Julia Sets Converging to Filled Quadratic Julia Sets *

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March 23, 2012

*This work was partially supported by grant #208780 to Robert L. Devaney from the Simons Foundation.
In this paper we consider singular perturbations of the quadratic polynomial $F(z) = z^2 + c$ where $c$ is the center of a hyperbolic component of the Mandelbrot set. These perturbations are obtained by replacing the critical point of $F$ with a pole. For reasons we will explain later, we will concentrate on the case where the pole has order two, as this is by far the most interesting case. So we will consider the family of maps

$$F_\lambda(z) = z^2 + c + \frac{\lambda}{z^2}. $$

Our goal is to investigate how the Julia sets of these maps evolve as $\lambda$ tends to 0.

When $\lambda = 0$, we have the quadratic polynomial $z^2 + c$ whose dynamics are completely understood. But as soon as $\lambda$ becomes non-zero, the degree of the map jumps to four and the Julia set of the map explodes. In Figure 1, we display the Julia set of $z^2 - 1$ (the basilica) as well as the Julia set for a singular perturbation of this map. Note that the outer boundary of the Julia set of the perturbed map is close to the Julia set of $z^2 - 1$, but there is a lot of other structure in the perturbed Julia set. Our main goal in this paper is to prove that, as $\lambda \to 0$, the Julia set of $F_\lambda$ converges in the Hausdorff topology to the filled Julia set of $z^2 + c$, i.e., the Julia set of $F$ together with all of its internal Fatou components.

In Figure 2, we display the filled Julia set known as the Douady rabbit (for the function $z^2 - 0.122 + 0.745i$) as well as its singular perturbation. Note that a similar phenomenon occurs.

The fact that Julia sets can converge to filled Julia sets of quadratic polynomials is somewhat surprising since it is well known that, if a Julia set contains an open set, then the Julia set must be the entire complex plane. Here we find Julia sets coming arbitrarily close to an open set that is not the entire plane. Of course, when $\lambda = 0$, the Julia set degenerates into the much
Figure 1: The Julia sets for $z^2 - 1 + \lambda/z^2$ where $\lambda = 0$ and $\lambda = -.00001$.

simpler Julia set of the unperturbed map, $z^2 + c$.

We choose $c$ to be the center of a hyperbolic component of the Mandelbrot set because the Julia sets are vastly different for all other values of $c$ in the interior of this component. When $c$ is chosen to be in a hyperbolic component of the Mandelbrot set (but not at its center), again the Julia set explodes when $\lambda$ becomes nonzero, but now the structure is much simpler (basically a countable collection of preimages of the original Julia set plus collections of Cantor sets accumulating on these sets). And it is known that these Julia sets do not converge to the filled Julia set of $z^2 + c$. See [10]. What happens when $c$ is chosen to be on the boundary of such a hyperbolic component of the Mandelbrot set is still not known.

In this paper we shall only consider the case where $c$ is the center of a hyperbolic component of period $n > 1$. The reason for this is that, in [6], it has been shown that the Julia sets of the maps $z^2 + \lambda/z^2$ converge to the closed unit disk (i.e., to the filled Julia set of $z^2$) as $\lambda \to 0$. In this case
the proof of convergence is much simpler. We also do not consider the case where $F$ is a higher degree polynomial of the form $z^k + c$. One reason for this is that, for the family of maps $G_\lambda(z) = z^k + \lambda/z^d$ with $k, d \geq 2$ but $k$ and $d$ not both equal to 2, then, as also shown in [6], the Julia sets of these maps do not converge to the unit disk. Indeed, for $\lambda$ sufficiently small, it is known [11] that the Julia sets of $G_\lambda$ are always Cantor sets of closed curves centered around the origin, and, using the results from [3], one of the complementary annuli must contain a round annulus of some given width. One other special case is the family $z^k + \lambda/z$ where $k \geq 2$. For this family we again have that the Julia sets converge to the unit disk, but this only occurs as $\lambda$ tends to 0 along $n-1$ special rays. See [8].

In [2] the family of maps of the form $z^n + c + \lambda/z^d$ with $n, d > 2$ and $c$ the center of a hyperbolic component of the Multibrot set is considered. There it is shown that the Julia sets also contain Cantor sets of closed curves when $\lambda$ is small just as in the case $c = 0$. The difference here is that countably
many of these curves have small, homeomorphic copies of the Julia set of the corresponding polynomial attached. Presumably there is an annulus of some given width in the Fatou set for these maps just as in the case where \( c = 0 \), but this is still an open question. The major difference between these families of maps and the family we consider (i.e., \( k = d = 2 \)), is that all Fatou components in our family are simply connected when the parameter \( \lambda \) is sufficiently small; that is, there are no annuli in the Fatou set. We then show that the size of all these disks in the Fatou set shrinks to zero as \( \lambda \to 0 \).

1 Preliminaries

We consider the family of maps

\[
F_\lambda(z) = z^2 + c + \frac{\lambda}{z^2}
\]

where \( c \) is a fixed parameter that lies at the the center of a hyperbolic component with period \( n > 1 \) in the Mandelbrot set, i.e., a parameter for which the orbit of the critical point 0 of \( z^2 + c \) is periodic with period \( n \). We will generally choose the parameter \( \lambda \) to be close to 0 so that \( F_\lambda \) is a singular perturbation of the quadratic polynomial \( z^2 + c \). We denote the unperturbed map (when \( \lambda = 0 \)) by \( F \).

These maps each have four free critical points located at \( c_\lambda = \lambda^{1/4} \) when \( \lambda \neq 0 \). There are two other critical points at \( \infty \) and at 0, but these are not free since \( \infty \) is fixed and 0 is mapped immediately to \( \infty \). There are only two free critical values for \( F_\lambda \) which are given by \( \pm v_\lambda = c \pm 2\sqrt{\lambda} \); two of the critical points are mapped to \( +v_\lambda \), the other two to \( -v_\lambda \).

The point at \( \infty \) is a superattracting fixed point since \( F_\lambda \approx z^2 + c \) near \( \infty \). Hence we have an immediate basin of \( \infty \) which we denote by \( B_\lambda \). Since 0 is a pole, there is an open set containing 0 that is mapped into \( B_\lambda \). If this
set is disjoint from $B_\lambda$ (which it is when $\lambda$ is small), we call this set the trap door and denote it by $T_\lambda$.

The Julia set of $F_\lambda$, denoted by $J(F_\lambda)$, is the set of points in the plane at which the family of iterates of $F_\lambda$ is not a normal family in the sense of Montel. Equivalently, $J(F_\lambda)$ is the closure of the set of repelling periodic points of $F_\lambda$ and also the set of points on which $F_\lambda$ behaves chaotically. The Fatou set is the complement of the Julia set. Both $B_\lambda$ and $T_\lambda$ lie in the Fatou set.

There are several symmetries in the dynamical planes of these maps. First, we have $F_\lambda(-z) = F_\lambda(z)$, so $J(F_\lambda), B_\lambda,$ and $T_\lambda$ are all symmetric under $z \mapsto -z$. Second, let $H_\lambda$ be one of the two involutions given by $z \mapsto \pm \sqrt{\lambda}/z$. Then we have $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, so $J(F_\lambda)$ is also symmetric under the involution $H_\lambda$. Note that $H_\lambda$ interchanges $B_\lambda$ and $T_\lambda$.

The circle surrounding the origin given by $|z| = |\lambda|^{1/4}$ is called the critical circle and is denoted by $C_\lambda$. This circle is mapped four-to-one onto the straight line connecting $\pm v_\lambda = c \pm 2\sqrt{\lambda}$ and passing through $c$. One checks easily that any other circle centered at the origin is mapped two-to-one onto an ellipse centered at $c$ with foci at $\pm v_\lambda$.

When $\lambda = 0$ and $c$ is the center of a hyperbolic component of the Mandelbrot set, the Julia set is well understood since the map $F(z) = z^2 + c$ is hyperbolic on $J(F)$. It is well known that $J(F)$ is a connected set which is the boundary the immediate basin of attraction of $\infty$. For $\lambda$ close to 0, it can be shown exactly as in [6] using a holomorphic motions argument that the boundary of $B_\lambda$, denoted by $\partial B_\lambda$, is homeomorphic to the Julia set of $J(F)$ and that $\partial B_\lambda$ varies analytically with $\lambda$. As a consequence, for these $\lambda$-values, both $B_\lambda$ and $T_\lambda$ are open, simply connected sets in the Riemann sphere.
2 Behavior of the Critical Orbits

The most important part of the proof of the convergence of $J(F_\lambda)$ to the filled Julia set of $F$ involves the fact that all the components of the Fatou set are simply connected, at least when $\lambda$ is small. As mentioned earlier, non-simply connected components do arise in other singularly perturbed families. For example, if $k > 2$, then it is known that the Julia set of $z^k + \lambda/z^k$ is a Cantor set of simple closed curves surrounding the origin when $|\lambda|$ is small [11]. So there are infinitely many Fatou components in this case that are annuli. For the family $z^k + c + \lambda/z^k$ where again $k > 2$ and $c$ is the center of the Multibrot set, a similar situation arises for $|\lambda|$ small. Here we again have a Cantor set of curves surrounding the origin, but countably many of them have “decorations,” i.e., infinitely many small copies of $\partial B_\lambda$ are attached. Still, there are infinitely many components of the Fatou set that are annuli. See [2].

What causes this type of behavior for $z^k + \lambda/z^k$ is that, when $k > 2$ and $|\lambda|$ is small, the critical values all lie in the trap door and hence the second iterate of the critical points all lie in the immediate basin of $\infty$. In order to eliminate this type of behavior in our family, we therefore have to show that the $n^{th}$ iterates of the critical points do not lie in the trap door. So our goal in this section is to describe the behavior of the critical orbits of $F_\lambda$, at least when $|\lambda|$ is sufficiently close to 0.

For the special case of $z^2 + \lambda/z^2$, this is easy. One computes that the second iterate of each critical point is given by $4\lambda + 1/4$, so the second iterate of the critical points tends to $1/4$ as $\lambda \to 0$. Therefore these second iterates are definitely not in $B_\lambda$ (which is approximately the exterior of the unit disk) when $\lambda$ is small, and so the first iterates of the critical points do not lie in $T_\lambda$. 
In our case we need a more complicated calculation to show that the $n^{th}$ iterates of the critical points are not in the trap door. Recall that, when $|\lambda|$ is small, $\partial B_\lambda$ is a homeomorphic copy of the Julia set of $F$ that varies analytically with $\lambda$. For the remainder of this paper we assume that $\lambda$ is chosen so that this is the case. Let $D_0$ denote the Fatou component in $\mathbb{C} - J(F)$ that contains 0 and let $D_j$ be the Fatou component that contains $F^j(0)$. Then there are analogous components $D_j^\lambda$ for $F_\lambda$ in $\mathbb{C} - \partial B_\lambda$ and each $D_j^\lambda$ contains $F^j(0)$. Note that, when $\lambda \neq 0$, $D_j^\lambda$ is no longer a Fatou component. Also note that $F_\lambda$ maps $\partial D_0^\lambda$ two-to-one onto $\partial D_1^\lambda$. Since the pole now lies in $D_0^\lambda$, there is another preimage of $\partial D_1^\lambda$ that lies inside $D_0^\lambda$; call this curve $\tau_\lambda$. $F_\lambda$ maps $\tau_\lambda$ two-to-one onto $\partial D_1^\lambda$, and $\tau_\lambda$ is the curve that surrounds $T_\lambda$ (though it is not the boundary of $T_\lambda$ since there are preimages of $\partial B_\lambda - \partial D_1^\lambda$ attached to the inside of $\tau_\lambda$). In particular, it follows that all four of the preimages of $\partial D_1^\lambda$ lie in $D_0^\lambda$.

In Figure 3, we display the Julia set of $z^2 - 1 - .001/z^2$ together with a magnification of the trap door. Note that the boundary of the trap door is a doubly inverted copy of the basilica.

Recall that the two critical values of $F_\lambda$ are given by $\pm v_\lambda = c \pm 2\sqrt{\lambda}$, so these critical values lie in $D_1^\lambda$ when $\lambda$ is small. We will be interested in the approximate location of the pair of points that are the $n^{th}$ images of the critical points. To determine these, we compute the approximate locations of the first $n - 1$ iterates of the orbit of $\pm v_\lambda$. We have

$$F_\lambda(c \pm 2\sqrt{\lambda}) = (c \pm 2\sqrt{\lambda})^2 + c + \frac{\lambda}{(c \pm 2\sqrt{\lambda})^2}$$
$$= c^2 + c \pm 4c\sqrt{\lambda} + O(\lambda)$$
$$= c^2 + c \pm (2c)(2\sqrt{\lambda}) + O(\lambda)$$
$$\approx F(c) + F'(F(0))(\pm 2\sqrt{\lambda})$$
Figure 3: The Julia set $z^2 - 1 - .001/z^2$ and a magnification showing in the center the region bounded by $\tau_\lambda$.

where, by assumption, $\lambda$ is close to 0. Continuing in this fashion, we compute

$$F^2_\lambda(c \pm 2\sqrt{\lambda}) \approx F^2(c) + F'(F(0))F''(F^2(0))(\pm 2\sqrt{\lambda})$$

and so forth until

$$F^{n-1}_\lambda(c \pm 2\sqrt{\lambda}) \approx F^{n-1}(c) + \prod_{i=1}^{n-1} F'(F^i(0))(\pm 2\sqrt{\lambda}).$$

Since $F^{n-1}(c) = 0$ and $F'(z) = 2z$, we therefore have

$$F^{n-1}_\lambda(c \pm 2\sqrt{\lambda}) \approx 2^{n-1}(\pm 2\sqrt{\lambda}) \prod_{i=1}^{n-1} F^i(0).$$

Thus it follows that the orbit of the critical values returns very close to 0 after $n - 1$ iterations.

So the question is whether or not these orbits now enter the trap door at this iteration. This in fact does not happen as we shall now show that the
next iterate, \( F^n_\lambda(v_\lambda) \), moves a bounded distance away from \( F(0) = c \) but still lies in \( D_\lambda^1 \), assuming \( \lambda \) is small enough.

To show this, we compute

\[
F^n_\lambda(v_\lambda) \approx \left( 2^{n-1} \left( \prod_{i=1}^{n-1} F^i(0) \right) (\pm 2\sqrt{\lambda}) \right)^2 + c + \frac{\lambda}{\left( 2^{n-1} \left( \prod_{i=1}^{n-1} F^i(0) \right) (\pm 2\sqrt{\lambda}) \right)^2} \\
\approx c + \frac{1}{\left( 2^n \prod_{i=1}^{n-1} F^i(0) \right)^2} \\
= c + \frac{\lambda}{4^n \left( \prod_{i=1}^{n-1} F^i(0) \right)^2}.
\]

Let

\[
\kappa = \left( 4^n \left( \prod_{i=1}^{n-1} F^i(0) \right)^2 \right)^{-1} \neq 0.
\]

So \( F^n_\lambda(c \pm 2\sqrt{\lambda}) \approx c + \kappa \). Thus, for small \( \lambda \) values, the \( n^{th} \) iterates of the critical values always end up a bounded distance away from \( c = F^n(c) \) and these iterates tend to \( c + \kappa \) as \( \lambda \to 0 \). For example, when \( c = -1 \), we have

\[
F^n_\lambda(-1 \pm 2\sqrt{\lambda}) \approx -1 + (2^4(-1)^2)^{-1} = -15/16.
\]

Next we claim that, despite the fact that \( F^n_\lambda(\pm v_\lambda) \) is bounded away from \( c \), these points nevertheless do lie in \( D_\lambda^1 \) when \( \lambda \) is small. As is well known, \( F^n \mid D_0 \) is analytically conjugate to \( z \mapsto z^2 \) on the unit disk \( \mathbb{D} \). Let this conjugacy be given by \( h: D_0 \to \mathbb{D} \) with \( h(0) = 0 \). Then we have \( h(F^n(z)) = (h(z))^2 \). Suppose \( h(z) = a_1 z + a_2 z^2 + \ldots \) so that \( h'(0) = a_1 \). We can compute \( a_1 \) explicitly. On the right side of the above conjugacy equation, the leading term in the expansion of \( (h(z))^2 \) is \( a_1^2 z^2 \). On the left side, we claim that the leading term is

\[
\left( \prod_{i=1}^{n-1} F'(F^i(0)) \right) a_1 z^2.
\]
To see this, we compute
\[ F^2(z) = (z^2 + c)^2 + c = c^2 + c + F'(c)z^2 + \text{h.o.t.}, \]
\[ F^3(z) = (c^2 + c)^2 + c + 4(c^2 + c)z^2 + \ldots = F^3(0) + F'(c)F'(F(c))z^2 + \ldots, \]
and continuing, we find
\[
F^n(z) = F^n(0) + F'(c) \cdot F'(F(c)) \cdots \cdot F'(F^{n-1}(c))z^2 + \ldots \\
= F^n(0) + \prod_{i=1}^{n-1} F'(F^i(0))z^2 + \ldots.
\]
Since \( F^n(0) = 0 \), the leading term in the expansion of \( h(F^n(z)) \) is thus given by
\[
\prod_{i=1}^{n-1} F'(F^i(0))a_1 z^2.
\]
Comparing these leading coefficients, we have
\[
\prod_{i=1}^{n-1} F'(F^i(0))a_1 = a_1^2
\]
so that
\[
h'(0) = a_1 = \prod_{i=1}^{n-1} F'(F^i(0)).
\]
Hence, for \( h^{-1}: \mathbb{D} \to D_0 \), we have
\[
(h^{-1})'(0) = \left( \prod_{i=1}^{n-1} (F'(F^i(0))) \right)^{-1}.
\]
Note that \( h^{-1} \) is the Riemann map from \( \mathbb{D} \) to the disk \( D_0 \).

We can construct a similar Riemann map for the disk \( D_1 \) since we know that \( F^{-1}: D_{j+1} \to D_j \) is an analytic homeomorphism for \( j = 1, 2, \ldots, n - 1 \). So we have \( F^{-1} \circ h^{-1}: \mathbb{D} \to D_1 \) is a univalent and surjective analytic map that takes 0 to \( c \). Call this map \( \Phi \). Then
\[
\Phi'(0) = (h^{-1})'(0) \cdot (F^{-1})'(0) \cdot (F^{-1})'(F^{-1}(0)) \cdots \cdot (F^{-1})'(F^{2}(0))
\]
\[
\begin{align*}
&= \left( \prod_{i=1}^{n-1} F'(F^i(0)) \right)^{-1} \cdot \frac{1}{F'(F^{n-1}(0))} \cdots \frac{1}{F'(F(0))} \\
&= \left( \prod_{i=1}^{n-1} F'(F^i(0)) \right)^{-2} \cdot 1 \\
&= \frac{1}{4^{n-1} \left( \prod_{i=1}^{n-1} F'(F^i(0)) \right)^2} = 4\kappa
\end{align*}
\]

So \( \Phi \) maps \( \mathbb{D} \) univalently onto \( D_1 \) and takes 0 to \( c \). Since \( \Phi'(0) = 4\kappa \), the Koebe 1/4 Theorem implies that \( D_1 \) must contain a round disk of radius greater than or equal to \( \kappa \). Since the Riemann map is not equivalent to the classical Koebe map, we in fact have that \( D_1 \) contains a round disk of radius strictly larger than \( \kappa \) and centered at \( c \). So, for \( \lambda \) small enough, \( c + \kappa \) must lie inside \( D_1^\lambda \). Therefore we have shown:

**Theorem:** If \( |\lambda| \) is small enough, then \( F_\lambda^n(\pm v_\lambda) \) both lie in the set \( D_1^\lambda \). As \( \lambda \to 0 \), \( F_\lambda^n(\pm v_\lambda) \to c + \kappa \). Consequently, the \( n \)th iterates of the critical points do not lie in \( T_\lambda \) for these \( \lambda \)-values.

In particular, it follows that \( F_\lambda^{n-1}(\pm v_\lambda) \) do not lie in \( T_\lambda \), so we do not have a Cantor set of closed and decorated curves in the Julia set.

### 3 Invariant Circles in the Julia Set

For later purposes, we need to construct an infinite collection of simple closed curves that surround the origin in \( D_0^\lambda \) and also lie in the Julia set. Recall that the critical circle for \( F_\lambda \) is the circle of radius \( |\lambda|^{1/4} \) centered at the origin. Let us denote the critical circle \( C_\lambda \) by \( C_0 \). As discussed earlier, \( F_\lambda \) maps \( C_0 \) onto the straight line segment connecting \( c + 2\sqrt{\lambda} \) and \( c - 2\sqrt{\lambda} \). Hence, when \( \lambda \) is small, we have \( F_\lambda^n(c_\lambda) \approx \alpha \sqrt{\lambda} \) for some constant \( \alpha \). So \( |F_\lambda^n(c_\lambda)| \) tends to zero faster than \( |\lambda|^{1/4} \) as \( \lambda \to 0 \). Therefore we may choose \( |\lambda| \) small enough so that \( F_\lambda^n(C_0) \) lies strictly inside the critical circle.
Consequently, $F^n_\lambda$ maps the region in $D^\lambda_0$ that lies in the exterior of $C_0$ as a two-to-one covering over itself. Therefore there is a curve $C_1$ lying in $D^\lambda_0$ but outside $C_0$ that is mapped two-to-one onto $C_0$ by $F^n_\lambda$. Then the region in $D^\lambda_0$ in the exterior of $C_1$ is mapped as a two-to-one covering over the exterior of $C_1$ in $D^\lambda_0$, so there exists another curve $C_2$ lying in $D^\lambda_0$ but outside $C_1$ that is mapped two-to-one onto $C_1$. Continuing in this fashion, we find a sequence of closed curves $C_j, j \geq 1$ having the property that $C_j$ lies outside $C_{j-1}$ and is mapped two-to-one onto $C_{j-1}$. Now let $C_{-j} = H_\lambda(C_j)$. Then we have $F^n_\lambda(C_{-j}) = F^n_\lambda(C_j) = C_{j-1}$ and $C_{-j-1}$ lies strictly inside $C_{-j}$ for $j = 0, 1, 2, \ldots$.

**Proposition.** The closed curves $C_j$ accumulate on at least some points in $\partial D^\lambda_0$ as $j \to \infty$ and, similarly, the curves $C_{-j}$ accumulate on some points on the curve $\tau_\lambda$ surrounding the trap door.

**Proof:** Suppose the $C_j$ do not accumulate on some points in $\partial D^\lambda_0$ as $j \to \infty$. Then these curves must accumulate on some set $\Lambda_\lambda$ which is necessarily invariant under $F^n_\lambda$ and surrounds the origin. Hence there is an open domain contained between $\partial D^\lambda_0$ and $\Lambda_\lambda$ which is invariant under $F^n_\lambda$ and hence lies in the Fatou set. This open domain is an annulus. Now this Fatou component contains no critical points of $F^n_\lambda$ and so cannot be an attracting or parabolic domain. Since the unperturbed map $F^n$ takes certain simple closed curves near $\partial D_0$ strictly inside themselves, the same must be true for $F^n_\lambda$ for $|\lambda|$ small. Hence it follows that this domain cannot be a Herman ring. Therefore there is no such Fatou domain and so the $C_j$ must in fact accumulate on some points in $\partial B_\lambda$. By symmetry, the same is true for the $C_{-j}$.

$\square$

From now on we assume that $\lambda$ is chosen even smaller so that $F^n_\lambda$ maps the critical circle to a curve that lies strictly inside the smaller circle $C_{-1}$.
Proposition. If $|\lambda|$ is sufficiently small, there is a closed curve in the Julia set that is invariant under $F^n_\lambda$ and that lies strictly between the curves $C_0$ and $C_{-1}$.

Proof: Let $A$ be the annulus bounded by $C_0$ and $C_{-1}$. Assuming $\lambda$ is such that $F^n_\lambda$ maps the critical circle strictly inside $C_{-1}$, it follows that there is a simple closed curve $\eta_0$ that lies in $A$, wraps once around $A$, and is mapped two-to-one onto $C_{-1}$. Since $F^n_\lambda$ maps $C_{-1}$ onto $C_0$ and hence outside $\eta_0$, there is another simple closed curve $\eta_1$ lying in the region between $C_{-1}$ and $\eta_0$ that is mapped two-to-one onto $\eta_0$ by $F^n_\lambda$. Let $\tilde{A}$ denote the annular region bounded by $\eta_0$ and $\eta_1$. Note that $\tilde{A}$ is strictly contained inside $A$. Then $F^{2n}_\lambda$ maps $\tilde{A}$ as a four-to-one covering of the annulus $A$ with $\eta_0$ mapped to $C_0$ and $\eta_1$ to $C_{-1}$. Then standard arguments involving quasiconformal surgery show that the set of points whose orbits remain for all time in $\tilde{A}$ under iteration of $F^{2n}_\lambda$ is a quasicircle $\gamma_0$ that surrounds the origin. Moreover, $F^{2n}_\lambda$ is quasiconformally conjugate to $z \mapsto z^4$ on $\gamma_0$. Since $F^n_\lambda$ maps $\eta_0$ inside $\eta_1$ and $\eta_1$ to $\eta_0$, it follows that $F^n_\lambda$ is conjugate to $z \mapsto z^{-2}$ on $\gamma_0$.

□

Now we may construct a sequence of preimages of $\gamma_0$ much the same as the preimages $C_j$ of the critical circle. We have that $F^n_\lambda$ maps the annular region between $C_0$ and $C_1$ as a two-to-one covering of a region that contains the annulus $\tilde{A}$, so there is a simple closed curve $\gamma_1$ lying in this annulus that is mapped two-to-one onto $\gamma_0$. Similarly, there is another simple closed curve $\gamma_2$ lying between $C_1$ and $C_2$ and mapped two-to-one by $F^n_\lambda$ onto $\gamma_1$. Continuing, we find another sequence of closed curves $\gamma_j$ for $j > 0$ with $\gamma_j$ lying outside $\gamma_{j-1}$ and $F^n_\lambda$ maps $\gamma_j$ as a two-to-one covering of $\gamma_{j-1}$. Let $\gamma_{-j} = H_\lambda(\gamma_j)$. Then $F^n_\lambda$ maps $\gamma_{-j}$ two-to-one onto $\gamma_j$ (not $\gamma_{j-1}$, as was the case with the $C_j$’s).

Remark. Assuming that $F^n_\lambda$ maps the critical circle strictly inside $C_{-j}$ for
some $j \geq 2$, one can show that there is in fact an Cantor set of simple closed curves in the Julia set of $F_\lambda$ that is invariant under $F_\lambda^j$. See [5] for this construction for the family $z^k + \lambda/z^k$. The extension to $z^2 + c + \lambda/z^2$ then proceeds as above. We will not use this result in the sequel, however.

4 Convergence to the Filled Julia Set of $F$

In this section we prove that the Julia sets of $F_\lambda$ converge to the filled Julia set of $z^2 + c$ as $\lambda \to 0$. The main tool to be used in proving this convergence is the following result.

**Theorem.** If $\lambda$ is sufficiently small, then all of the Fatou components of $F_\lambda$ are simply connected.

In Figure 4 we display a several magnifications of the region $D_0^\lambda$ for different perturbations of the basilica. Here we see that all of the Fatou components appear to be very small disks.

![Figure 4](image_url)

Figure 4: A magnification of the central region $D_0^\lambda$ for $\lambda = -0.001$ and $\lambda = -0.00001$ showing that the Fatou components are small disks.
As there a number of different types of multiply connected Fatou domains (infinite connected basins of attraction, Herman rings, annular preimages of disks, etc.), we shall deal with each different type in a series of Propositions. We first show that all Fatou components that are preimages of $B_{\lambda}$ are simply connected. More generally, we have:

**Proposition:** Suppose $V$ is a simply connected Fatou domain of $F_{\lambda}$. Then all of the preimages of $V$ are also simply connected.

**Proof:** Suppose that $U$ is a preimage of $V$ that is not simply connected. We may assume without loss of generality that $F_{\lambda}(U) = V$. Also, since $V$ is simply connected, all of the forward images of $V$ under $F_{\lambda}^j$ are also simply connected, so $U$ is not a forward image of $V$ under some iterate of $F_{\lambda}$.

Let $N(U)$ be the number of boundary components of $U$, so $N(U) - 2 \geq 0$. Then the Riemann-Hurwitz formula says that

$$0 \leq N(U) - 2 = \# \text{ of critical points of } F_{\lambda} \mid U - \deg F_{\lambda} \mid U.$$

Since $\deg F_{\lambda} \mid U \geq 1$, the number of critical points in $U$ must be at least one. But then the degree of $F_{\lambda}$ on $U$ must be at least two, so the above formula implies that the number of critical points in $U$ must also be at least two.

Now, if the set $U$ surrounds the origin, then $U$ must be symmetric under $z \mapsto -z$. Therefore we must have two symmetrically located critical points $\pm c_{\lambda}$ in $U$, each of which is mapped to the same critical value by $F_{\lambda}$, and so the degree of $F_{\lambda} \mid U$ must in fact be four. Then the number of critical points in $U$ must also be four. But this cannot happen, since there must be at least one other critical point associated to the forward orbit of the Fatou domain $V$. But $F_{\lambda}$ only has four critical points, and they all lie in $U$, which is not a forward image of $V$.

If the set $U$ does not surround the origin, then there is another preimage of $V$ given by $-U$ and we have $U \cap -U = \emptyset$. But then both $U$ and $-U$
contain two critical points as above and so, again, all four critical points lie
in a preimage of the Fatou domain $V$ that is not periodic. This completes
the proof.

Thus we know that a multiply connected component of the Fatou set
cannot be a preimage of $B_\lambda$ when $|\lambda|$ is small. Hence it must be a different
type of eventually periodic domain. First of all, this domain cannot be a
Herman ring.

Proposition. The Fatou set of $F_\lambda$ never contains a Herman ring.
Proof: Suppose $U$ is a Herman ring in the Fatou set of $F_\lambda$. Then we claim
that one of the iterates of $U$ must be an annulus surrounding the origin.
If this does not happen, let $U_k$ be the union of $F_\lambda^k(U)$ together with the
complementary domain of $F_\lambda^k(U)$ that does not contain 0. So $U_k$ is a bounded,
open disk in $\mathbb{C}$. Then, for each $k$, since there are no poles in each $U_k$, the
complementary domain portion of $U_k$ is mapped to the corresponding domain
in $U_{k+1}$. Hence the family of maps $F_\lambda^k$ is a normal family on $U = U_0$. But
this means that $U$ cannot be a Herman ring since its internal boundary is in
the Julia set.

So suppose that $U$ is the Herman ring that surrounds the origin. There
must be a pair of points $z$ and $-z$ that lie in $U$. Then, by the $z \mapsto -z$
symmetry, we have $F_\lambda(z) = F_\lambda(-z)$. Hence $F_\lambda$ cannot be one-to-one on $U$
and so $U$ is not a Herman ring.

So let $U$ be some other type of periodic, multiply connected Fatou com-
ponent. Then $U$ cannot be a Siegel disk, so $U$ is either the basin of an
attracting or a parabolic cycle. As in the above proof, there is at least one
forward iterate of $U$ that surrounds the origin so we assume at the outset that
$U$ has this property. Thus $U$ must be symmetric under $z \mapsto -z$. Moreover
$U$ must lie in one of the annuli bounded by the simple closed curves $\gamma_j$ and $\gamma_{j+1}$ that lie in the Julia set and surround the origin. Denote this annulus by $A_j$.

If $j < 0$, then $F^m_\lambda$ maps $A_j$ as a two-to-one covering of $A_{j-1}$. If $j > 0$, then $F^m_\lambda$ maps $A_j$ as a two-to-one covering of $A_{j-1}$. And if $j = 0$, then $F^m_\lambda$ maps $A_0$ four-to-one onto the disk containing 0 and bounded by $\gamma_0$. Since the critical circle lies in $A_0$, we have that $A_0$ contains all of the critical points and prepoles of $F_\lambda$. Therefore, if $U$ lies in $A_j$ with $j \neq 0$, then $F^m_\lambda$ maps $U$ as a two-to-one covering onto its image, which also must surround the origin. Continuing in this manner, there must be a first integer $j$ such that $F^m_\lambda(U)$ lies in $A_0$, surrounds the origin, and hence is symmetric under $z \mapsto -z$.

There are then two possibilities: either $H_\lambda(F^m_\lambda(U)) = F^m_\lambda(U)$ or else $H_\lambda(F^m_\lambda(U)) \cap F^m_\lambda(U) = \emptyset$. In the second case we have that $F^m_\lambda$ is two-to-one on $F^m_\lambda(U)$. Thus there are no critical points in $F^m_\lambda(U)$, since, by the $z \mapsto -z$ symmetry, if there were one critical point in this region, its negative would also lie in the region, and so the map would be four-to-one on $F^m_\lambda(U)$. Now each $H_\lambda$ fixes a pair of critical points of the form $\pm c_\lambda$ and inverts $F^m_\lambda(U)$ about these points. So it follows that, in this second case, $F^m_\lambda(U)$ must lie either strictly inside all four critical points or strictly outside these points.

And its image under $H_\lambda$ then lies on the opposite side of the critical points.

Consequently, both of these sets are mapped as a two-to-one covering onto the region $F^{(j+1)m}_\lambda(U)$ that lies in some $A_k$ where $k < 0$ and, as before, $F^{(j+1)m}_\lambda(U)$ surrounds the origin. Therefore, we have that $F^{(j+1)m}_\lambda$ maps $U$ as a $2^{(j+1)m}$-fold covering onto its image that surrounds the origin.

We may then continue iterating $F^m_\lambda$ on $U$ and there must be some subsequent iterate for which $F^{kn}_\lambda(U)$ is a Fatou domain that again lies in $A_0$, surrounds the origin, but this time $H_\lambda(F^{kn}_\lambda(U)) = F^{kn}_\lambda(U)$. Hence $F^m_\lambda$ is now four-to-one on the set $F^{kn}_\lambda(U)$. So we may assume at the outset that $U$
is the Fatou domain that has this property.

Now $U$ has a unique complementary domain that contains the origin; let $\partial_{\text{in}}(U)$ denote the boundary of this complementary domain. We call this the inner boundary of $U$. Similarly, let $\partial_{\text{out}}(U)$, the outer boundary of $U$, be the boundary component of the complementary domain that contains $\infty$. Then the involutions $H_\lambda$ each interchange the inner and outer boundaries of $U$, and consequently $F^n_\lambda$ maps $\partial_{\text{in}}(U)$ and $\partial_{\text{out}}(U)$ to the same set.

Let $\mathcal{A}$ denote the open annulus bounded by $\partial_{\text{in}}(U)$ and $\partial_{\text{out}}(U)$. So $\mathcal{A}$ contains $U$ as well as (presumably) many other complementary domains. Then $F^n_\lambda(\mathcal{A})$ is bounded by $F^n_\lambda(\partial_{\text{in}}(U)) = F^n_\lambda(\partial_{\text{out}}(U))$. Since no other points in $\mathcal{A}$ can be mapped to this boundary curve, it follows that $F^n_\lambda(\mathcal{A})$ is a disk $\mathcal{D}$. Moreover $F^n_\lambda$ maps $\mathcal{A}$ four-to-one onto $\mathcal{D}$. Since we are assuming $\lambda$ is small, we have that the disk $\mathcal{D}$ lies well inside one of the annuli $\mathcal{A}_{-i}$ where $i > 0$ is large. By the Riemann-Hurwitz formula, we have

$$0 = \# \text{ of critical points in } \mathcal{A} - \deg F^n_\lambda|\mathcal{A}$$

and it therefore follows that all four critical points lie in $\mathcal{A}$. (As a remark, these critical points could lie in $U$ or they could lie in some domain in $\mathcal{A}$ that lies in the complement of $U$; however, they do all lie in $\mathcal{A}$.) Since $\partial_{\text{in}}(U)$ lies inside the four critical points and $\partial_{\text{out}}(U)$ lies outside, it follows that $F^n_\lambda(\partial_{\text{in}}(U))$ surrounds the origin. Hence the disk $\mathcal{D}$ contains the origin.

We now prove that such a multiply connected Fatou domain $U$ surrounding the origin does not exist. Let $U = U_0$. Let $U_1 = F^n_\lambda(U_0)$ be the Fatou component lying in $\mathcal{D}$. Let $V$ denote the component of the complement of $U_1$ in $\mathcal{D}$ that contains $T_\lambda$. Then there are two possibilities for the preimages of $V$. One possibility is that there are four disjoint closed sets $V_1, \ldots, V_4$ in $\mathcal{A}$ that are mapped to $V$ by $F^n_\lambda$. This happens when none of the critical points are mapped into $V$. The other possibility is that a pair of symmetric
critical points map into $V$ (but not into $T_{\lambda}$) in which case there are only two preimages of $V$. We shall deal with the first case; the situation in the second case is similar, though the numbers change a bit.

Let $B$ be the open set $\mathcal{A} - \cup V_j$. So $B$ is an open annulus with four holes removed and $F^n_{\lambda}(B)$ is an annulus surrounding the origin. Since $|\lambda|$ is small, this annulus lies inside the annulus $A_A$ which is well inside the critical circle. Hence $F^2_{\lambda}(B)$ is an annulus lying well outside the critical circle and near the boundary of the region $D_0$ and $F^n_{\lambda}$ maps $F^2_{\lambda}(B)$ two-to-one onto $F^2_{\lambda}(B)$ as earlier. Now applying $F^n_{\lambda}$ we see that the successive images of $F^n_{\lambda}(B)$ are always annuli and, at each iteration, $F^n_{\lambda}$ is two-to-one. Since we know that $U_0$ is a periodic domain, there must be a first integer $\ell$ such that $F^\ell_{\lambda}(B) = \mathcal{A}$. By choosing $|\lambda|$ very small, we may assume that $\ell$ is large. So $F^n_{\lambda}$ takes $B$ onto $\mathcal{A}$ with degree $\beta = 4 \cdot 2^\ell - 1$ since $F^n_{\lambda}$ is four-to-one on $B$ while each subsequent iteration of $F^n_{\lambda}$ is two-to-one.

Choose a continuous curve $\xi$ in the annulus $\mathcal{A}$ that connects the inner and outer boundaries of $\mathcal{A}$ and does not pass through any point on a critical orbit. Then the preimage of $\xi$ under $F^\ell_{\lambda}$ in $B$ consists of $\beta$ disjoint curves, each of which connects one of the four sets $V_j$ to either the inner or outer boundaries of $B$. By symmetry, there are exactly $\beta/8$ such preimages of $\xi$ that connect a given $V_j$ to the inner boundary of $B$ and the same number that connect the outer boundary of $B$ to each $V_j$. These preimages divide $B$ into $\beta - 4$ distinct regions. Four of these regions contain a critical point and hence these regions are mapped two-to-one onto $\mathcal{A}$; the remaining regions are mapped one-to-one onto $\mathcal{A}$ (except along the curve $\xi$). See Figure 5.

Now we may choose a curve $\gamma$ that lies in the Fatou component $U_0$ in $B$ and wraps once around $\mathcal{A}$. Then $\gamma$ passes through at least half of the regions bounded by the preimages of $\xi$. Therefore $F^\ell_{\lambda}(\gamma)$ is a closed curve in $U_0$ that now wraps at least $\beta/2$ times around $\mathcal{A}$. Continuing, we see that subsequent
images of \( \gamma \) always lie in \( U_0 \) but wrap more and more often about \( A \). But \( U_0 \) is a Fatou component so there must either be an attracting or parabolic fixed point for \( F_\lambda^n \) in \( U_0 \) and all points in \( \gamma \) must have orbits that tend to this point. Indeed, given a small neighborhood of this fixed point, since \( \gamma \) is compact, there must be an integer \( \alpha \) such that \( F_\lambda^{\alpha n}(\gamma) \) is contained in this neighborhood. This, however, contradicts the fact that iterates of \( F_\lambda^n \) wrap \( \gamma \) more and more about the annulus \( A \). This proves that there cannot be a multiply connected Fatou component for \( F_\lambda \).

\[ \square \]

We now prove the main Theorem of this paper.

**Theorem:** Let \( c \) be the center of a hyperbolic component of period \( n > 1 \) in the Mandelbrot set and let

\[ F_\lambda(z) = z^2 + c + \frac{\lambda}{z^2}. \]

As \( \lambda \to 0 \), the Julia sets of \( F_\lambda \) converge to the filled Julia set of the quadratic
polynomial $F(z) = z^2 + c$ in the Hausdorff metric.

**Proof:** We assume at the outset that $|\lambda|$ is small enough so that all of the previous results hold and, in particular, that $F_\lambda$ has no multiply connected Fatou components. Now suppose that the Julia sets of $F_\lambda$ do not converge to the filled Julia set of $F$ as $\lambda \to 0$. Then there exists a $\delta > 0$ and a sequence $\lambda_j \to 0$ such that the portion of the Fatou set of $F_{\lambda_j}$ lying inside $\partial B_{\lambda_j}$ contains an open disk of some fixed radius $\delta$ centered at some point $z_j$ for each $j$. Note that these open disks do not lie in the trap door as one checks easily that the diameter of $T_{\lambda}$ tends to 0 as $\lambda \to 0$.

Since the filled Julia set of $F$ is compact and $\partial B_{\lambda_j}$ converges to $J(F)$ as $\lambda \to 0$, we may find a subsequence of the $\lambda_j$ converging to 0 such that the corresponding points $z_j$ accumulate on some special point $z_*$ and so the Fatou set of $F_{\lambda_j}$ always contains an open disk of radius $\delta/2$ centered at $z_*$. So this open disk does not depend on the parameter $\lambda_j$; call this disk $D_*$. So we may assume without loss of generality that the entire sequence $\lambda_j$ has the property that the disk $D_*$ lies in the Fatou set of $F_{\lambda_j}$ for each $j$ and, by the above, does not contain the origin.

Now $z_*$ may not lie in the Fatou component $D_0$ of $F$ that surrounds the origin. However, there is a smallest integer $\ell$ such that $F^\ell(z_*)$ does lie in $D_0$. So too does $F^\ell(D_*)$. Then there exists $\epsilon > 0$ such that $F^\ell(D_*)$ contains a disk of fixed radius $\epsilon$ surrounding $F^\ell(z_*)$. Then, as above, this disk also does not contain the origin. For $j$ sufficiently large in our sequence, we have that $F_{\lambda_j}^\ell \approx F^\ell$ on $D_*$. Hence, we may assume that, for each sufficiently large $j$, there is a disk of radius $\epsilon/2$ surrounding $F^\ell(z_*)$ that lies in the Fatou set of each $F_{\lambda_j}$. Call this disk $\Omega$.

Now $F^n$ is conjugate to $z^2$ on the region $D_0$. Since the map $z \mapsto z^2$ doubles angles, any disk inside the unit circle that does not contain the origin is eventually mapped onto an annular region by some high iterate of $z^2$. Hence
some higher iterate of $F^n$ takes $\Omega$ onto an annular region surrounding the origin. By choosing $j$ even larger, since $F^n_{\lambda_j} \approx F^n$, we have that, for such $j$-values, $F^n_{\lambda_j}$ also maps $\Omega$ onto an annular region. It follows that, for these large $j$-values, the Fatou set of $F_{\lambda_j}$ always contains a multiply connected region surrounding the origin, which we have shown cannot happen. Therefore, there cannot be such a sequence of $\lambda$-values with large disks in the Fatou set, so $J(F_{\lambda})$ does indeed converge to the filled Julia set of $F$ as $\lambda \to 0$. This completes the proof.

References


