Dynamics of $z^n + \lambda/z^n$; Why the Case $n = 2$ is Crazy

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Abstract

In this paper, we survey some recent results concerning the family of rational maps $F_\lambda(z) = z^n + \lambda/z^n$ where $n \geq 2$. We shall show that there are several reasons why the case $n = 2$ is by far the most difficult to understand. The first reason for this is that there is no McMullen domain in the parameter plane when $n = 2$. Secondly, there is an extraordinary amount of structure around the McMullen domain when $n > 2$, namely an infinite collection of "Mandelpinski" necklaces. This structure is absent in the parameter plane when $n = 2$. And, finally, the Julia sets converge to the closed unit disk as $\lambda \to 0$ when $n = 2$, and so the structure of these sets becomes extremely complicated as λ approaches 0. However, the Julia sets for $n \geq 3$ and $|\lambda|$ small are all the same and they do not limit on the closed unit disk.

1 Introduction

In recent years there have been a number of papers dealing with singular perturbations of complex dynamical systems. Most of these papers deal with maps of the form $z^n + \lambda/z^d$ where $n \geq 2$ and $d \geq 1$, though a few have tackled more general families of the form

$$F_{\lambda,a,c}(z) = z^n + c + \frac{\lambda}{(z-a)^d}$$

where c is the center of a hyperbolic component of the Multibrot set. These maps are called singular perturbations because, when $\lambda = 0$, the map is just $z \mapsto z^n$ (or $z \mapsto z^n + c$), and the dynamics here are completely understood. However, when $\lambda \neq 0$, the degree of the map goes up and the dynamical behavior explodes.

In this paper, for simplicity, we will restrict attention to the case

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

where $n \geq 2$. The reason for this is that this family possesses some symmetries that make the results much easier to state as well as to visualize. The main theme of this paper is to describe several ways that, strangely enough, the case $n = 2$ is much more complicated than the case $n > 2$.

There are three major reasons why the family with $n = 2$ differs from the higher degree families. The first reason is that, when $n > 2$, there is always a McMullen domain $\mathcal{M}$ surrounding $\lambda = 0$ in the parameter plane (the λ-plane) for these families. The McMullen domain is an open disk containing parameters for which the Julia set is always a Cantor set of concentric simple closed curves. Moreover, each of the maps corresponding to parameters in $\mathcal{M}$ have conjugate dynamics, so while the Julia set changes dramatically when $\lambda \neq 0$, the fact is that all of the nearby singularly perturbed maps have the same dynamics. When $n = 2$ no such region exists around 0. In fact, there are uncountably many non-conjugate maps in any neighborhood of 0 in the parameter plane in this case.

The second major difference between the two cases involves the structure in the parameter plane around the McMullen domain. When $n > 2$, the McMullen domain is surrounded by infinitely many "Mandelpinski" necklaces $\mathcal{S}^k$ for $k = 1, 2, \ldots$. The Mandelpinski necklaces have the property that:

1. Each necklace $\mathcal{S}^k$ is a simple closed curve that surrounds $\mathcal{M}$ as well as $\mathcal{S}^{k+1}$, and the $\mathcal{S}^k$ accumulate on the boundary of the McMullen domain as $k \to \infty$;

2. The curve $\mathcal{S}^k$ meets the centers of $(n-2)n^{k-1}+1$ Sierpinski holes;

3. The curve $\mathcal{S}^k$ also passes through the same number of centers of baby Mandelbrot sets, and these Mandelbrot sets and Sierpinski holes alternate as the parameter winds around $\mathcal{S}^k$.

A Sierpinski hole is a region in the parameter plane for which the associated maps have Julia sets that are Sierpinski curves, i.e., they are homeomorphic to the Sierpinski carpet fractal. The center of a baby Mandelbrot set is a parameter lying in the main cardioid of the associated Mandelbrot set so there is a superattracting cycle for the corresponding map. It is known [6] that the Julia sets corresponding to parameters drawn from this main cardioid are also Sierpinski curves (as long as the Mandelbrot set is "buried"). When $n = 2$, not only do we not have a McMullen domain, but there is also none of this interesting structure around the parameter 0.

The third major difference between the cases $n = 2$ and $n > 2$ concerns the Julia sets near $\lambda = 0$. When $n = 2$, we shall show that, as $\lambda \to 0$, the Julia sets of F_λ converge to the closed unit disk. This is surprising since it is well known that, if a Julia set contains an open set, then the Julia set is necessarily the entire Riemann sphere. Here we find Julia sets getting closer and closer to the closed unit disk, but of course, in the actual limit, the Julia set when $\lambda = 0$ is simply the unit circle. Moreover, there are uncountably many parameters with different dynamical behavior in any neighborhood of $\lambda = 0$. When $n > 2$, the Julia sets when $|\lambda| > 0$ and small are always Cantor sets of simple closed curves, so they are all the same topologically as well as dynamically. It can be shown that, in any neighborhood of the origin, the complements of the Julia sets for these maps always contains a round annulus of some fixed width inside the unit disk, so the Julia sets here do not converge to the closed unit disk.

For the family of maps $G_\lambda(z) = z^n + \lambda/z^d$ with $n \geq 2, d \geq 1$, we have a similar situation. The most complicated case as above is when $n = d = 2$. When $d = 1$ the situation is also very different. Again there is no McMullen domain and no Mandelpinski necklaces. However, the Julia sets converge to the closed unit disk only as λ approaches 0 along $n-1$ special rays [14]; away from these rays, the dynamical behavior is relatively tame.

This paper is dedicated to Linda Keen, who has been a great colleague, collaborator, and friend over the years. I also wish to thank the referee for catching many infelicities in the original text and especially for providing much better versions of many of the figures included herein.

2 Elementary Mapping Properties

In this paper we restrict attention to the family of rational maps given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}.$$

In the dynamical plane, the object of principal interest is the *Julia set* of F_λ, which we denote by $J(F_\lambda)$. The Julia set is the set of points at which the family of iterates $\{F_\lambda^n\}$ fails to be a normal family in the sense of Montel. It is known that $J(F_\lambda)$ is also the closure of the set of repelling periodic points for F_λ as well as the boundary of the set of points whose orbits escape to ∞ under iteration of F_λ. See [12].

The point at ∞ is a superattracting fixed point for F_λ and we denote the immediate basin of ∞ by B_λ. It is well known that F_λ is conjugate to $z \mapsto z^n$ in a neighborhood of ∞ in B_λ [16]. There is also a pole of order n for F_λ at the origin, so there is a neighborhood of 0 that is mapped into B_λ by F_λ. If this preimage of B_λ is disjoint from B_λ, then we denote this preimage of B_λ by T_λ. So F_λ maps both B_λ and T_λ in n-to-one fashion onto B_λ. We call T_λ the *trap door* since any orbit that eventually enters the immediate basin of ∞ must "fall through" T_λ en route to B_λ.

The map F_λ has $2n$ free critical points given by $c_\lambda = \lambda^{1/2n}$. (We say "free" here since ∞ is also a critical point, but it is fixed, and 0 is also a critical point, but 0 is immediately mapped to ∞.) There are, however, only two critical values, and these are given by $v_\lambda = \pm 2\sqrt{\lambda}$. The map also has $2n$ prepoles given by $(-\lambda)^{1/2n}$. Note that all of the critical points and prepoles lie on the circle of radius $|\lambda|^{1/2n}$ centered at the origin. We call this circle the *critical circle* and denote it by C_λ.

The map F_λ has some very special properties when restricted to circles centered at the origin. The following are straightforward computations:

1. F_λ takes the critical circle $2n$-to-one onto the straight line connecting the two critical values $\pm 2\sqrt{\lambda}$ and passing through 0;

2. F_λ takes any other circle centered at the origin to an ellipse whose foci are the critical values.

We call the image of the critical circle the *critical segment*. Also, the straight ray connecting the origin to ∞ and passing through one of the critical points is called a *critical point ray*. Similar straightforward computations show that each of the critical point rays is mapped in two-to-one fashion onto one of the two straight line segments of the form tv_λ, where $t \geq 1$ and v_λ is the image of the critical point on this ray. So the image of a critical point ray is one of two straight rays connecting $\pm v_\lambda$ to ∞. Therefore the critical segment together with these two rays forms a straight line through the origin.

We now turn to the symmetry properties of F_λ in both the dynamical and parameter planes. Let ν be the primitive $2n^{\text{th}}$ root of unity given by $\exp(\pi i/n)$. Then, for each j, we have $F_\lambda(\nu^j z) = (-1)^j F_\lambda(z)$. Hence, if n is even, we have $F_\lambda^2(\nu^j z) = F_\lambda(z)$. Therefore the points z and $\nu^j z$ land on the same orbit after two iterations and so have the same eventual behavior for each j. If n is odd, the orbits of $F_\lambda(z)$ and $F_\lambda(\nu^j z)$ are either the same or else they are the negatives of each other. In either case it follows that the orbits of $\nu^j z$ behave symmetrically under $z \mapsto -z$ for each j. Hence the Julia set of F_λ is symmetric under $z \mapsto \nu z$. In particular, each of the free critical points eventually maps onto the same orbit (in case n is even) or onto one of two symmetric orbits (in case n is odd). Thus these orbits all have the same behavior (up to the symmetry) and so the λ-plane is a natural parameter plane for each of these families. That is, like the well-studied quadratic family $z^2 + c$, there is only one free critical orbit for this family up to symmetry.

Let $H_\lambda(z)$ be one of the n involutions given by $H_\lambda(z) = \lambda^{1/n}/z$. Then we have $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, so the Julia set is also preserved by each of these involutions. Note that each H_λ maps the critical circle to itself and also fixes a pair of critical points $\pm\sqrt{\lambda^{1/n}}$. H_λ also maps circles centered at the origin outside the critical circle to similar circles inside the critical circle and vice versa. It follows that two such circles, one inside and one outside the critical circle, are mapped onto the same ellipse by F_λ.

Since there is only one free critical orbit, we may use the orbit of any critical point to plot the picture of the parameter plane. In Figure 1 we have plotted the parameter planes in the cases $n = 3$ and $n = 4$. The parameter planes for F_λ also possess several symmetries. First of all, we have

$$\overline{F_\lambda(z)} = F_{\overline{\lambda}}(\overline{z})$$

so that F_λ and $F_{\overline{\lambda}}$ are conjugate via the map $z \mapsto \overline{z}$. Therefore the parameter plane is symmetric under complex conjugation.

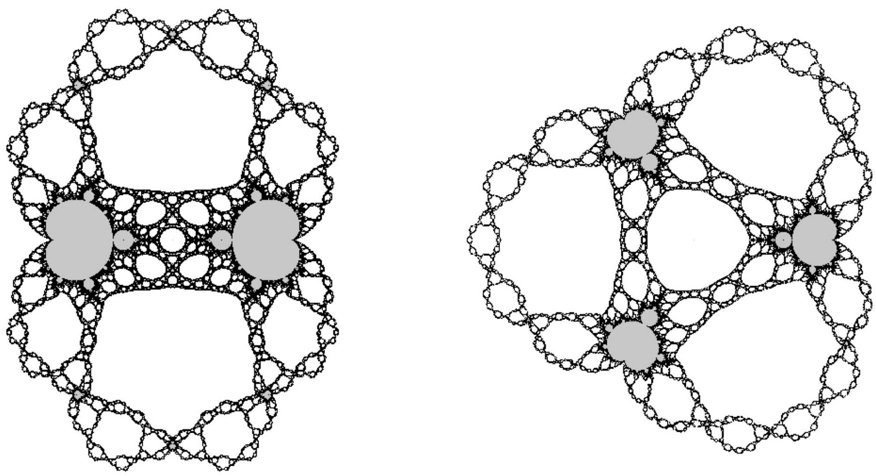

Figure 1: The parameter planes for the cases $n = 3$ and $n = 4$.

We also have $(n-1)$-fold symmetry in the parameter plane for F_λ. To see this, let ω be the primitive $(n-1)^{\text{st}}$ root of unity given by $\exp(2\pi i/(n-1))$. Then, if n is even, we compute that

$$F_{\lambda\omega}(\omega^{n/2}z) = \omega^{n/2}(F_\lambda(z)).$$

As a consequence, for each $\lambda \in \mathbb{C}$, the maps F_λ and $F_{\lambda\omega}$ are conjugate under the linear map $z \mapsto \omega^{n/2}z$. When n is odd, the situation is a little different. We now have

$$F_{\lambda\omega}(\omega^{n/2}z) = -\omega^{n/2}(F_\lambda(z)).$$

Since $F_\lambda(-z) = -F_\lambda(z)$, we therefore have that $F_{\lambda\omega}^2$ is conjugate to F_λ^2 via the map $z \mapsto \omega^{n/2}z$. This means that the dynamics of F_λ and $F_{\lambda\omega}$ are "essentially" the same, though subtly different. For example, if F_λ has a fixed point, then under complex conjugation, this fixed point and its negative are mapped to a 2-cycle for $F_{\lambda\omega}$. To summarize the symmetry properties of F_λ, we have:

Proposition (Symmetries in the dynamical and parameter plane). *The dynamical plane for F_λ is symmetric under the map $z \mapsto \nu z$ where ν is a primitive $(2n)^{\text{th}}$ root of unity as well as the involution $z \mapsto \lambda^{1/n}/z$ The parameter plane is symmetric under both $z \mapsto \overline{z}$ and $z \mapsto \omega z$ where ω is a primitive $(n-1)^{\text{st}}$ root of unity.*

Recall that, for the quadratic family, if the critical orbit escapes to ∞, the Julia set is always a Cantor set. For F_λ, it turns out that there are three different possibilities for the Julia sets when the free critical orbit escapes. The following result is proved in [7].

Theorem (The Escape Trichotomy). *For the family of functions*

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

with $n \geq 2$ and $\lambda \in \mathbb{C}$:

1. *If the critical values lie in B_λ, then the Julia set is a Cantor set.*
2. *If the critical values lie in T_λ, then the Julia set is a Cantor set of simple closed curves.*
3. *If the critical values lie in any other preimage of T_λ, then the Julia set is a Sierpinski curve.*

A *Sierpinski curve* is a planar set that is characterized by the following five properties: it is a compact, connected, locally connected and nowhere dense set whose complementary domains are bounded by simple closed curves that are pairwise disjoint. It is known from work of Whyburn [18] that any two Sierpinski curves are homeomorphic. In fact, they are homeomorphic to the well-known Sierpinski carpet fractal. From the point of view of topology, a Sierpinski curve is a universal set in the sense that it contains a homeomorphic copy of any planar, compact, connected, one-dimensional set [15]. The first example of a Sierpinski curve Julia set was given by Milnor and Tan Lei [13].

Case 2 of the Escape Trichotomy was first observed by McMullen [11], who showed that this phenomenon occurs in each family provided that $n \neq 2$ and $|\lambda|$ is sufficiently small.

In the parameter plane pictures, the white regions consist of parameters for which the critical orbit escapes to ∞. The external white region is the set of parameters for which the Julia set is a Cantor set. The small central disk is the region containing parameters for which the Julia set is a Cantor set of simple closed curves. This is the McMullen domain, $\mathcal{M}$. And all of the other white regions contain parameters whose Julia sets are Sierpinski curves. These are the Sierpinski holes.

In Figure 2 we display three Julia sets drawn from the family $F_\lambda(z) = z^4 + \lambda/z^4$, one corresponding to each of the three cases in the Escape Trichotomy.

3 The McMullen domain

One of the big differences between the cases $n = 2$ and $n > 2$ is that there is no McMullen domain $\mathcal{M}$ when $n = 2$. To see this, recall that $\lambda \in \mathcal{M}$ if the critical values $\pm v_\lambda = \pm 2\sqrt{\lambda}$ lie in the trap door T_λ. So when does this happen? First let $\lambda^* = 4^{-n/(n-1)}$. Then one checks easily that, if $|\lambda| = \lambda^*$, then $|v_\lambda| = |c_\lambda|$ so both the critical points and critical values lie on the critical circle. We call the circle of radius λ^* centered at 0 in the parameter plane the *dividing circle*. Then, if $|\lambda| < \lambda^*$, we have $|v_\lambda| < |c_\lambda|$, and so F_λ maps the critical circle strictly inside itself. So a slightly larger circle is mapped to an ellipse that lies strictly inside this circle. Then, using quasiconformal surgery, one can glue the map $z \mapsto z^2$ into the disk bounded by this circle. See [1] for details. It follows that B_λ is bounded by a simple closed curve lying strictly outside this disk. And, in particular, there is a disjoint preimage of B_λ surrounding the origin inside this circle. This is the trap door T_λ which is therefore disjoint from B_λ.

Next we compute that

$$F_\lambda(v_\lambda) = 2^n \lambda^{n/2} + \frac{1}{2^n \lambda^{n/2-1}}.$$

When $n > 2$, as $\lambda \to 0$, we have $v_\lambda \to 0$ and so $F_\lambda(v_\lambda) \to \infty$. Thus, when $|\lambda|$ is small, v_λ does indeed lie inside the trap door when $n > 2$. But when $n = 2$, $F_\lambda(v_\lambda) \to 1/4$ as $\lambda \to 0$. The point $1/4$ is not in B_λ for $|\lambda|$ small since the boundary of B_λ is close to the unit circle in this case. Hence v_λ does not lie in T_λ in this case.

There is another way to see this. Suppose both critical values lie in T_λ. It is easy to see that T_λ is an open disk, so the question is: what is the preimage of T_λ? A natural first thought would be that the preimage of T_λ is a collection of open disks, one surrounding each preimage of $\pm v_\lambda$. But there

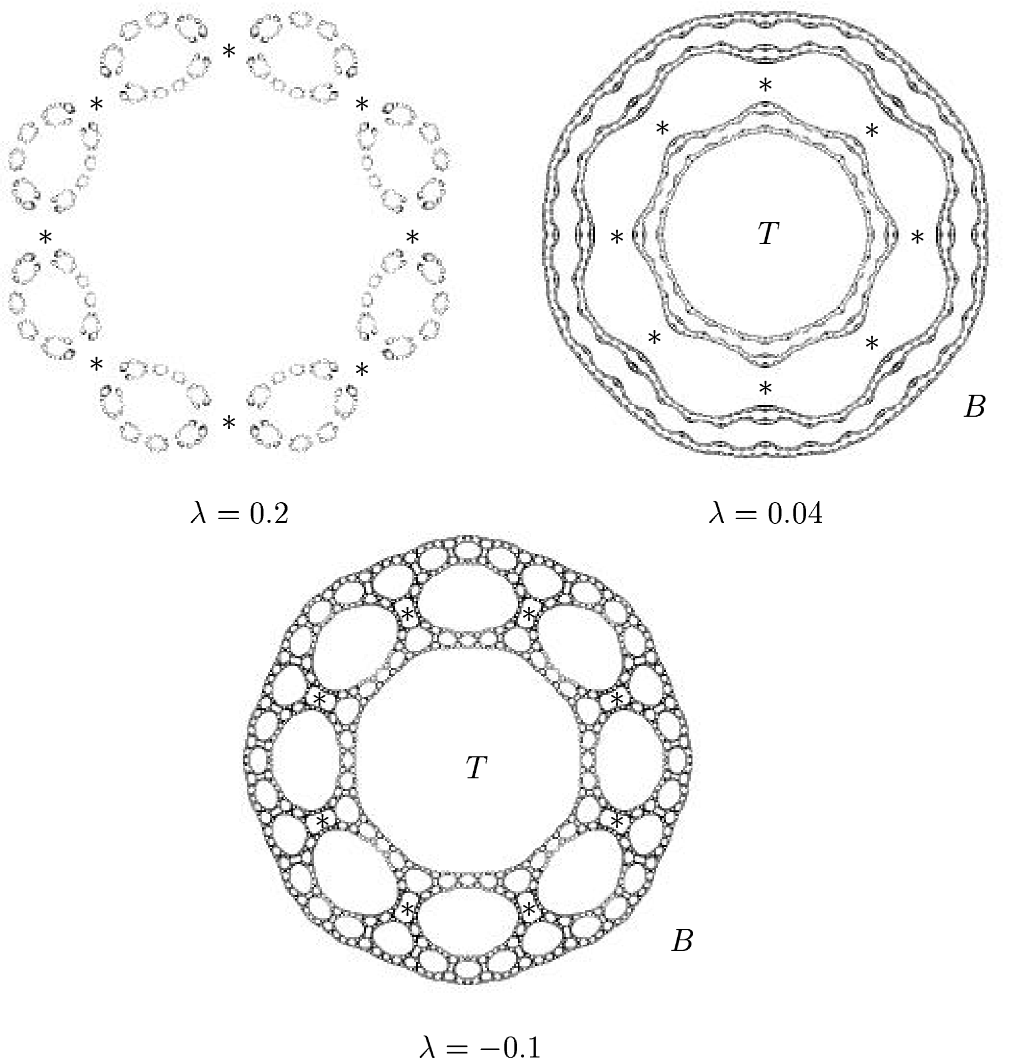

Figure 2: Some Julia sets for $z^4 + \lambda/z^4$: if $\lambda = 0.2$, $J(F_\lambda)$ is a Cantor set; if $\lambda = 0.04$, $J(F_\lambda)$ is a Cantor set of circles; and if $\lambda = -0.1$, $J(F_\lambda)$ is a Sierpinski curve. Asterisks indicate the location of critical points.

are $2n$ such preimages, namely the critical points, and so each of these disks would then necessarily be mapped two-to-one onto T_λ. But this would then mean that the map would have degree $4n$. But the degree of F_λ is $2n$, so the preimages of T_λ cannot be a collection of disjoint disks. Therefore some of the preimages of T_λ must overlap. But then, by the symmetries discussed earlier, all of these preimages must overlap, and so the preimage of T_λ is a connected set. By the Riemann-Hurwitz formula, we have

$$\mathrm{conn}\,(\mathrm{F}_\lambda^{-1}(\mathrm{T}_\lambda)) - 2 = (\deg \mathrm{F}_\lambda)(\mathrm{conn}\,(\mathrm{T}_\lambda) - 2) + (\text{number of critical points})$$

where $\mathrm{conn}(X)$ denotes the number of boundary components of the set X. But both the degree and the number of critical points in this formula is $2n$, and $\mathrm{conn}(T_\lambda) = 1$. So it follows that the preimage of T_λ has two boundary components. That is, $F_\lambda^{-1}(T_\lambda)$ is an annulus.

This then is the beginning of McMullen's proof that the Julia set in this case is a Cantor set of simple closed curves [11]. We know that the complement of the Julia set contains the disks B_λ and T_λ as well as the annulus $F_\lambda^{-1}(T_\lambda)$. The entire preimage of B_λ is the union of B_λ and T_λ, while the entire preimage of T_λ is the annulus $F_\lambda^{-1}(T_\lambda)$. So what is the preimage of $F_\lambda^{-1}(T_\lambda)$? This preimage must lie in the two annular regions between $F_\lambda^{-1}(T_\lambda)$ and B_λ or T_λ. Call these annuli A_{in} and A_{out}. See Figure 3. Since the preimage cannot contain a critical point, it follows that the preimage must be mapped as a covering onto $F_\lambda^{-1}(T_\lambda)$, in fact, an n-to-one covering since F_λ is n-to-one on both B_λ and T_λ. So the preimage consists of a pair of disjoint annuli. Then the preimages of these annuli consist of four annuli, and so forth. What McMullen shows is that, when you remove all these preimage annuli, what is left is a Cantor set of simple closed curves, each surrounding the origin.

Here then is another reason why there is no McMullen domain when $n = 2$. From the above, we have that each of the annuli A_{in} and A_{out} is mapped as an n-to-one covering onto the annulus A which is the union of $F_\lambda^{-1}(T_\lambda)$, A_{in}, and A_{out}. Then the modulus of A_{in} is equal to $\mathrm{mod}\,(A)/n$ and similarly for the modulus of A_{out}. But then, when $n = 2$, we have

$$\mathrm{mod}\,A_{\mathrm{in}} + \mathrm{mod}\,A_{\mathrm{out}} = \mathrm{mod}\,A.$$

So this leaves no room for the intermediate annulus, $F_\lambda^{-1}(T_\lambda)$, so this picture cannot occur when $n = 2$.

One final remark: in [2] it is shown that the McMullen domain is a single open disk surrounding the origin whose boundary is a simple closed curve.

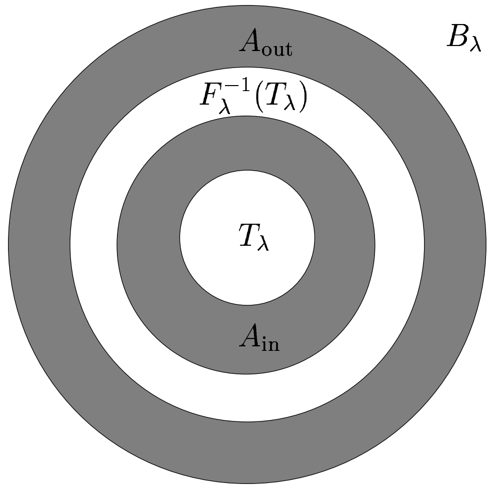

Figure 3: The annuli A_{in} and A_{out}.

4 Mandelpinski Necklaces

In this section, we describe the very orderly structure in the region surrounding the McMullen domain $\mathcal{M}$ when $n \geq 3$. In Figure 4, we display several magnifications of this region when $n = 3$. In each case, $\mathcal{M}$ is the central disk. In the first picture, $\mathcal{M}$ seems to be surrounded by a closed curve that passes through four Sierpinski holes and a smaller closed curve passing through ten Sierpinski holes. In the magnification you can see smaller closed curves passing through 28 and 82 Sierpinski holes. You also see some black "regions" between each of these Sierpinski holes; these are actually baby Mandelbrot sets. In [8] the following Theorem is proved:

Theorem. (Rings Around the McMullen Domain.) *For each $n \geq 3$, the McMullen domain for the family $z^n + \lambda/z^n$ is surrounded by infinitely many simple closed curves $\mathcal{S}^k$ for $k = 1, 2, \ldots$ having the property that:*

1. *Each curve $\mathcal{S}^k$ surrounds the McMullen domain as well as $\mathcal{S}^{k+1}$, and the $\mathcal{S}^k$ accumulate on the boundary of the McMullen domain as $k \to \infty$;*

2. *The curve $\mathcal{S}^k$ meets the centers of τ_k^n Sierpinski holes, each with escape time $k+2$, where*
$$\tau_k^n = (n-2)n^{k-1} + 1.$$

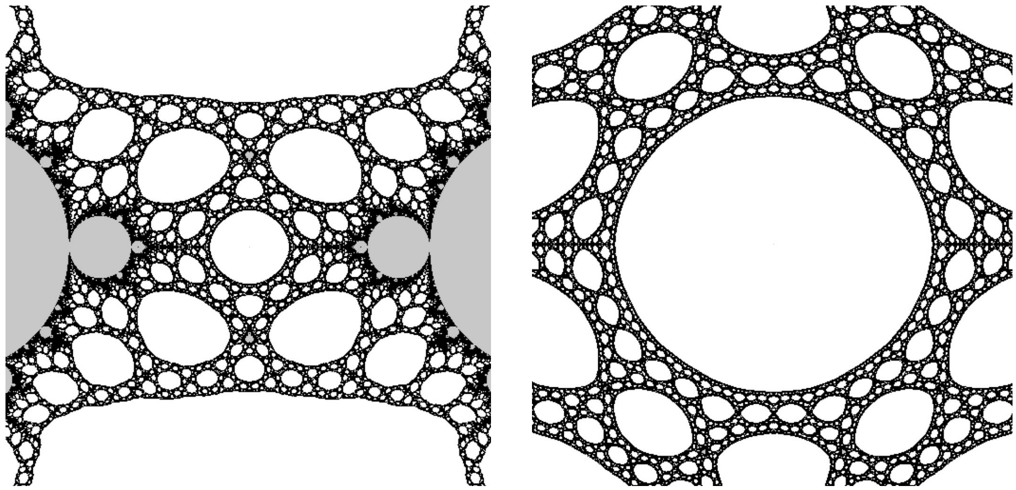

Figure 4: Magnifications of the parameter plane for the family $z^3 + \lambda/z^3$ around the McMullen domain.

The escape time is the number of iterations that it takes for the orbit of a critical point to first land in B_λ;

3. *The curve $\mathcal{S}^k$ also passes through τ_k^n superstable parameter values where a critical point is periodic of period k or $2k$.*

We call the ring $\mathcal{S}^k$ a *Mandelpinski necklace* since it contains so many centers of Sierpinski holes and baby Mandelbrot sets.

Remarks:

1. In [3] it was shown that each of these superstable parameter values is actually the center of the main cardioid of a small Mandelbrot set.

2. There is one slight exception to the above. This involves the ring $\mathcal{S}^2$. This ring passes through the centers of $\tau_2^n - (n-1)$ baby Mandelbrot sets and $n-1$ centers of period 2 bulbs off larger Mandelbrot sets (whose centers are on $\mathcal{S}^1$). In Figure 3 the curve $\mathcal{S}^2$ meets two small Mandelbrot sets and two period 2 bulbs as well as four Sierpinski holes.

3. There really is an amazing amount of structure here; for example, in the case $n = 3$, the ring $\mathcal{S}^{13}$ passes through exactly 1,594,324 centers of Sierpinski holes and baby Mandelbrot sets.

The existence of the first Mandelpinski necklace $\mathcal{S}^1$ is easy to prove. Recall that on the circle of radius λ^* in the parameter plane (the dividing circle), the critical points, critical values, and prepoles all lie on the same circle in the dynamical plane, namely the circle given by $|z| = 2\sqrt{\lambda^*}$. As λ rotates once around the dividing circle, then $v_\lambda = 2\sqrt{\lambda}$ rotates exactly half way around this circle, while the critical points and prepoles rotate exactly $1/2n$ of a turn in the same direction. As a consequence, the critical values meet exactly $n-1$ critical points and the same number of prepoles as they wind around the circle in the dynamical plane, so this defines $\mathcal{S}^1$. See Figure 5 for the dividing circle in the case $n = 4$.

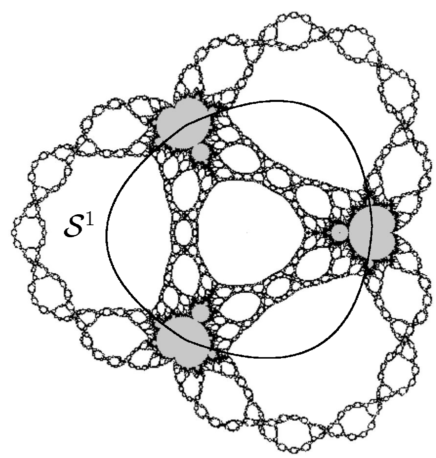

Figure 5: The curve $\mathcal{S}^1$ in the parameter plane for $n = 4$.

To define $\mathcal{S}^k$ for $k > 1$, we first revisit the dynamical plane. If λ lies inside the dividing circle, then we know that $|v_\lambda| < |c_\lambda|$ so the critical segment connecting $\pm v_\lambda$, i.e., $F_\lambda(C_\lambda)$, lies strictly inside the critical circle, C_λ. Then there is a preimage $C^1 = C^1_\lambda$ of C_λ which is a simple closed curve that lies strictly outside C_λ and which is mapped by F_λ as an n-to-one covering onto C_λ. Then, similarly, there is a simple closed curve C_2 which lies outside C_1 and is mapped as an n-to-one covering onto C_1. Continuing in this fashion, we find an increasing sequence of simple closed curves C_k that have the property that F^k_λ maps C_k as an n^k-to-one covering of the critical circle. Since the critical circle contains $2n$ critical points and $2n$ prepoles, it follows that there are exactly $2n^{k+1}$ points on C_k that are mapped to critical points by F_λ^k and the same number that are mapped to prepoles. Note that, since these maps are covering maps, on the curve C_k the points that land on critical points and prepoles after k iterations alternate as they wind around C_k.

Recall that the involution $H_\lambda(z) = \lambda^{1/n}/z$ has the property that $F_\lambda(H_\lambda(z)) = F_\lambda(z)$. Furthermore, H_λ maps the exterior (resp., interior) of C_λ univalently onto the interior (resp., exterior) of C_λ. Thus we find a similar collection of simple closed curves C_{-k} lying inside C_λ where $C_{-k} = H_\lambda(C_k)$. So the curves C_{-k} contain the exact same number of points that are mapped under k iterations to the critical points or prepoles as are contained in C_k.

To produce the Mandelpinski necklaces, we can produce a "natural" parametrization of each C_{-k} of the form $C_{-k}^\lambda(\theta)$. For fixed k and θ, the map $\lambda \mapsto C_{-k}^\lambda(\theta)$ can be chosen to vary analytically with λ, at least in one of the $n-1$ symmetric regions in the parameter plane that lies outside of $\mathcal{M}$. We also have the map $V(\lambda) = v_\lambda$, which is also analytic in one of the symmetry regions outside $\mathcal{M}$. But the map V is invertible in this region, so we can consider the composition $\lambda \mapsto V^{-1}(C_{-k}^\lambda(\theta))$ on this region. Then, as shown in [8], using the Schwarz Lemma, for each θ and k, there is a unique fixed point, λ_{-k}^θ, for this map. This fixed point is a parameter for which the critical value lands on the given point on the curve C_{-k}. Moreover, this fixed point varies continuously with θ, and this produces the Mandelpinski necklace $\mathcal{S}^k$.

So the exterior of the McMullen domain contains quite a bit of interesting structure. When $n = 2$, however, this structure disappears. In Figure 6 we display the parameter plane in the case $n = 2$ as well as a magnification around the origin. The grey regions in these pictures are now the Sierpinski holes, so we know that there are infinitely many of them in any neighborhood of the origin. But the natural question is how are these holes arranged? Are there any Mandelpinski necklaces here? This is an open question. As we showed in Section 3, $F_\lambda(v_\lambda) \to 1/4$ as $\lambda \to 0$, so the critical values never come too close to the boundary of T_λ.

Remarks:

1. There are obviously many more Sierpinski holes in the parameter planes than just those on the Mandelpinski necklaces. Based on some work of Roesch [17] on the case $n = 2$, we were able to give in [3] a complete count of the number of Sierpinski holes in the parameter plane in general: there are exactly $(n-1)(2n)^{\kappa-3}$ Sierpinski holes with escape time κ in the parameter

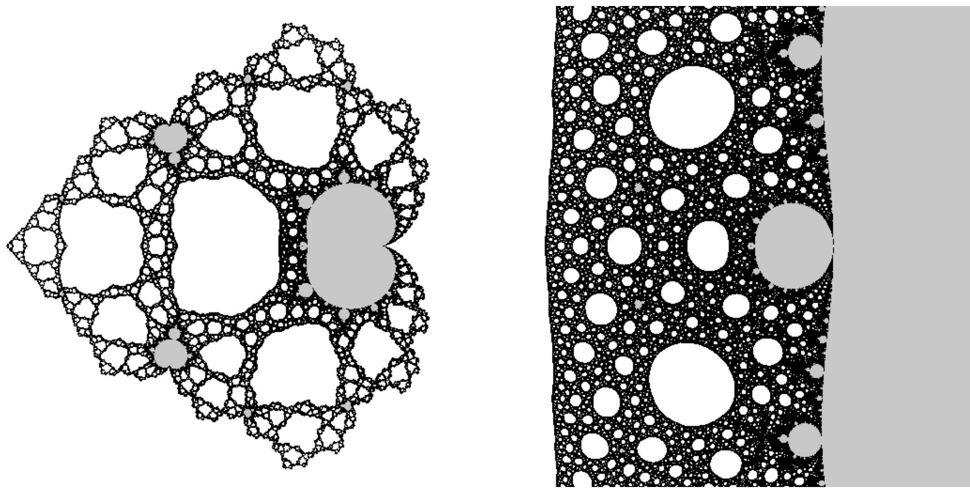

Figure 6: The parameter plane and a magnification around the origin for the family $z^2 + \lambda/z^2$. The large central disk is a Sierpinski hole, not the McMullen domain.

plane.

2. All of the Julia sets whose parameters lie in Sierpinski holes are homeomorphic, so the natural question is: are the dynamics on these sets the same? It is easy to show that two Sierpinski curve Julia sets for which the escape times are different have non-conjugate dynamics. This follows from the fact that the disks containing the critical points are the only ones that are mapped two-to-one onto their images, so they must be mapped to similar disks by the conjugacy. Also, it is easy to show using quasiconformal surgery that if λ_1 and λ_2 lie in the same Sierpinski hole, then F_{λ_1} and F_{λ_2} are topologically conjugate on their Julia sets. So the only question that remains is what is the situation when two parameters are drawn from different Sierpinski holes that have the same escape time. In joint work with K. Pilgrim [9], we showed that parameters from two distinct Sierpinski holes have conjugate dynamics if and only if the holes are symmetric under either complex conjugation or under the map $z \mapsto \alpha^2 z$ where α is a primitive $(n-1)^{\text{st}}$ root of unity. This then allows us to give an exact count of the number of conjugacy classes of Sierpinski curve Julia sets with escape time κ. This number is $(2n)^{\kappa-3}$ if n is odd and $(2n)^{\kappa-3}/2 + 2^{\kappa-4}$ if n is even. The discrepancy between n even and n odd arises because there are no Sierpinski holes lying along the real

axis when n is odd (and so every Sierpinski hole has a complex conjugate hole that is different from it), while there are infinitely many Sierpinski holes straddling the real axis when n is even.

5 Julia Sets Converging to the Unit Disk

The final (and perhaps most interesting) difference between the cases $n = 2$ and $n > 2$ concerns the behavior of the Julia sets of F_λ as $\lambda \to 0$. When $n = 2$ these Julia sets converge to the closed unit disk as λ tends to the origin. But when $n > 2$, we have already seen that, for $|\lambda|$ sufficiently small, $J(F_\lambda)$ is always a Cantor set of simple closed curves surrounding the origin. Hence there are countably many annuli separating these components of the Julia sets. At least one of these annuli must contain a round annulus of some definite width for every parameter in a neighborhood of the origin, so these Julia sets do not converge to the unit disk. More precisely, in [5] we have shown:

Theorem:

1. *Suppose* $n = 2$. *If* λ_j *is a sequence of parameters converging to* 0, *then the Julia sets of* F_{λ_j} *converge as sets to the closed unit disk.*

2. *If* $n > 2$, *this is not the case. Specifically, for a given punctured neighborhood* U *of* 0 *in* $\mathcal{M}$, *there exists* $\delta > 0$ *such that, for each* $\lambda \in U$, *there is a round annulus (i.e., bounded by circles centered at the origin) in the complement of the Julia set inside the unit circle whose internal and external radii differ by at least* δ.

In Figure 7, we display several Julia sets of F_λ when $n = 2$ and λ is close to 0. As λ decreases, note how the preimages of the trap door (the white regions) get smaller. On the other hand, in Figure 8, n is 3 and the Julia sets are Cantor sets of circles and there is an annulus of at least some given width in the complement of the Julia set.

The proof that the Julia sets converge to the unit disk as $\lambda \to 0$ when $n = 2$ is straightforward. It is known that if c_λ does not lie in B_λ (or T_λ), then $J(F_\lambda)$ is a connected set [4]. It has also been proved in that paper that, if $|\lambda| < 1/16$, then the Julia set always contains an invariant *Cantor necklace.* A Cantor necklace is a set that is homeomorphic to the following subset of

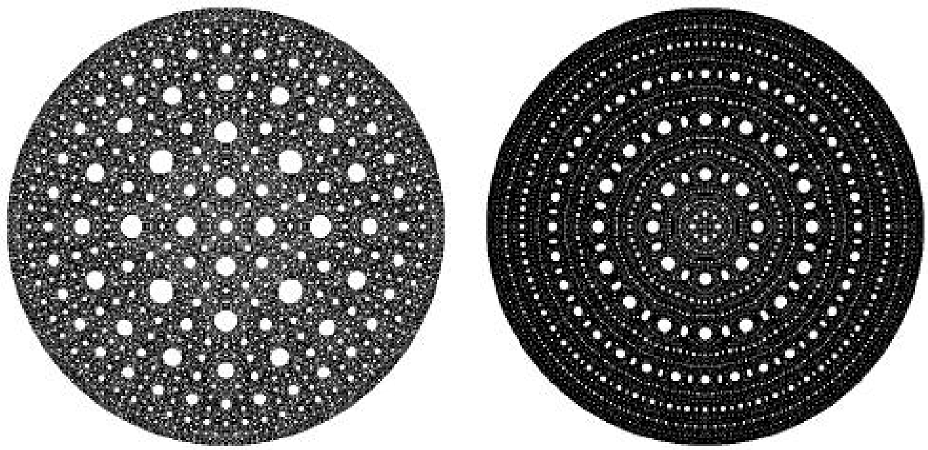

Figure 7: The Julia sets for $n = 2$ and $\lambda = -0.001$ and $\lambda = -0.00001$

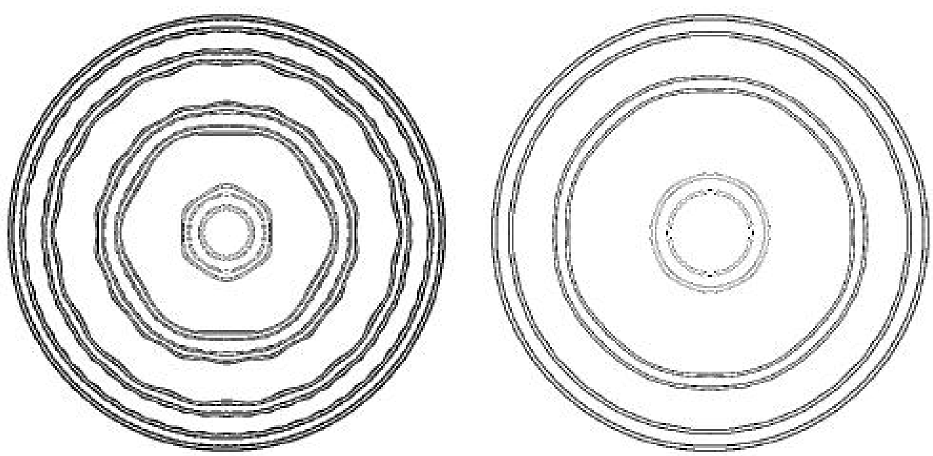

Figure 8: The Julia sets for $z^3 - 0.001/z^3$ and $z^4 - 0.001/z^4$ are both Cantor sets of circles.

the plane: Place the Cantor middle thirds set on the real axis. Then adjoin a circle of radius $1/3^j$ in place of each of the 2^j removed intervals at the j^{th} level of the construction of the Cantor middle thirds set. The union of the Cantor set and the adjoined circles is the model for the Cantor necklace. See Figure 9. We remark that this result holds for any λ for which $J(F_\lambda)$ is connected, not just those with $|\lambda| < 1/16$. The only difference is that the boundaries of the open regions now need not be simple closed curves — they may just be the boundary of a disk (which need not be a simple closed curve).

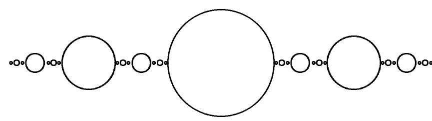

Figure 9: The Cantor middle-thirds necklace.

In the Julia set of F_λ, the invariant Cantor necklace has the following properties: the simple closed curve corresponding to largest circle in the model is the boundary of the trap door. All of the closed curves corresponding to the circles at level j correspond to the boundaries of preimages of ∂B_λ that map to this set after j iterations. The Cantor set portion of the necklace is an invariant set on which F_λ is hyperbolic and, in fact, conjugate to the one-sided shift map on two symbols. The two extreme points in this set correspond to a fixed point and its negative, both of which lie in ∂B_λ. Hence the Cantor necklace stretches completely "across" $J(F_\lambda)$. Moreover, it is known that the Cantor necklace is located in a particular subset of the Julia set. Specifically, let $c_0(\lambda)$ be the critical point of F_λ that lies in the sector $0 \leq \operatorname{Arg} z < \pi/2$ when $0 \leq \operatorname{Arg} \lambda < 2\pi$. Let c_j be the other critical points arranged in the clockwise direction around the origin as j increases. Let I_0 denote the sector bounded by the two critical point rays connecting the origin to ∞ and passing through c_0 and c_1. Let I_1 be the negative of this sector. Then, as shown in [4], the Cantor set portion of the necklace is the set of points whose orbits remain in $I_0 \cup I_1$ for all λ with $0 \leq \operatorname{Arg} \lambda < 2\pi$.

We saw earlier that, when λ is small, the boundary of B_λ is close to the unit circle, so $J(F_\lambda)$ is contained in a region close to the unit disk. We now show that, when $n = 2$, the Julia sets of F_λ actually converge to the closed unit disk $\mathbb{D}$ as $\lambda \to 0$. By converges to the unit disk we mean convergence in the Hausdorff metric:

Proposition. *Let $\epsilon > 0$ and denote the disk of radius ϵ centered at z by $B_\epsilon(z)$. There exists $\mu > 0$ such that, for any λ with $0 < |\lambda| \leq \mu$, $J(F_\lambda) \cap B_\epsilon(z) \neq \emptyset$ for all $z \in \mathbb{D}$.*

Proof: Suppose that this is not the case. Then, given any $\epsilon > 0$, we may find a sequence of parameters $\lambda_j \to 0$ and another sequence of points $z_j \in \mathbb{D}$ such that $J(F_{\lambda_j}) \cap B_{2\epsilon}(z_j) = \emptyset$ for each j. Since $\mathbb{D}$ is compact, there is a subsequence of the z_j that converges to some point $z^* \in \mathbb{D}$. This point $z^* \neq 0$ since one checks easily that T_λ shrinks to the origin as $\lambda \to 0$. For each parameter in the corresponding subsequence, we then have $J(F_{\lambda_j}) \cap B_\epsilon(z^*) = \emptyset$. Hence we may assume at the outset that we are dealing with a subsequence $\lambda_j \to 0$ such that $J(F_{\lambda_j}) \cap B_\epsilon(z^*) = \emptyset$.

Now consider the circle of radius $|z^*|$ centered at the origin. This circle meets $B_\epsilon(z^*)$ in an arc γ of length ℓ. Choose k so that $2^k \ell \geq 2\pi$.

Since $\lambda_j \to 0$, we may choose j large enough so that $|F^i_{\lambda_j}(z) - z^{2^i}|$ is very small for $1 \leq i \leq k$, provided z lies outside the circle of radius $|z^*|/2$ centered at the origin. In particular, it follows that $F^k_{\lambda_j}(\gamma)$ is a curve whose argument increases by approximately 2π, i.e., the curve $F^k_{\lambda_j}(\gamma)$ wraps at least once around the origin. As a consequence, the curve $F^k_{\lambda_j}(\gamma)$ must meet the Cantor necklace in the dynamical plane. But this necklace lies in $J(F_{\lambda_j})$. Hence $J(F_{\lambda_j})$ must intersect this curve. Since the Julia set is backward invariant, it follows that $J(F_{\lambda_j})$ must intersect $B_\epsilon(z^*)$. This then yields a contradiction, and so the result follows.

$\square$

Remarks:

1. A similar result concerning the convergence to the unit disk occurs in the family of maps $G_\lambda(z) = z^n + \lambda/z$. See [8]. The difference here is that the Julia sets only converge to the unit disk if λ approaches the origin along the straight rays given by

$$\operatorname{Arg} \lambda = \frac{(2k+1)\pi}{n-1}$$

In Figure 10 we display the parameter plane for the family $z^5 + \lambda/z$. Note that there are four accesses to the origin where the parameter plane is "interesting." It is along these rays that the Julia sets converge to the unit disk.

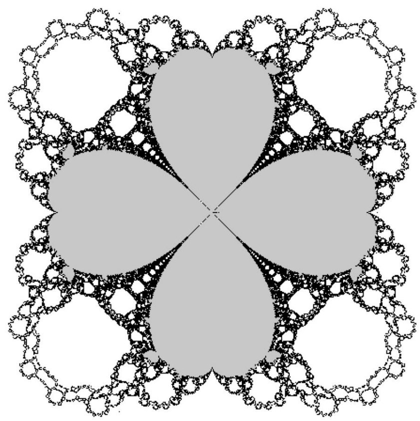

Figure 10: The parameter plane for $z^5 + \lambda/z$.

2. There has been some recent work on the convergence of other Julia sets to filled Julia sets of quadratic polynomials. For example, in [10], it is shown that, for the family

$$G_\lambda(z) = z^2 - 1 + \frac{\lambda}{z^2},$$

as $\lambda \to 0$, $J(G_\lambda)$ converges to the filled "basilica," i.e., the filled Julia set of the polynomial $z^2 - 1$. This same result goes over to other maps of the form $z^2 + c + \lambda/z^2$ where c is real and has the property that the polynomial $z^2 + c$ has a superattracting cycle. The case where c is a complex parameter with this property is still open.

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