Julia Sets Converging to Filled Quadratic Julia Sets *

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In this paper we consider singular perturbations of the quadratic polynomial $F(z) = z^2 + c$ where c is the center of a hyperbolic component of the Mandelbrot set. These perturbations are obtained by replacing the critical point of F with a pole. For reasons we will explain later, we will concentrate on the case where the pole has order two, as this is by far the most interesting case. So we will consider the family of maps

$$F_\lambda(z) = z^2 + c + \frac{\lambda}{z^2}.$$

Our goal is to investigate how the Julia sets of these maps evolve as λ tends to 0.

When $\lambda = 0$, we have the quadratic polynomial $z^2 + c$ whose dynamics are completely understood. But as soon as λ becomes non-zero, the degree of the map jumps to four and the Julia set of the map explodes. In Figure 1, we display the Julia set of $z^2 - 1$ (the basilica) as well as the Julia set for a singular perturbation of this map. Note that the outer boundary of the Julia set of the perturbed map is close to the Julia set of $z^2 - 1$, but there is a lot of other structure in the perturbed Julia set. Our main goal in this paper is to prove that, as $\lambda \to 0$, the Julia set of F_λ converges in the Hausdorff topology to the filled Julia set of $z^2 + c$, i.e., the Julia set of F together with all of its internal Fatou components.

In Figure 2, we display the filled Julia set known as the Douady rabbit (for the function $z^2 - 0.122 + 0.745i$) as well as its singular perturbation. Note that a similar phenomenon occurs.

The fact that Julia sets can converge to filled Julia sets of quadratic polynomials is somewhat surprising since it is well known that, if a Julia set contains an open set, then the Julia set must be the entire complex plane. Here we find Julia sets coming arbitrarily close to an open set that is not the entire plane. Of course, when $\lambda = 0$, the Julia set degenerates into the much

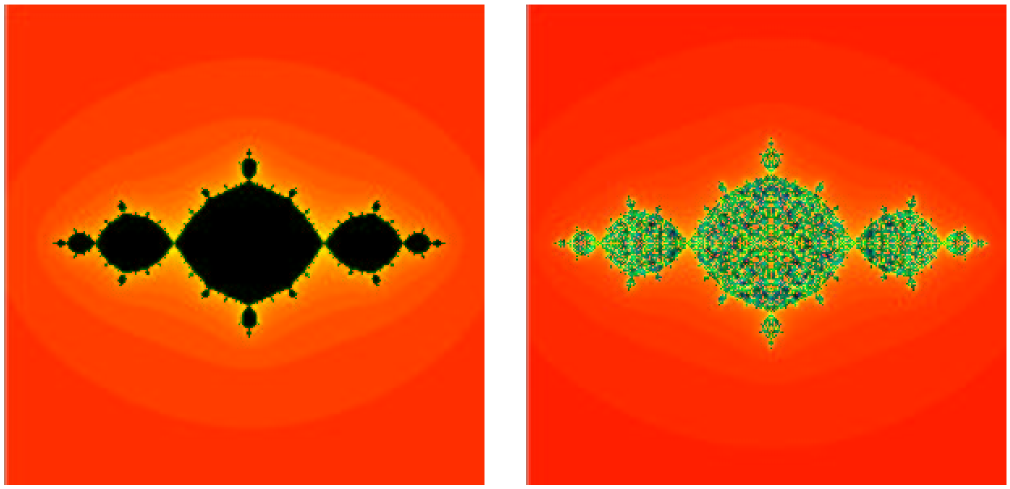


Figure 1: The Julia sets for $z^2 - 1 + \lambda/z^2$ where $\lambda = 0$ and $\lambda = -.00001$.

simpler Julia set of the unperturbed map, $z^2 + c$.

We choose c to be the center of a hyperbolic component of the Mandelbrot set because the Julia sets are vastly different for all other values of c in the interior of this component. When c is chosen to be in a hyperbolic component of the Mandelbrot set (but not at its center), again the Julia set explodes when λ becomes nonzero, but now the structure is much simpler (basically a countable collection of preimages of the original Julia set plus collections of Cantor sets accumulating on these sets). And it is known that these Julia sets do not converge to the filled Julia set of $z^2 + c$. See [10]. What happens when c is chosen to be on the boundary of such a hyperbolic component of the Mandelbrot set is still not known.

In this paper we shall only consider the case where c is the center of a hyperbolic component of period $n > 1$. The reason for this is that, in [6], it has been shown that the Julia sets of the maps $z^2 + \lambda/z^2$ converge to the closed unit disk (i.e., to the filled Julia set of z^2) as $\lambda \to 0$. In this case

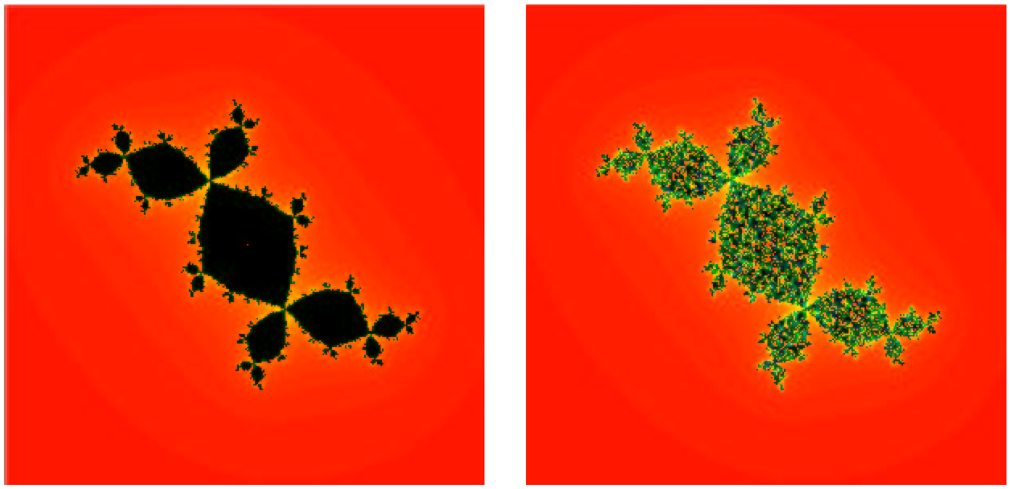


Figure 2: The Julia sets for $z^2 - 0.122 + 0.745i + \lambda/z^2$ where $\lambda = 0$ and $\lambda = -.000001$

the proof of convergence is much simpler. We also do not consider the case where F is a higher degree polynomial of the form $z^k + c$. One reason for this is that, for the family of maps $G_\lambda(z) = z^k + \lambda/z^d$ with $k, d \geq 2$ but k and d not both equal to 2, then, as also shown in [6], the Julia sets of these maps do not converge to the unit disk. Indeed, for λ sufficiently small, it is known [11] that the Julia sets of G_λ are always Cantor sets of closed curves centered around the origin, and, using the results from [3], one of the complementary annuli must contain a round annulus of some given width. One other special case is the family $z^k + \lambda/z$ where $k \geq 2$. For this family we again have that the Julia sets converge to the unit disk, but this only occurs as λ tends to 0 along $n - 1$ special rays. See [8].

In [2] the family of maps of the form $z^n + c + \lambda/z^d$ with $n, d > 2$ and c the center of a hyperbolic component of the Multibrot set is considered. There it is shown that the Julia sets also contain Cantor sets of closed curves when λ is small just as in the case $c = 0$. The difference here is that countably many of these curves have small, homeomorphic copies of the Julia set of the corresponding polynomial attached. Presumably there is an annulus of some given width in the Fatou set for these maps just as in the case where $c = 0$, but this is still an open question. The major difference between these families of maps and the family we consider (i.e., $k = d = 2$), is that all Fatou components in our family are simply connected when the parameter λ is sufficiently small; that is, there are no annuli in the Fatou set. We then show that the size of all these disks in the Fatou set shrinks to zero as $\lambda \to 0$.

1 Preliminaries

We consider the family of maps

$$F_\lambda(z) = z^2 + c + \frac{\lambda}{z^2}$$

where c is a fixed parameter that lies at the the center of a hyperbolic component with period $n > 1$ in the Mandelbrot set, i.e., a parameter for which the orbit of the critical point 0 of $z^2 + c$ is periodic with period n. We will generally choose the parameter λ to be close to 0 so that F_λ is a singular perturbation of the quadratic polynomial $z^2 + c$. We denote the unperturbed map (when $\lambda = 0$) by F.

These maps each have four free critical points located at $c_\lambda = \lambda^{1/4}$ when $\lambda \neq 0$. There are two other critical points at ∞ and at 0, but these are not free since ∞ is fixed and 0 is mapped immediately to ∞. There are only two free critical values for F_λ which are given by $\pm v_\lambda = c \pm 2\sqrt{\lambda}$; two of the critical points are mapped to $+v_\lambda$, the other two to $-v_\lambda$.

The point at ∞ is a superattracting fixed point since $F_\lambda \approx z^2 + c$ near ∞. Hence we have an immediate basin of ∞ which we denote by B_λ. Since 0 is a pole, there is an open set containing 0 that is mapped into B_λ. If this set is disjoint from B_λ (which it is when λ is small), we call this set the trap door and denote it by T_λ.

The Julia set of F_λ, denoted by $J(F_\lambda)$, is the set of points in the plane at which the family of iterates of F_λ is not a normal family in the sense of Montel. Equivalently, $J(F_\lambda)$ is the closure of the set of repelling periodic points of F_λ and also the set of points on which F_λ behaves chaotically. The Fatou set is the complement of the Julia set. Both B_λ and T_λ lie in the Fatou set.

There are several symmetries in the dynamical planes of these maps. First, we have $F_\lambda(-z) = F_\lambda(z)$, so $J(F_\lambda), B_\lambda$, and T_λ are all symmetric under $z \mapsto -z$. Second, let H_λ be one of the two involutions given by $z \mapsto \pm\sqrt{\lambda}/z$. Then we have $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, so $J(F_\lambda)$ is also symmetric under the involution H_λ. Note that H_λ interchanges B_λ and T_λ.

The circle surrounding the origin given by $|z| = |\lambda|^{1/4}$ is called the *critical circle* and is denoted by C_λ. This circle is mapped four-to-one onto the straight line connecting $\pm v_\lambda = c \pm 2\sqrt{\lambda}$ and passing through c. One checks easily that any other circle centered at the origin is mapped two-to-one onto an ellipse centered at c with foci at $\pm v_\lambda$.

When $\lambda = 0$ and c is the center of a hyperbolic component of the Mandelbrot set, the Julia set is well understood since the map $F(z) = z^2 + c$ is hyperbolic on $J(F)$. It is well known that $J(F)$ is a connected set which is the boundary the immediate basin of attraction of ∞. For λ close to 0, it can be shown exactly as in [6] using a holomorphic motions argument that the boundary of B_λ, denoted by ∂B_λ, is homeomorphic to the Julia set of $J(F)$ and that ∂B_λ varies analytically with λ. As a consequence, for these λ-values, both B_λ and T_λ are open, simply connected sets in the Riemann sphere.

2 Behavior of the Critical Orbits

The most important part of the proof of the convergence of $J(F_\lambda)$ to the filled Julia set of F involves the fact that all the components of the Fatou set are simply connected, at least when λ is small. As mentioned earlier, non-simply connected components do arise in other singularly perturbed families. For example, if $k > 2$, then it is known that the Julia set of $z^k + \lambda/z^k$ is a Cantor set of simple closed curves surrounding the origin when $|\lambda|$ is small [11]. So there are infinitely many Fatou components in this case that are annuli. For the family $z^k + c + \lambda/z^k$ where again $k > 2$ and c is the center of the Multibrot set, a similar situation arises for $|\lambda|$ small. Here we again have a Cantor set of curves surrounding the origin, but countably many of them have "decorations," i.e., infinitely many small copies of ∂B_λ are attached. Still, there are infinitely many components of the Fatou set that are annuli. See [2].

What causes this type of behavior for $z^k + \lambda/z^k$ is that, when $k > 2$ and $|\lambda|$ is small, the critical values all lie in the trap door and hence the second iterate of the critical points all lie in the immediate basin of ∞. In order to eliminate this type of behavior in our family, we therefore have to show that the n^{th} iterates of the critical points do not lie in the trap door. So our goal in this section is to describe the behavior of the critical orbits of F_λ, at least when $|\lambda|$ is sufficiently close to 0.

For the special case of $z^2 + \lambda/z^2$, this is easy. One computes that the second iterate of each critical point is given by $4\lambda + 1/4$, so the second iterate of the critical points tends to $1/4$ as $\lambda \to 0$. Therefore these second iterates are definitely not in B_λ (which is approximately the exterior of the unit disk) when λ is small, and so the first iterates of the critical points do not lie in T_λ.

In our case we need a more complicated calculation to show that the n^{th} iterates of the critical points are not in the trap door. Recall that, when $|\lambda|$ is small, ∂B_λ is a homeomorphic copy of the Julia set of F that varies analytically with λ. For the remainder of this paper we assume that λ is chosen so that this is the case. Let D_0 denote the Fatou component in $\mathbb{C} - J(F)$ that contains 0 and let D_j be the Fatou component that contains $F^j(0)$. Then there are analogous components D_j^λ for F_λ in $\mathbb{C} - \partial B_\lambda$ and each D_j^λ contains $F^j(0)$. Note that, when $\lambda \neq 0$, D_j^λ is no longer a Fatou component. Also note that F_λ maps ∂D_0^λ two-to-one onto ∂D_1^λ. Since the pole now lies in D_0^λ, there is another preimage of ∂D_1^λ that lies inside D_0^λ; call this curve τ_λ. F_λ maps τ_λ two-to-one onto ∂D_1^λ, and τ_λ is the curve that surrounds T_λ (though it is not the boundary of T_λ since there are preimages of $\partial B_\lambda - \partial D_1^\lambda$ attached to the inside of τ_λ). In particular, it follows that all four of the preimages of ∂D_1^λ lie in D_0^λ.

In Figure 3, we display the Julia set of $z^2 - 1 - .001/z^2$ together with a magnification of the trap door. Note that the boundary of the trap door is a doubly inverted copy of the basilica.

Recall that the two critical values of F_λ are given by $\pm v_\lambda = c \pm 2\sqrt{\lambda}$, so these critical values lie in D_1^λ when λ is small. We will be interested in the approximate location of the pair of points that are the n^{th} images of the critical points. To determine these, we compute the approximate locations of the first $n-1$ iterates of the orbit of $\pm v_\lambda$. We have

$$\begin{aligned}
F_\lambda(c \pm 2\sqrt{\lambda}) &= (c \pm 2\sqrt{\lambda})^2 + c + \frac{\lambda}{(c \pm 2\sqrt{\lambda})^2} \\
&= c^2 + c \pm 4c\sqrt{\lambda} + O(\lambda) \\
&= c^2 + c \pm (2c)(2\sqrt{\lambda}) + O(\lambda) \\
&\approx F(c) + F'(F(0))(\pm 2\sqrt{\lambda})
\end{aligned}$$

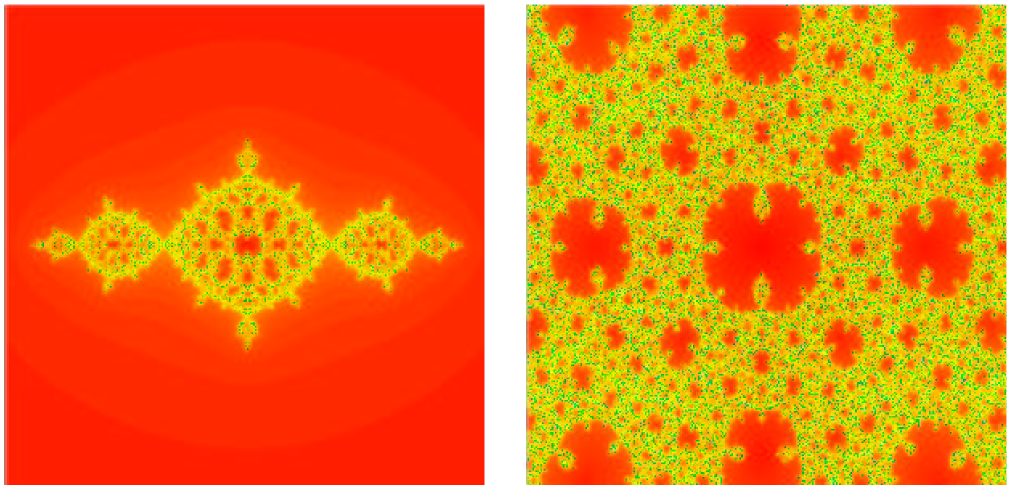


Figure 3: The Julia set $z^2 - 1 - .001/z^2$ and a magnification showing in the center the region bounded by τ_λ.

where, by assumption, λ is close to 0. Continuing in this fashion, we compute

$$F_\lambda^2(c \pm 2\sqrt{\lambda}) \approx F^2(c) + F'(F(0))F'(F^2(0))(\pm 2\sqrt{\lambda})$$

and so forth until

$$F_\lambda^{n-1}(c \pm 2\sqrt{\lambda}) \approx F^{n-1}(c) + \prod_{i=1}^{n-1} F'(F^i(0))(\pm 2\sqrt{\lambda}).$$

Since $F^{n-1}(c) = 0$ and $F'(z) = 2z$, we therefore have

$$F_\lambda^{n-1}(c \pm 2\sqrt{\lambda}) \approx 2^{n-1}(\pm 2\sqrt{\lambda}) \prod_{i=1}^{n-1} F^i(0).$$

Thus it follows that the orbit of the critical values returns very close to 0 after $n-1$ iterations.

So the question is whether or not these orbits now enter the trap door at this iteration. This in fact does not happen as we shall now show that the next iterate, $F_\lambda^n(v_\lambda)$, moves a bounded distance away from $F(0) = c$ but still lies in D_1^λ, assuming λ is small enough.

To show this, we compute

$$\begin{aligned} F_\lambda^n(v_\lambda) &\approx \left(2^{n-1}\left(\prod_{i=1}^{n-1} F^i(0)\right)(\pm 2\sqrt{\lambda})\right)^2 + c + \frac{\lambda}{\left(2^{n-1}\left(\prod_{i=1}^{n-1} F^i(0)\right)(\pm 2\sqrt{\lambda})\right)^2} \\ &\approx c + \frac{1}{\left(2^n \prod_{i=1}^{n-1} F^i(0)\right)^2} \\ &= c + \frac{1}{4^n\left(\prod_{i=1}^{n-1} F^i(0)\right)^2}. \end{aligned}$$

Let

$$\kappa = \left(4^n\left(\prod_{i=1}^{n-1} F^i(0)\right)^2\right)^{-1} \neq 0.$$

So $F_\lambda^n(c \pm 2\sqrt{\lambda}) \approx c + \kappa$. Thus, for small λ values, the n^{th} iterates of the critical values always end up a bounded distance away from $c = F^n(c)$ and these iterates tend to $c + \kappa$ as $\lambda \to 0$. For example, when $c = -1$, we have

$$F_\lambda^2(-1 \pm 2\sqrt{\lambda}) \approx -1 + (2^4(-1)^2)^{-1} = -15/16.$$

Next we claim that, despite the fact that $F_\lambda^n(\pm v_\lambda)$ is bounded away from c, these points nevertheless do lie in D_1^λ when λ is small. As is well known, $F^n \mid D_0$ is analytically conjugate to $z \mapsto z^2$ on the unit disk $\mathbb{D}$. Let this conjugacy be given by $h\colon D_0 \to \mathbb{D}$ with $h(0) = 0$. Then we have $h(F^n(z)) = (h(z))^2$. Suppose $h(z) = a_1 z + a_2 z^2 + \ldots$ so that $h'(0) = a_1$. We can compute a_1 explicitly. On the right side of the above conjugacy equation, the leading term in the expansion of $(h(z))^2$ is $a_1^2 z^2$. On the left side, we claim that the leading term is

$$\left(\prod_{i=1}^{n-1} F'(F^i(0))\right) a_1 z^2.$$

To see this, we compute

$$F^2(z) = (z^2+c)^2 + c = c^2 + c + F'(c)z^2 + \text{h.o.t.},$$

$$F^3(z) = (c^2+c)^2 + c + 4c(c^2+c)z^2 + \ldots = F^3(0) + F'(c)F'(F(c))z^2 + \ldots,$$

and continuing, we find

$$\begin{aligned} F^n(z) &= F^n(0) + F'(c)\cdot F'(F(c))\cdot \ldots \cdot F'(F^{n-1}(c))z^2 + \ldots \\ &= F^n(0) + \prod_{i=1}^{n-1} F'(F^i(0))z^2 + \ldots . \end{aligned}$$

Since $F^n(0) = 0$, the leading term in the expansion of $h(F^n(z))$ is thus given by

$$\prod_{i=1}^{n-1} F'(F^i(0))a_1 z^2.$$

Comparing these leading coefficients, we have

$$\prod_{i=1}^{n-1} F'(F^i(0))a_1 = a_1^2$$

so that

$$h'(0) = a_1 = \prod_{i=1}^{n-1} F'(F^i(0)).$$

Hence, for $h^{-1}\colon \mathbb{D} \to D_0$, we have

$$(h^{-1})'(0) = \left(\prod_{i=1}^{n-1}(F'(F^i(0))\right)^{-1}.$$

Note that h^{-1} is the Riemann map from $\mathbb{D}$ to the disk D_0.

We can construct a similar Riemann map for the disk D_1 since we know that $F^{-1} : D_{j+1} \to D_j$ is an analytic homeomorphism for $j = 1, 2, \ldots, n-1$. So we have $F^{-(n-1)} \circ h^{-1}\colon \mathbb{D} \to D_1$ is a univalent and surjective analytic map that takes 0 to c. Call this map Φ. Then

$$\Phi'(0) = (h^{-1})'(0) \cdot (F^{-1})'(0) \cdot (F^{-1})'(F^{n-1}(0)) \cdot \ldots \cdot (F^{-1})'(F^2(0))$$

$$
\begin{aligned}
&= \left(\prod_{i=1}^{n-1} F'(F^i(0))\right)^{-1} \cdot \frac{1}{F'(F^{n-1}(0))} \cdot \dots \cdot \frac{1}{F'(F(0))} \\
&= \left(\prod_{i=1}^{n-1} F'(F^i(0))\right)^{-2} \\
&= \frac{1}{4^{n-1}} \cdot \frac{1}{\left(\prod_{i=1}^{n-1} F^i(0)\right)^2} = 4\kappa
\end{aligned}
$$

So Φ maps $\mathbb{D}$ univalently onto D_1 and takes 0 to c. Since $\Phi'(0) = 4\kappa$, the Koebe 1/4 Theorem implies that D_1 must contain a round disk of radius greater than or equal to κ. Since the Riemann map is not equivalent to the classical Koebe map, we in fact have that D_1 contains a round disk of radius strictly larger than κ and centered at c. So, for λ small enough, $c + \kappa$ must lie inside D_1^λ. Therefore we have shown:

Theorem: *If $|\lambda|$ is small enough, then $F_\lambda^n(\pm v_\lambda)$ both lie in the set D_1^λ. As $\lambda \to 0$, $F_\lambda^n(\pm v_\lambda) \to c + \kappa$. Consequently, the n^{th} iterates of the critical points do not lie in T_λ for these λ-values.*

In particular, it follows that $F_\lambda^{n-1}(\pm v_\lambda)$ do not lie in T_λ, so we do not have a Cantor set of closed and decorated curves in the Julia set.

3 Invariant Circles in the Julia Set

For later purposes, we need to construct an infinite collection of simple closed curves that surround the origin in D_0^λ and also lie in the Julia set. Recall that the critical circle for F_λ is the circle of radius $|\lambda|^{1/4}$ centered at the origin. Let us denote the critical circle C_λ by C_0. As discussed earlier, F_λ maps C_0 onto the straight line segment connecting $c + 2\sqrt{\lambda}$ and $c - 2\sqrt{\lambda}$. Hence, when λ is small, we have $F_\lambda^n(c_\lambda) \approx \alpha\sqrt{\lambda}$ for some constant α. So $|F_\lambda^n(c_\lambda)|$ tends to zero faster than $|\lambda|^{1/4}$ as $\lambda \to 0$. Therefore we may choose $|\lambda|$ small enough so that $F_\lambda^n(C_0)$ lies strictly inside the critical circle.

Consequently, F_λ^n maps the region in D_0^λ that lies in the exterior of C_0 as a two-to-one covering over itself. Therefore there is a curve C_1 lying in D_0^λ but outside C_0 that is mapped two-to-one onto C_0 by F_λ^n. Then the region in D_0^λ in the exterior of C_1 is mapped as a two-to-one covering over the exterior of C_1 in D_0^λ, so there exists another curve C_2 lying in D_0^λ but outside C_1 that is mapped two-to-one onto C_1. Continuing in this fashion, we find a sequence of closed curves $C_j, j \geq 1$ having the property that C_j lies outside C_{j-1} and is mapped two-to-one onto C_{j-1}. Now let $C_{-j} = H_\lambda(C_j)$. Then we have $F_\lambda^n(C_{-j}) = F_\lambda^n(C_j) = C_{j-1}$ and C_{-j-1} lies strictly inside C_{-j} for $j = 0, 1, 2, \ldots$.

Proposition. *The closed curves C_j accumulate on at least some points in ∂D_0^λ as $j \to \infty$ and, similarly, the curves C_{-j} accumulate on some points on the curve τ_λ surrounding the trap door.*

Proof: Suppose the C_j do not accumulate on some points in ∂D_0^λ as $j \to \infty$. Then these curves must accumulate on some set Λ_λ which is necessarily invariant under F_λ^n and surrounds the origin. Hence there is an open domain contained between ∂D_0^λ and Λ_λ which is invariant under F_λ^n and hence lies in the Fatou set. This open domain is an annulus. Now this Fatou component contains no critical points of F_λ^n and so cannot be an attracting or parabolic domain. Since the unperturbed map F^n takes certain simple closed curves near ∂D_0 strictly inside themselves, the same must be true for F_λ^n for $|\lambda|$ small. Hence it follows that this domain cannot be a Herman ring. Therefore there is no such Fatou domain and so the C_j must in fact accumulate on some points in ∂B_λ. By symmetry, the same is true for the C_{-j}.

□

From now on we assume that λ is chosen even smaller so that F_λ^n maps the critical circle to a curve that lies strictly inside the smaller circle C_{-1}.

Proposition. *If $|\lambda|$ is sufficiently small, there is a closed curve in the Julia set that is invariant under F_λ^n and that lies strictly between the curves C_0 and C_{-1}.*

Proof: Let A be the annulus bounded by C_0 and C_{-1}. Assuming λ is such that F_λ^n maps the critical circle strictly inside C_{-1}, it follows that there is a simple closed curve η_0 that lies in A, wraps once around A, and is mapped two-to-one onto C_{-1}. Since F_λ^n maps C_{-1} onto C_0 and hence outside η_0, there is another simple closed curve η_1 lying in the region between C_{-1} and η_0 that is mapped two-to-one onto η_0 by F_λ^n. Let $\tilde{A}$ denote the annular region bounded by η_0 and η_1. Note that $\tilde{A}$ is strictly contained inside A. Then F_λ^{2n} maps $\tilde{A}$ as a four-to-one covering of the annulus A with η_0 mapped to C_0 and η_1 to C_{-1}. Then standard arguments involving quasiconformal surgery show that the set of points whose orbits remain for all time in $\tilde{A}$ under iteration of F_λ^{2n} is a quasicircle γ_0 that surrounds the origin. Moreover, F_λ^{2n} is quasiconformally conjugate to $z \mapsto z^4$ on γ_0. Since F_λ^n maps η_0 inside η_1 and η_1 to η_0, it follows that F_λ^n is conjugate to $z \mapsto z^{-2}$ on γ_0.

□

Now we may construct a sequence of preimages of γ_0 much the same as the preimages C_j of the critical circle. We have that F_λ^n maps the annular region between C_0 and C_1 as a two-to-one covering of a region that contains the annulus $\tilde{A}$, so there is a simple closed curve γ_1 lying in this annulus that is mapped two-to-one onto γ_0. Similarly, there is another simple closed curve γ_2 lying between C_1 and C_2 and mapped two-to-one by F_λ^n onto γ_1. Continuing, we find another sequence of closed curves γ_j for $j > 0$ with γ_j lying outside γ_{j-1} and F_λ^n maps γ_j as a two-to-one covering of γ_{j-1}. Let $\gamma_{-j} = H_\lambda(\gamma_j)$. Then F_λ^n maps γ_{-j} two-to-one onto γ_j (not γ_{j-1}, as was the case with the C_j's).

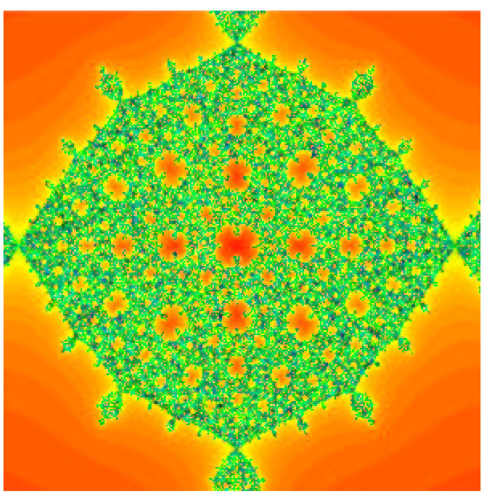
Remark. Assuming that F_λ^n maps the critical circle strictly inside C_{-j} for some $j \geq 2$, one can show that there is in fact an Cantor set of simple closed curves in the Julia set of F_λ that is invariant under F_λ^n. See [5] for this construction for the family $z^k + \lambda/z^k$. The extension to $z^2 + c + \lambda/z^2$ then proceeds as above. We will not use this result in the sequel, however.

4 Convergence to the Filled Julia Set of F

In this section we prove that the Julia sets of F_λ converge to the filled Julia set of $z^2 + c$ as $\lambda \to 0$. The main tool to be used in proving this convergence is the following result.

Theorem. *If λ is sufficiently small, then all of the Fatou components of F_λ are simply connected.*

In Figure 4 we display a several magnifications of the region D_0^λ for different perturbations of the basilica. Here we see that all of the Fatou components appear to be very small disks.



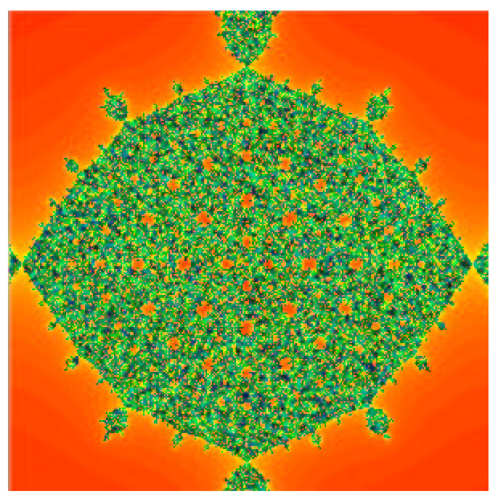


Figure 4: A magnification of the central region D_0^λ for $\lambda = -0.001$ and $\lambda = -0.00001$ showing that the Fatou components are small disks.

As there a number of different types of multiply connected Fatou domains (infinitely connected basins of attraction, Herman rings, annular preimages of disks, etc.), we shall deal with each different type in a series of Propositions. We first show that all Fatou components that are preimages of B_λ are simply connected. More generally, we have:

Proposition: *Suppose V is a simply connected Fatou domain of F_λ. Then all of the preimages of V are also simply connected.*

Proof: Suppose that U is a preimage of V that is not simply connected. We may assume without loss of generality that $F_\lambda(U) = V$. Also, since V is simply connected, all of the forward images of V under F_λ^j are also simply connected, so U is not a forward image of V under some iterate of F_λ.

Let $N(U)$ be the number of boundary components of U, so $N(U) - 2 \geq 0$. Then the Riemann-Hurwitz formula says that

$$0 \leq N(U) - 2 = \{\#\,\text{of critical points of}\, F_\lambda \,|\, U\} - \deg F_\lambda \,|\, U.$$

Since $\deg F_\lambda \,|\, U \geq 1$, the number of critical points in U must be at least one. But then the degree of F_λ on U must be at least two, so the above formula implies that the number of critical points in U must also be at least two.

Now, if the set U surrounds the origin, then U must be symmetric under $z \mapsto -z$. Therefore we must have two symmetrically located critical points $\pm c_\lambda$ in U, each of which is mapped to the same critical value by F_λ, and so the degree of $F_\lambda \,|\, U$ must in fact be four. Then the number of critical points in U must also be four. But this cannot happen, since there must be at least one other critical point associated to the forward orbit of the Fatou domain V. But F_λ only has four critical points, and they all lie in U, which is not a forward image of V.

If the set U does not surround the origin, then there is another preimage of V given by $-U$ and we have $U \cap -U = \emptyset$. But then both U and $-U$

contain two critical points as above and so, again, all four critical points lie in a preimage of the Fatou domain V that is not periodic. This completes the proof.

$\square$

Thus we know that a multiply connected component of the Fatou set cannot be a preimage of B_λ when $|\lambda|$ is small. Hence it must be a different type of eventually periodic domain. First of all, this domain cannot be a Herman ring.

Proposition. *The Fatou set of F_λ never contains a Herman ring.*

Proof: Suppose U is a Herman ring in the Fatou set of F_λ. Then we claim that one of the iterates of U must be an annulus surrounding the origin. If this does not happen, let U_k be the union of $F_\lambda^k(U)$ together with the complementary domain of $F_\lambda^k(U)$ that does not contain 0. So U_k is a bounded, open disk in $\mathbb{C}$. Then, for each k, since there are no poles in each U_k, the complementary domain portion of U_k is mapped to the corresponding domain in U_{k+1}. Hence the family of maps F_λ^k is a normal family on $U = U_0$. But this means that U cannot be a Herman ring since its internal boundary is in the Julia set.

So suppose that U is the Herman ring that surrounds the origin. There must be a pair of points z and $-z$ that lie in U. Then, by the $z \mapsto -z$ symmetry, we have $F_\lambda(z) = F_\lambda(-z)$. Hence F_λ cannot be one-to-one on U and so U is not a Herman ring.

$\square$

So let U be some other type of periodic, multiply connected Fatou component. Then U cannot be a Siegel disk, so U is either the basin of an attracting or a parabolic cycle. As in the above proof, there is at least one forward iterate of U that surrounds the origin so we assume at the outset that U has this property. Thus U must be symmetric under $z \mapsto -z$. Moreover

U must lie in one of the annuli bounded by the simple closed curves γ_j and γ_{j+1} that lie in the Julia set and surround the origin. Denote this annulus by A_j.

If $j < 0$, then F_λ^n maps A_j as a two-to-one covering of $A_{|j|-1}$. If $j > 0$, then F_λ^n maps A_j as a two-to-one covering of A_{j-1}. And if $j = 0$, then F_λ^n maps A_0 four-to-one onto the disk containing 0 and bounded by γ_0. Since the critical circle lies in A_0, we have that A_0 contains all of the critical points and prepoles of F_λ. Therefore, if U lies in A_j with $j \neq 0$, then F_λ^n maps U as a two-to-one covering onto its image, which also must surround the origin. Continuing in this manner, there must be a first integer j such that $F_\lambda^{jn}(U)$ lies in A_0, surrounds the origin, and hence is symmetric under $z \mapsto -z$.

There are then two possibilities: either $H_\lambda(F_\lambda^{jn}(U)) = F_\lambda^{jn}(U)$ or else $H_\lambda(F_\lambda^{jn}(U)) \cap F_\lambda^{jn}(U) = \emptyset$. In the second case we have that F_λ^n is two-to-one on $F_\lambda^{jn}U$. Thus there are no critical points in $F_\lambda^{jn}(U)$, since, by the $z \mapsto -z$ symmetry, if there were one critical point in this region, its negative would also lie in the region, and so the map would be four-to-one on $F_\lambda^{jn}(U)$. Now each H_λ fixes a pair of critical points of the form $\pm c_\lambda$ and inverts $F_\lambda^{jn}(U)$ about these points. So it follows that, in this second case, $F_\lambda^{jn}(U)$ must lie either strictly inside all four critical points or strictly outside these points. And its image under H_λ then lies on the opposite side of the critical points. Consequently, both of these sets are mapped as a two-to-one covering onto the region $F_\lambda^{(j+1)n}(U)$ that lies in some A_k where $k < 0$ and, as before, $F_\lambda^{(j+1)n}(U)$ surrounds the origin. Therefore, we have that $F_\lambda^{(j+1)n}$ maps U as a $2^{(j+1)n}$-fold covering onto its image that surrounds the origin.

We may then continue iterating F_λ^n on U and there must be some subsequent iterate for which $F_\lambda^{kn}(U)$ is a Fatou domain that again lies in A_0, surrounds the origin, but this time $H_\lambda(F_\lambda^{kn}(U)) = F_\lambda^{kn}(U))$. Hence F_λ^n is now four-to-one on the set $F_\lambda^{kn}(U)$. So we may assume at the outset that U

is the Fatou domain that has this property.

Now U has a unique complementary domain that contains the origin; let $\partial_{\text{in}}(U)$ denote the boundary of this complementary domain. We call this the inner boundary of U. Similarly, let $\partial_{\text{out}}(U)$, the outer boundary of U, be the boundary component of the complementary domain that contains ∞. Then the involutions H_λ each interchange the inner and outer boundaries of U, and consequently F_λ^n maps $\partial_{\text{in}}(U)$ and $\partial_{\text{out}}(U)$ to the same set.

Let $\mathcal{A}$ denote the open annulus bounded by $\partial_{\text{in}}(U)$ and $\partial_{\text{out}}(U)$. So $\mathcal{A}$ contains U as well as (presumably) many other complementary domains. Then $F_\lambda^n(\mathcal{A})$ is bounded by $F_\lambda^n(\partial_{\text{in}}(U)) = F_\lambda^n(\partial_{\text{out}}(U)$. Since no other points in $\mathcal{A}$ can be mapped to this boundary curve, it follows that $F_\lambda^n(\mathcal{A})$ is a disk $\mathcal{D}$. Moreover F_λ^n maps $\mathcal{A}$ four-to-one onto $\mathcal{D}$. Since we are assuming λ is small, we have that the disk $\mathcal{D}$ lies well inside one of the annuli A_{-i} where $i > 0$ is large. By the Riemann-Hurwitz formula, we have

$$0 = \{\# \text{ of critical points in } \mathcal{A}\} - \deg \mathrm{F}_\lambda^\mathrm{n} \,|\, \mathcal{A}$$

and it therefore follows that all four critical points lie in $\mathcal{A}$. (As a remark, these critical points could lie in U or they could lie in some domain in $\mathcal{A}$ that lies in the complement of U; however, they do all lie in $\mathcal{A}$.) Since $\partial_{\text{in}}(U)$ lies inside the four critical points and $\partial_{\text{out}}(U)$ lies outside, it follows that $F_\lambda^n(\partial_{\text{in}}(U)$ surrounds the origin. Hence the disk $\mathcal{D}$ contains the origin.

We now prove that such a multiply connected Fatou domain U surrounding the origin does not exist. Let $U = U_0$. Let $U_1 = F_\lambda^n(U_0)$ be the Fatou component lying in $\mathcal{D}$. Let V denote the component of the complement of U_1 in $\mathcal{D}$ that contains T_λ. Then there are two possibilities for the preimages of V. One possibility is that there are four disjoint closed sets $V_1, \ldots, V_4$ in $\mathcal{A}$ that are mapped to V by F_λ^n. This happens when none of the critical points are mapped into V. The other possibility is that a pair of symmetric critical points map into V (but not into T_λ) in which case there are only two preimages of V. We shall deal with the first case; the situation in the second case is similar, though the numbers change a bit.

Let B be the open set $\mathcal{A} - \cup V_j$. So B is an open annulus with four holes removed and $F_\lambda^n(B)$ is an annulus surrounding the origin. Since $|\lambda|$ is small, this annulus lies inside the annulus A_{-i} which is well inside the critical circle. Hence $F_\lambda^{2n}(B)$ is an annulus lying well outside the critical circle and near the boundary of the region D_0 and F_λ^n maps $F_\lambda^n(B)$ two-to-one onto $F_\lambda^{2n}(B)$ as earlier. Now applying F_λ^n we see that the successive images of $F_\lambda^n(B)$ are always annuli and, at each iteration, F_λ^n is two-to-one. Since we know that U_0 is a periodic domain, there must be a first integer ℓ such that $F_\lambda^{\ell n}(B) = \mathcal{A}$. By choosing $|\lambda|$ very small, we may assume that ℓ is large. So $F_\lambda^{\ell n}$ takes B onto $\mathcal{A}$ with degree $\beta = 4 \cdot 2^{\ell-1}$ since F_λ^n is four-to-one on B while each subsequent iteration of F_λ^n is two-to-one.

Choose a continuous curve ξ in the annulus $\mathcal{A}$ that connects the inner and outer boundaries of $\mathcal{A}$ and does not pass through any point on a critical orbit. Then the preimage of ξ under $F_\lambda^{\ell n}$ in B consists of β disjoint curves, each of which connects one of the four sets V_j to either the inner or outer boundaries of B. By symmetry, there are exactly $\beta/8$ such preimages of ξ that connect a given V_j to the inner boundary of B and the same number that connect the outer boundary of B to each V_j. These preimages divide B into $\beta - 4$ distinct regions. Four of these regions contain a critical point and hence these regions are mapped two-to-one onto $\mathcal{A}$; the remaining regions are mapped one-to-one onto A (except along the curve ξ). See Figure 5.

Now we may choose a curve γ that lies in the Fatou component U_0 in B and wraps once around $\mathcal{A}$. Then γ passes through at least half of the regions bounded by the preimages of ξ. Therefore $F_\lambda^{\ell n}(\gamma)$ is a closed curve in U_0 that now wraps at least $\beta/2$ times around A. Continuing, we see that subsequent

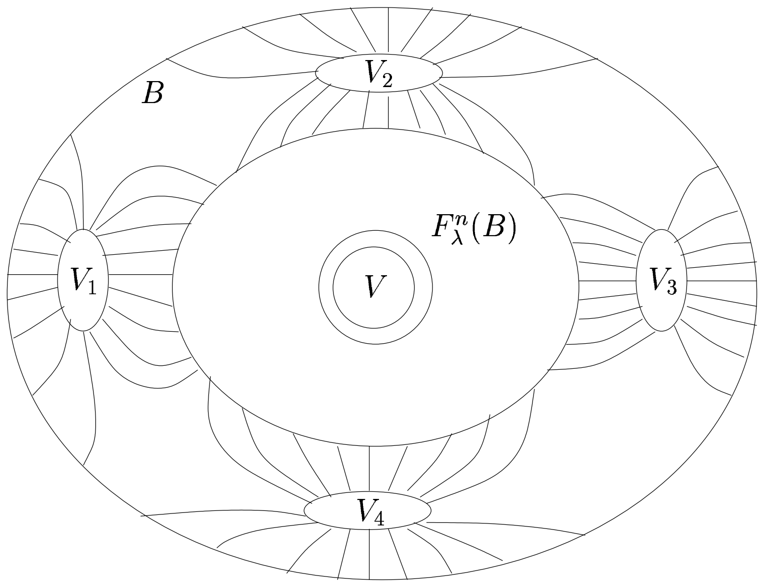


Figure 5: The region B and the preimages of ξ.

images of γ always lie in U_0 but wrap more and more often about $\mathcal{A}$. But U_0 is a Fatou component so there must either be an attracting or parabolic fixed point for $F_\lambda^{\ell n}$ in U_0 and all points in γ must have orbits that tend to this point. Indeed, given a small neighborhood of this fixed point, since γ is compact, there must be an integer α such that $F_\lambda^{\alpha \ell n}(\gamma)$ is contained in this neighborhood. This, however, contradicts the fact that iterates of $F_\lambda^{\ell n}$ wrap γ more and more about the annulus A. This proves that there cannot be a multiply connected Fatou component for F_λ.

□

We now prove the main Theorem of this paper.

Theorem: *Let c be the center of a hyperbolic component of period $n > 1$ in the Mandelbrot set and let*

$$F_\lambda(z) = z^2 + c + \frac{\lambda}{z^2}.$$

As $\lambda \to 0$, the Julia sets of F_λ converge to the filled Julia set of the quadratic *polynomial $F(z) = z^2 + c$ in the Hausdorff metric.*

Proof: We assume at the outset that $|\lambda|$ is small enough so that all of the previous results hold and, in particular, that F_λ has no multiply connected Fatou components. Now suppose that the Julia sets of F_λ do not converge to the filled Julia set of F as $\lambda \to 0$. Then there exists a $\delta > 0$ and a sequence $\lambda_j \to 0$ such that the portion of the Fatou set of F_{λ_j} lying inside ∂B_{λ_j} contains an open disk of some fixed radius δ centered at some point z_j for each j. Note that these open disks do not lie in the trap door as one checks easily that the diameter of T_λ tends to 0 as $\lambda \to 0$.

Since the filled Julia set of F is compact and ∂B_{λ_j} converges to $J(F)$ as $\lambda \to 0$, we may find a subsequence of the λ_j converging to 0 such that the corresponding points z_j accumulate on some special point z_* and so the Fatou set of F_{λ_j} always contains an open disk of radius $\delta/2$ centered at z_*. So this open disk does not depend on the parameter λ_j; call this disk D_*. So we may assume without loss of generality that the entire sequence λ_j has the property that the disk D_* lies in the Fatou set of F_{λ_j} for each j and, by the above, does not contain the origin.

Now z_* may not lie in the Fatou component D_0 of F that surrounds the origin. However, there is a smallest integer ℓ such that $F^\ell(z_*)$ does lie in D_0. So too does $F^\ell(D_*)$. Then there exists $\epsilon > 0$ such that $F^\ell(D_*)$ contains a disk of fixed radius ϵ surrounding $F^\ell(z_*)$. Then, as above, this disk also does not contain the origin. For j sufficiently large in our sequence, we have that $F_\lambda^\ell \approx F^\ell$ on D_*. Hence, we may assume that, for each sufficiently large j, there is a disk of radius $\epsilon/2$ surrounding $F^\ell(z_*)$ that lies in the Fatou set of each F_{λ_j}. Call this disk Ω.

Now F^n is conjugate to z^2 on the region D_0. Since the map $z \mapsto z^2$ doubles angles, any disk inside the unit circle that does not contain the origin is eventually mapped onto an annular region by some high iterate of z^2. Hence some higher iterate of F^n takes Ω onto an annular region surrounding the origin. By choosing j even larger, since $F^n_{\lambda_j} \approx F^n$, we have that, for such j-values, $F^n_{\lambda_j}$ also maps Ω onto an annular region. It follows that, for these large j-values, the Fatou set of F_{λ_j} always contains a multiply connected region surrounding the origin, which we have shown cannot happen. Therefore, there cannot be such a sequence of λ-values with large disks in the Fatou set, so $J(F_\lambda)$ does indeed converge to the filled Julia set of F as $\lambda \to 0$. This completes the proof.

References

[1] Blanchard, P., Devaney, R. L., Look, D. M., Seal, P., and Shapiro, Y. Sierpinski Curve Julia Sets and Singular Perturbations of Complex Polynomials. *Ergodic Theory and Dynamical Systems* **25** (2005), 1047-1055.

[2] Blanchard, P., Devaney, R. L., Garijo, A., and Russell, E. D. A generalized Version of the McMullen Domain. *Int'l J. Bifurcation and Chaos* **18** (2008), 2309-2318.

[3] Ble, G., Douady, A., and Henriksen, C. Round Annuli.*Contemporary Mathematics.* **355** (2004), 71-76.

[4] Devaney, R. L. Cantor Necklaces and Structurally Unstable Sierpinski Curve Julia Sets for Rational Maps. *Qual. Theory Dynamical Systems* **5** (2006), 337-359.

[5] Devaney, R. L. Cantor Sets of Circles of Sierpinski Curve Julia Sets. *Ergodic Theory and Dynamical Systems* **27** (2007), 1525-1539.

[6] Devaney, R. L. and Garijo, A. Julia Sets Converging to the Unit Disk. *Proceedings of the Amer. Math. Soc.* **136** (2008), 981-988.

[7] Devaney, R. L., Look, D. M., and Uminsky, D. The Escape Trichotomy for Singularly Perturbed Rational Maps. *Indiana University Mathematics Journal* **54** (2005), 1621-1634.

[8] Devaney, R. L. and Morabito, M. Limiting Julia Sets for Singularly Perturbed Rational Maps. *Int'l J. Bifurcation and Chaos* **18** (2008), 3175-3181.

[9] Garijo, A., Marotta, S., and Russell, E. Singular Perturbations in the Quadratic Family with Multiple Poles. To appear.

[10] Marotta, S. Singular Perturbations in the Quadratic Family. *J. Difference Equations and Applications* **4** (2008), 581-595.

[11] McMullen, C. Automorphisms of Rational Maps. *Holomorphic Functions and Moduli*. Vol. 1. Math. Sci. Res. Inst. Publ. **10**. Springer, New York, 1988.

[12] Milnor, J. *Dynamics in One Complex Variable.* Vieweg, 1999.