-1} \frac{|\text{Im } E_\lambda(E^i_\lambda(z))|}{|\text{Im } (E^i_\lambda(z))|} \\ &\le \prod_{i=1}^{n-1} |E'_\lambda(E^i_\lambda(z))|. \end{aligned}$$

Since $|\text{Im } (E_\lambda(z))| \le |E_\lambda(z)| = |E'_\lambda(z)|$ we may write

$$\begin{aligned} |\text{Im } (E^n_\lambda(z))| &\le \prod_{i=0}^{n-1} |E'_\lambda(E^i_\lambda(z))| \\ &= |(E^n_\lambda)'(z)|. \end{aligned}$$

□

Now let $\Gamma(k_1 k_2 \ldots k_j)$ denote the set of points whose orbits visit S_0 and S_1 according to the portion of the itinerary

$$1 0_{k_1} 1 0_{k_2} 1 \ldots 0_{k_j} 1.$$

Note that $\Gamma(k_1 \ldots k_j)$ is properly contained in $\Gamma(k_1 \ldots k_{j-1})$. We also have that $E^\ell|\Gamma(k_1 \ldots k_j)$ is a strict expansion onto S_1 where $\ell = k_1 + \ldots + k_j + j + 1$. It follows that

$$\bigcap_{j=1}^{\infty} \Gamma(k_1 \ldots k_j)$$

is a unique point. Since the orbit of this point remains in S_0 and S_1, the orbit of this point is bounded. This is our point $z_{\hat{s}}$.

To see that $z_{\hat{s}}$ is the only point with itinerary $\hat{s}$ whose orbit is bounded we modify the above argument to enlarge the strips S_0 and S_1.

For $j \ge 4$, let $x_j = E^j(0) + 100$. It is easy to check that

$$E^j(0) < x_j < E^{j+1}(0)$$

for each such j. Now define $S'_0(j)$ to be the strip $-x_j \le \text{Re}\, z \le x_j$, $0 \le \text{Im}\, z \le \pi$ and $S_1(j)$ the strip $-x_j \le \text{Re } z \le x_j$, $2\pi \le \text{Im} z \le 3\pi$. Let B^j_0 denote the open ball of radius $\exp(-x_j)$ about 0 and $B_i^j = E^i(B_0^j)$. Then, just as above, each B_i^j for $0 \leq i \leq j$ is a topological disk about $E^i(0)$ whose diameter is less than one. When $0 \leq i < j$, each B_i^j meets S_0' in an open half disk, and B_j^j lies strictly to the right of $S_0'(j)$. Let $S_0(j) = S_0' - \bigcup_{i=0}^{j-1}(B_i^j)$ as before. Then the exact same arguments as above show that $z_{\hat{s}}$ is the only point whose orbit remains for all time in the enlarged strips $S_0(j)$ and $S_1(j)$. This proves uniqueness of $z_{\hat{s}}$.

As a consequence, the orbit of any $z \in I(\hat{s})$, $z \neq z_{\hat{s}}$, must leave $S_0(j) \cup S_1(j)$ for any j. If z lies on the hair $\gamma(\hat{s})$, then clearly $\omega(z)$ is just ∞. In all other cases, the orbit must visit the left half strip Re $z < -x_j$ for arbitrarily large j. Hence the orbit of z must accumulate on 0 and its forward orbit, together with ∞. This completes the proof of the theorem in the special case of sequences of the form $10_{k_1}10_{k_2}10_{k_3}\ldots$.

The general case of sequences of the form $\hat{s} = t_{m_1}0_{k_1}t_{m_2}0_{k_2}t_{m_3}\ldots$ with $\hat{s} \in \Sigma_M$ follows simimlarly. We need only define additional strips S_j corresponding to digits j with $-M \leq j \leq M$. We must also divide the strip $|\text{Im}\, z| \leq \pi$ into two substrips, S_0^+ given by $0 \leq \text{Im} z \leq \pi$ and S_0^- where $-\pi \leq \text{Im} z \leq 0$. The remainder of the proof follows as above. We leave the details to the reader.

Finally, we complete the proof of Theorem 5.1. All that remains to show is that the closure of $\gamma(\hat{s})$ is equal to $I(\hat{s})$. Suppose this is not the case. Then there is a point z_0 in $I(\hat{s})$ and a neighborhood U of z_0 that misses $\gamma(\hat{s})$. Given the sequence $\hat{s}$, there is a sequence of points $z_{n_j} = E^{n_j}(z_0)$ which satisfy $|\text{Im}\, z_{n_j}| > \pi$. Using the above Lemma, it follows that $|(E^{n_j})'(z_0)| \to \infty$ as $j \to \infty$. This follows from the fact that we may write

$$E^{n_j}(z_0) = E^{n_j - n_{j-1}} \circ \cdots \circ E^{n_2 - n_1} \circ E^{n_1}(z_0)$$

and the fact that each of the derivatives in this composition is larger than π in magnitude. Thus we may find an open set $U_{n_j} \subset U$ which is mapped univalently by E^{n_j} onto a rectangle that connects the upper and lower boundaries of the strip containing z_{n_j}. Now z_{n_j} lies in either the far right half plane or in the far left. In the former case, this rectangle certainly meets the tail of the appropriate image of $\gamma(\hat{s})$. Hence $\gamma(\hat{s})$ meets U and we get a contradiction.

In the latter case, this rectangle will not meet the tails of hairs, which lie in the far right half plane. However, consider $z_{n_j - 1}$. This point does lie in the far right half plane and it may or may not lie in the 0-strip. If not, then we have $|(E^{n_j-1})'(z_0)| >> 1$ as before and the rectangle about z_{n_j-1} now meets the tail of the appropriate image of $\gamma(\hat{s})$. If on the other hand z_{n_j-1} lies in the 0-strip, then this point has imaginary part close to $\pm\pi$ and again, using the Lemma, we have $|(E^{n_j-1})'(z_0)| >> 1$. As above, the rectangle about z_{n_j-1} thus meets the tail of the appropriate image of $\gamma(\hat{s})$. This gives a contradiction and completes the proof.

7 Open Questions

We conclude this paper with several open questions.

1. We have shown that $I(s)$ is an indecomposable continuum if s is a sequence that contains blocks of zeroes whose lengths increase sufficiently rapidly. Does $I(s)$ have this property for any sequence that contains blocks of zeroes whose length increases without bound?

2. Are there sequences for which $I(s)$ is neither a hair nor an indecomposable continuum?

3. If $I(s)$ and $I(t)$ are indecomposable continua with $s \neq t$, are $I(s)$ and $I(t)$ homeomorphic? What if $t = 000\ldots$?

4. Is it possible to construct indecomposable continua in $J(E_\lambda)$ which have n accessible curves where $n > 2$?

5. Do such indecomposable continua exist in the Julia sets of other entire functions such as $\sin z, \cos z$, or the standard family? It is known that such maps admit itineraries s for which $I(s)$ is a hair (see [8], [5]).

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