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## A CRITERION FOR SIERPINSKI CURVE JULIA SETS FOR RATIONAL MAPS

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ABSTRACT. This paper presents a criterion for Sierpinski curve Julia sets in the family of rational maps  $F_\lambda(z) = z^2 + \lambda/z^2$ .

### 1. INTRODUCTION

In this paper we discuss the one-parameter family of rational maps given by  $F_\lambda(z) = z^2 + \lambda/z^2$  where  $\lambda \neq 0$  is a complex parameter. Our goal is to give a criterion for the Julia set of such a map to be a Sierpinski curve. A Sierpinski curve is a rather interesting topological space that is homeomorphic to the well known Sierpinski carpet fractal. The interesting topology arises from the fact that a Sierpinski curve contains a homeomorphic copy of any one-dimensional plane continuum. Hence any such set is a universal planar continuum.

When  $\lambda$  is small, this family of maps may be regarded as a singular perturbation of the map  $z \mapsto z^2$ . The Julia set of  $z^2$  is well understood: it is the unit circle in  $\overline{\mathbb{C}}$ , and the restriction of the map to the Julia set is just the angle doubling map on the circle. For  $\lambda \neq 0$ , the Julia set changes dramatically. In [1] it is shown that, in every neighborhood of  $\lambda = 0$  in the parameter plane, there are infinitely many disjoint open sets of parameters for which the Julia set is a Sierpinski curve. This result should be contrasted with the situation that occurs for the related family  $G_\lambda(z) = z^n + \lambda/z^m$  with  $1/n + 1/m < 1$ . McMullen [8] has shown that, provided  $\lambda$  is sufficiently small, the Julia set of  $G_\lambda$  is always a Cantor set of

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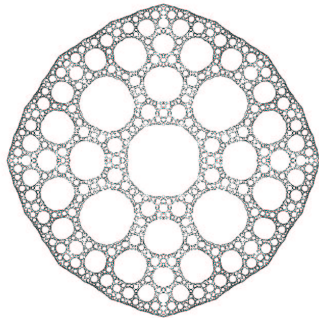
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circles. A dynamical criterion for this is given in [4]. On the other hand, Hawkins [7] has shown that very different phenomena arise in the family  $H_\lambda(z) = z + \lambda/z$ .

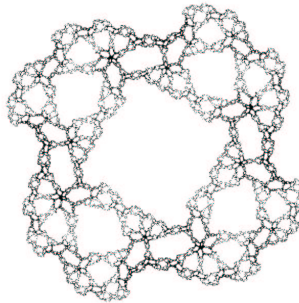
Our goal in this paper is to investigate the dynamics of the family  $F_\lambda$  for all  $\lambda$ -values, not just those close to the origin. Our main result is a criterion for the Julia set of  $F_\lambda$  to be a Sierpinski curve:

**Theorem.** *Suppose that the critical orbit of  $F_\lambda$  tends to  $\infty$  but the critical points of  $F_\lambda$  do not lie in the immediate basin of  $\infty$ . Then the Julia set of  $F_\lambda$  is a Sierpinski curve. In particular, any two Julia sets corresponding to an eventually escaping critical orbit are homeomorphic.*

In Figure 1, we display two Sierpinski curve Julia sets drawn from the family  $F_\lambda$ .



$$\lambda = -1/16$$



$$\lambda = 0.132 + 0.097i$$

FIGURE 1. The Sierpinski curve Julia sets for two values of  $\lambda$ .

We say *critical orbit* in this theorem because, despite the fact that this family consists of rational maps of degree 4, all of the free critical points for  $F_\lambda$  eventually land on the same orbit. This is reminiscent of the situation for the family of quadratic polynomials  $Q_c(z) = z^2 + c$ , where the orbit of the sole critical point 0 plays a significant role in determining the dynamics. As is well known, the Julia set of a quadratic polynomial is either a connected set or a Cantor set, and it is the behavior of the critical orbit that determines which case we are in. For if  $Q_c^n(0) \rightarrow \infty$ , then the Julia set of this map is a Cantor set, whereas if the orbit of 0 is

bounded, the Julia set is a connected set. This determines whether  $c$  lies outside or inside the Mandelbrot set [6]. For the family  $F_\lambda$ , we shall prove that there is a similar fundamental dichotomy, but there is a subtle but extremely important difference. Our result is:

**Theorem.** *If the entire critical orbit of  $F_\lambda$  lies in the immediate basin of attraction of  $\infty$ , then the Julia set of  $F_\lambda$  is a Cantor set. On the other hand, if the entire critical orbit does not lie in the immediate basin, then the Julia set is connected.*

The subtle difference here lies in our assumption that the entire critical orbit lies in the *immediate basin* of  $\infty$ . For quadratic polynomials, if the critical orbit escapes to  $\infty$ , then its entire orbit must lie in the immediate basin of  $\infty$ . However, for  $F_\lambda$ , it is possible that the critical orbit escapes to  $\infty$  but that the entire orbit does not lie in the immediate basin. That is, the critical points may lie in one of the (disjoint) preimages of the immediate basin, or, said another way, the critical orbit may jump around before entering the immediate basin of  $B$ . This is the case in which we find Sierpinski curve Julia sets.

There are other significant differences between the Julia sets of the family of rational maps and those of the quadratic polynomials. For example, in the case of connected quadratic Julia sets, it is often the case that the boundary of the basin at  $\infty$  has infinitely many pinch points. That is, the complement of the closure of the immediate basin of  $\infty$  consists of infinitely many disjoint open sets. For example, if  $Q_c$  admits an attracting periodic point of period  $n \geq 2$ , then the complement of the closure of the immediate basin of  $\infty$  always consists of infinitely many disjoint components made up of the various basins of attraction and their preimages. These are the Fatou components for the map.

For  $F_\lambda$ , a very different situation occurs. Let  $B$  denote the immediate basin of  $\infty$  for  $F_\lambda$ . Then we shall prove:

**Theorem.** *Suppose  $J(F_\lambda)$  is connected. Then  $\overline{\mathbb{C}} - \overline{B}$  is an open, connected, simply connected set.*

For “nice” simply connected open sets, the boundary of such sets is a simple closed curve, but as is well known, this need not be the case. For example, the topologists’ sine curve and other, non-locally connected sets may bound a simply connected open set in

the plane. In our case, however, we often have simple closed curves bounding the basin of  $\infty$ . We shall also show:

**Theorem.** *The boundary of the immediate basin of  $\infty$  is a simple closed curve in each of the following cases:*

- (1)  $|\lambda| < 1/16$ ;
- (2) *The critical orbits lie on the boundary of the basin of  $\infty$  but are preperiodic (the Misiurewicz case);*
- (3) *The critical points do not accumulate on the boundary of the basin of  $\infty$ , as in the special case where they eventually tend to  $\infty$  and we have a Sierpinski curve Julia set.*

## 2. PRELIMINARIES

In this section we describe some of the basic properties of the family  $F_\lambda(z) = z^2 + \lambda/z^2$  where, as always, we assume that  $\lambda \neq 0$ . Observe that  $F_\lambda(-z) = \overline{F_\lambda(z)}$  and  $F_\lambda(iz) = -F_\lambda(z)$  so that  $F_\lambda^2(iz) = F_\lambda^2(z)$  for all  $z \in \overline{\mathbb{C}}$ . Also note that 0 is the only pole for each function in this family. The points  $(-\lambda)^{1/4}$  are prepoles for  $F_\lambda$  since they are mapped directly to 0. The four critical points for  $F_\lambda$  occur at  $\lambda^{1/4}$ . Note that  $F_\lambda(\lambda^{1/4}) = \pm 2\lambda^{1/2}$  and  $F_\lambda^2(\lambda^{1/4}) = 1/4 + 4\lambda$ , so each of the four critical points lies on the same forward orbit after two iterations. We call the union of these orbits the *critical orbit* of  $F_\lambda$ .

Let  $J = J(F_\lambda)$  denote the *Julia set* of  $F_\lambda$ .  $J$  is the set of points at which the family of iterates of  $F_\lambda$  fails to be a normal family in the sense of Montel. Equivalently,  $J(F_\lambda)$  is the closure of the set of repelling periodic points of  $F_\lambda$  (see [9] for the basic properties of Julia sets).

The point at  $\infty$  is a superattracting fixed point for  $F_\lambda$ . Let  $B$  be the immediate basin of attraction of  $\infty$  and denote by  $\partial B$  the boundary of  $B$ . The map  $F_\lambda$  has degree 2 at  $\infty$  and so  $F_\lambda$  is conjugate to  $z \mapsto z^2$  on  $B$ , at least in a neighborhood of  $\infty$ . The basin  $B$  is a (forward) invariant set for  $F_\lambda$  in the sense that, if  $z \in B$ , then  $F_\lambda^n(z) \in B$  for all  $n \geq 0$ . The same is true for  $\partial B$ .

We denote by  $K = K(F_\lambda)$  the set of points whose orbit under  $F_\lambda$  is bounded.  $K$  is the *filled Julia set* of  $F_\lambda$ .  $K$  is given by  $\overline{\mathbb{C}} - \cup F^{-n}(B)$ . Both  $J$  and  $K$  are completely invariant subsets in the sense that if  $z \in J$  (resp.  $K$ ), then  $F_\lambda^n(z) \in J$  (resp.  $K$ ) for all  $n \in \mathbb{Z}$ . It is known that  $J(F_\lambda)$  is the boundary of  $K(F_\lambda)$  (see [9]).

**Proposition.** (Four-fold Symmetry) *The sets  $B$ ,  $\partial B$ ,  $J$ , and  $K$  are all invariant under  $z \mapsto iz$ .*

**Proof:** We prove this for  $B$ ; the other cases are similar. Let  $U = \{z \in B \mid iz \in B\}$ .  $U$  is an open subset of  $B$ . If  $U \neq B$ , there exists  $z_0 \in \partial U \cap B$ , where  $\partial U$  denotes the boundary of  $U$ . Hence  $z_0 \in B$  but  $iz_0 \in \partial B$ . It follows that  $F_\lambda^n(iz_0) \in \partial B$  for all  $n$ . But since  $F_\lambda^2(z_0) = F^2(iz_0)$ , it follows that  $z_0 \notin B$  as well. This contradiction establishes the result.  $\square$

There is a second symmetry present in this family. Consider the map  $H(z) = \sqrt{\lambda}/z$ . Note that we have two such maps depending upon which square root of  $\lambda$  we choose.  $H$  is an involution and we have  $F_\lambda(H(z)) = F_\lambda(z)$ . As a consequence,  $H$  preserves both  $J$  and  $K$ . The involution  $H$  also preserves the circle of radius  $\lambda^{1/4}$  and interchanges the interior and exterior of this circle. Hence both  $J$  and  $K$  are symmetric about this circle with respect to the action of  $H$ .

### 3. THE FUNDAMENTAL DICHOTOMY

We briefly recall a well known result for the family of quadratic polynomials  $Q_c(z) = z^2 + c$ . Each map  $Q_c$  has a single critical point at 0 and so, like  $F_\lambda$ ,  $Q_c$  has a single critical orbit. The fate of this orbit leads to the well known fundamental dichotomy for quadratic polynomials:

- (1) If  $Q_c^n(0) \rightarrow \infty$ , then  $J(Q_c)$  is a Cantor set;
- (2) But if  $Q_c^n(0) \not\rightarrow \infty$ , then  $J(Q_c)$  is a connected set.

The set of parameter values  $c$  for which the Julia set of  $Q_c$  is connected forms the well known Mandelbrot set. Our goal in this section is to prove a similar result in the case of  $F_\lambda$ .

Before stating this result, note that, unlike the quadratic case, there are two distinct ways that the critical orbit of  $F_\lambda$  may tend to  $\infty$ . One possibility is that one (and hence all) of the critical points lies in the immediate basin  $B$ . The other possibility is that these critical points lie in one of the preimages of  $B$  that is disjoint from  $B$ . For quadratic polynomials this second possibility does not occur.

Our goal in this section is to prove the fundamental dichotomy for the family  $F_\lambda$ :

**Theorem.** *If one and hence all of the critical points of  $F_\lambda$  lie in  $B$ , then  $J(F_\lambda)$  is a Cantor set; if the critical points of  $F_\lambda$  do not lie in  $B$ , then both  $J(F_\lambda)$  and  $K(F_\lambda)$  are compact, connected sets.*

**Proof:** Suppose first that no critical point lies in  $B$ . Then we may extend the conjugacy between  $F_\lambda$  and  $z^2$  to all of  $B$  and so  $B$  is a simply connected open set in  $\overline{\mathbb{C}}$ . Let  $U_0 = \overline{\mathbb{C}} - B$ .  $U_0$  is compact and connected with boundary  $\partial B$ . Let  $U_1 = U_0 - F_\lambda^{-1}(B)$ .  $F_\lambda^{-1}(B) - B$  is a simply connected open set containing 0 which is mapped two-to-one onto  $B$ . Hence  $F_\lambda^{-1}(B) - B$  lies in  $U_0$  and is disjoint from  $\partial B$  since orbits in  $\partial B$  remain bounded. Therefore  $U_1$  is compact and connected. Inductively,  $U_k$  is given by  $U_{k-1} - F_\lambda^{-k}(B)$ . Since  $F_\lambda^{-k}(B)$  is a collection of disjoint, simply connected, open sets which do not intersect the boundary of  $U_{k-1}$ , it follows that  $U_k$  is also compact and connected. Then  $K(F_\lambda) = \bigcap U_k$  is compact and connected. Since  $J$  is the boundary of  $K$ ,  $J$  too is compact and connected.

The proof that  $J(F_\lambda)$  is a Cantor set when all critical points lie in  $B$  is standard. See, for example, [9].

□

We emphasize again that the critical points for  $F_\lambda$  may eventually escape but not lie in  $B$ . In this case we still have a connected Julia set. In fact, we shall show in Section 5 that  $J(F_\lambda)$  is a Sierpinski curve in this case.

We denote the set of parameter values for which  $J(F_\lambda)$  is connected by  $\mathcal{M}$ ;  $\mathcal{M}$  is called the *connectedness locus* for this family. This set is the analogue of the Mandelbrot set for quadratic polynomials.

In the case where no critical points lie in  $B$ , we denote  $F_\lambda^{-1}(B) - B$  by  $T$ . Since  $F_\lambda|_B$  is only two to one,  $T$  is nonempty. Since 0 is a pole of order 2, it follows that  $T$  is an open set about 0 on which  $F_\lambda$  is two to one. We call  $T$  the *trap door*, since any orbit that enters  $T$  immediately “falls through” it and enters the basin at  $\infty$ . Just as in the four-fold symmetry proposition,  $T$  is invariant under  $z \mapsto iz$ . Also, the involution  $H$  interchanges  $B$  and  $T$ .

#### 4. THE CASE $|\lambda| < 1/16$

In this section we deal with the very special case where  $|\lambda| < 1/16$ . We first prove:

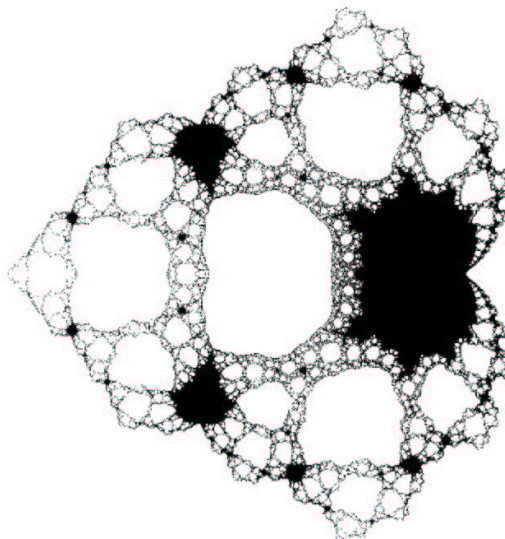


FIGURE 2. The parameter plane for the family  $z^2 + \lambda/z^2$ . White regions correspond to  $\lambda$ -values for which the critical orbit escapes to  $\infty$ .

**Theorem.** *Suppose that  $|\lambda| < 1/16$ . Then the boundary of  $B$  is a simple closed curve.*

**Proof:** Consider the *critical circle*  $S_\lambda$  given by  $r = |\lambda|^{1/4}$ . Note that  $S_\lambda$  contains all four critical points as well as the four prepoles. Write  $\lambda = \rho \exp(i\psi)$  and  $z = \rho^{1/4} \exp(i\theta) \in S_\lambda$ . Then we compute

$$\begin{aligned} F_\lambda(z) &= \rho^{1/2}(e^{2i\theta} + e^{i(\psi-2\theta)}) \\ &= \rho^{1/2}((\cos(2\theta) + \cos(\psi - 2\theta)) + i(\sin(2\theta) + \sin(\psi - 2\theta))) \end{aligned}$$

If we set  $x = \cos(2\theta) + \cos(\psi - 2\theta)$  and  $y = \sin(2\theta) + \sin(\psi - 2\theta)$ , then a computation shows that

$$\frac{d}{d\theta} \left( \frac{y}{x} \right) = 0.$$

Hence the image of  $S_\lambda$  under  $F_\lambda$  is a line interval passing through the origin.  $F_\lambda$  maps  $S_\lambda$  onto this line in four-to-one fashion, except at the two endpoints, which are the critical values  $\pm 2\sqrt{\lambda}$ . Note

that these two critical values lie inside  $S_\lambda$  provided we have

$$2|\sqrt{\lambda}| < |\lambda|^{1/4},$$

which occurs when  $|\lambda| < 1/16$ . Hence the condition  $|\lambda| < 1/16$  guarantees that the image of  $S_\lambda$  lies strictly inside  $S_\lambda$ .

Now if  $V_\lambda$  is another circle surrounding the origin whose radius is slightly larger than  $|\lambda|^{1/4}$ , then the image of  $V_\lambda$  also lies inside  $S_\lambda$  and hence inside  $V_\lambda$ . Moreover, the image of  $V_\lambda$  is a simple closed curve since there are no critical points or prepoles on  $V_\lambda$ . The involution  $H$  maps  $V_\lambda$  to a second circle  $W_\lambda$  that lies strictly inside the critical circle and we have  $F_\lambda(V_\lambda) = F_\lambda(W_\lambda)$ . The annular region between  $V_\lambda$  and  $W_\lambda$  is mapped in four-to-one fashion onto the disk surrounding the origin and bounded by  $F_\lambda(V_\lambda)$ . In particular, the image of this annulus is disjoint from the annulus provided that  $V_\lambda$  is sufficiently close to the critical circle.

We claim that the preimage of  $V_\lambda$  consists of a pair of disjoint simple closed curves, one lying inside the critical circle and one lying outside  $V_\lambda$ . This follows from the fact that  $F_\lambda$  maps the exterior of  $V_\lambda$  in two-to-one fashion onto the exterior of the curve  $F_\lambda(V_\lambda)$ . The interior of the circle  $W_\lambda$  is mapped in similar fashion onto the exterior of  $F_\lambda(V_\lambda)$ . Let  $U_\lambda$  denote the preimage of  $V_\lambda$  lying outside  $V_\lambda$ , and let  $A_\lambda$  denote the annular region bounded by  $V_\lambda$  and  $U_\lambda$ . Note that  $A_\lambda$  is mapped in two-to-one fashion onto the annulus bounded by  $V_\lambda$  and  $F_\lambda(V_\lambda)$ .

We now use quasiconformal surgery to modify  $F_\lambda$  to a new map  $G_\lambda$  which agrees with  $F_\lambda$  on the exterior of  $A_\lambda$  but which is conjugate to  $z \mapsto z^2$  in the interior of  $U_\lambda$  with a fixed point at the origin. To obtain  $G_\lambda$ , we first replace  $F_\lambda$  in the disk bounded by  $V_\lambda$  by a map which is a quasiconformal deformation of  $z \mapsto z^2$  on  $|z| < 1/2$ . Then we extend  $G_\lambda$  to  $A_\lambda$  so that the new map is quasiconformally conjugate to  $z^2$  on and inside  $A_\lambda$  and agrees with  $F_\lambda$  on the outer boundary  $U_\lambda$  of  $A_\lambda$ . The new map  $G_\lambda$  is continuous and has degree 2 with two superattracting fixed points, one at 0 and one at  $\infty$ . Hence  $G_\lambda$  is everywhere conjugate to  $z^2$ . Therefore the boundary of the basin of attraction of  $\infty$  for  $G_\lambda$  is a simple closed curve. Since  $G_\lambda$  agrees with  $F_\lambda$  in the exterior of  $A_\lambda$ , the same is true for  $F_\lambda$ . This proves that  $\partial B$  is a simple closed curve when  $|\lambda| < 1/16$ .  $\square$

We now use this result to prove:



**Theorem.** *Suppose that  $|\lambda| < 1/16$  and that the critical points of  $F_\lambda$  tend to  $\infty$  but do not lie in the immediate basin  $B$  of  $\infty$ . Then  $J(F_\lambda) = K(F_\lambda)$  is a Sierpinski curve.*

**Proof:** It is known [12] that any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by simple closed curves that are disjoint is homeomorphic to the Sierpinski carpet and is therefore a Sierpinski curve. In our case, the fact that both  $J$  and  $K$  are compact and connected was shown in the previous section. Since all of the critical orbits tend to  $\infty$ , it follows that  $J = K$  and hence, using standard properties of the Julia set,  $J$  is nowhere dense. Also, since no critical points accumulate on  $J$ , it is known [9] that  $J$  is locally connected.

It therefore suffices to show that the complementary domains are all bounded by disjoint simple closed curves. By the previous result,  $\partial B$  is bounded by a simple closed curve lying strictly outside the critical circle. Using the involution  $H$ , the boundary of the trap door is given by  $H(\partial B)$ , and so this region is bounded by a simple closed curve lying inside the critical circle and therefore disjoint from  $\partial B$ .

Now consider the preimage of the trap door. This preimage is an open set. It cannot consist of a single component, for if this were the case, this component would necessarily surround the origin (by four-fold symmetry) and thereby disconnect the Julia set. Hence each of the components of the preimage of  $T$  is an open set that is mapped in either one-to-one or two-to-one fashion onto  $T$  depending upon whether or not a critical point lies in the preimage. (In fact, the critical points cannot lie in the first preimage of  $T$ , but we do not need this fact here.)

It follows that each component of the preimage of  $T$  is a simply connected open set whose boundary is a simple closed curve that is mapped onto  $\partial T$ . The boundaries of these components are disjoint from  $\partial B$ , since this curve is invariant under  $F_\lambda$  and hence cannot be mapped to  $\partial T$ . They are also disjoint from  $\partial T$  since the boundary of the trap door is mapped to  $\partial B$  whereas the boundary of the components are mapped to  $\partial T$ , and we know that  $\partial T \cap \partial B = \emptyset$ . Finally, the boundary of each component is disjoint from any other such boundary for a point in the intersection would necessarily

be a critical point. If this were the case, then the critical orbit would eventually map to  $\partial B$ , contradicting our assumption that the critical orbit tends to  $\infty$ . Hence the first preimages of  $T$  are all bounded by simple closed curves that are disjoint from each other as well as the boundaries of  $B$  and  $T$ . Continuing in this fashion, we see that the preimages  $F_\lambda^{-n}(T)$  are similarly bounded by simple closed curves that are disjoint from all earlier preimages of  $\partial B$ . This gives the result.  $\square$

## 5. THE BOUNDARY OF $\mathbf{B}$

In this section we consider any  $\lambda$ -value for which the Julia set of  $F_\lambda$  is connected, not just those that satisfy  $|\lambda| < 1/16$ . Our aim is to show that the open set  $\overline{\mathbb{C}} - \overline{B}$  is a connected and simply connected set. This implies that the interior of the set containing the origin and bounded by the boundary of  $B$  has just one connected component. Moreover, we show below that if  $z$  lies in the intersection of the boundaries of both  $B$  and  $T$ , then  $z$  must be a critical point of  $F_\lambda$ . Hence there are at most four points in the intersection of these two boundaries.

**Proposition.** *The open set  $\overline{\mathbb{C}} - \overline{B}$  is connected and simply connected whenever  $B \cap T$  is empty.*

**Proof:** Let  $W_0$  denote the open connected component of  $\overline{\mathbb{C}} - \overline{B}$  that contains 0. Note that  $W_0$  contains all of  $T$  since the boundary of  $B$  does not meet  $T$ . Hence the closure of  $W_0$  also contains  $\partial T$ .

**Lemma.**  *$W_0$  is symmetric under  $z \mapsto iz$  and hence has four-fold symmetry.*

**Proof:** Let  $X$  denote the set of points  $z$  in  $W_0$  for which  $iz$  also lies in  $W_0$ . Note that  $X$  is an open subset of  $W_0$ . Note also that  $X \supset T$  since  $T$  possesses four-fold symmetry and lies in  $W_0$ . Hence  $X$  is nonempty. Now suppose that  $X \neq W_0$ . Then there must be a point  $z_1 \in \partial X \cap W_0$ . So  $z_1 \in W_0$  but  $iz_1 \notin W_0$ . Therefore  $iz_1$  lies in  $\partial W_0$ , which is contained in  $\partial B$ . Since  $iz_1 \in \partial B$  and it was earlier shown that  $\partial B$  has four-fold symmetry we know that  $z_1 \in \partial B$ , contradicting our assumption that  $z_1 \in W_0$ . This proves the lemma.  $\square$

**Lemma.**  $F_\lambda(W_0) \cap W_0$  is nonempty.

**Proof:** Since  $F_\lambda(\partial T) = \partial B$  and  $\partial W_0 \subset \partial B$ , it follows that if  $w_0 \in \partial W_0$ , then there exist two points  $\pm z_0 \in \partial T$  such that  $F_\lambda(\pm z_0) = w_0$ . Now either  $\pm z_0 \in \partial W_0$  or  $\pm z_0 \notin \partial W_0$ . In the latter case, there is a neighborhood  $\mathcal{O}$  of  $z_0$  in  $W_0$  with  $w_0 \in F_\lambda(\mathcal{O})$ . Since  $w_0$  also lies in  $\partial W_0$  we have  $F_\lambda(\mathcal{O}) \cap W_0 \neq \emptyset$  and we are done. Therefore, we may assume that, for any  $w_0 \in \partial W_0$ , there exists  $\pm z_0 \in \partial T \cap \partial W_0$  such that  $F_\lambda(\pm z_0) = w_0$ . Recall that  $H(z) = \sqrt{\lambda}/z$  is an involution that satisfies  $H(B) = T$  and  $H(T) = B$ . Therefore  $H(\partial B) = \partial T$  and  $H(\partial T) = \partial B$ . Since  $\pm z_0 \in \partial T \cap \partial W_0$  and  $\partial W_0 \subset \partial B$  we have  $H(\pm z_0) \in \partial T \cap \partial B$  as well. By the definition of  $W_0$ ,  $\partial B \cap W_0 = \emptyset$ . Since  $T \subset W_0$  we have  $H(\pm z_0) \in \partial T$ , implying that  $H(\pm z_0) \in W_0$  or  $H(\pm z_0) \in \partial W_0$ . Since  $H(\pm z_0) \in \partial B$  we know that  $H(\pm z_0) \notin W_0$ , implying that  $H(\pm z_0) \in \partial W_0$ . Therefore  $H(\pm z_0) \in \partial T \cap \partial W_0$ , and so all four preimages of  $w_0$ , namely  $\pm z_0, H(\pm z_0)$ , lie in  $\partial W_0 \cap \partial T$ .

Therefore,  $\pm z_0 \in \partial W_0$  and  $H(\pm z_0) \in \partial W_0$ . By the same arguments as above we see that all four preimages of  $\pm z_0$  and  $H(\pm z_0)$  are in  $\partial W_0 \cap \partial T$ . Continuing, we find that the entire backward orbit

$$\bigcup_{n \geq 0} F_\lambda^{-n}(w_0)$$

is contained in  $\partial W_0 \cap \partial T$ . But the Julia set is the closure of the backward orbit of any point in  $J(F_\lambda)$ , and so we have  $J(F_\lambda) \subset \partial W_0 \subset \partial B$ . Further, since  $B$  is a Fatou component we have  $\partial B \subset J(F_\lambda)$ . Hence,  $J(F_\lambda) = \partial B$ . Likewise,  $J(F_\lambda) \subset \partial T$  and since  $T$  is a Fatou component  $J(F_\lambda) \subset \partial T$  implying that  $J(F_\lambda) = \partial T$ . Hence,  $\partial T = \partial B$  implying that  $T = W_0$  and  $B = \overline{\mathbb{C}} - \overline{W_0}$ . But this implies that  $F_\lambda$  maps no points into  $T$ , which yields a contradiction and proves the lemma.  $\square$

So  $F_\lambda(W_0) \cap W_0 \neq \emptyset$  and  $F_\lambda(\partial W_0) \subset \partial B$ . Hence,  $F_\lambda(W_0)$  covers all of  $\overline{\mathbb{C}}$  or  $\partial F_\lambda(W_0) \subset \partial B$ . Since  $F_\lambda(\partial W_0) \subset \partial B$ ,  $\partial B \cap W_0 = \emptyset$ , and  $F_\lambda(W_0) \cap W_0 \neq \emptyset$  we know that  $\partial W_0 \subset F_\lambda(\partial W_0)$  which implies that  $W_0 \subset F_\lambda(W_0)$  by the maximum modulus principle. Also note that, since  $\partial W_0 \subset \partial B$ , we have  $H(\partial W_0) \subset \partial T \subset \overline{W_0}$ . Therefore  $H$  maps the exterior of  $\overline{W_0}$  into  $\overline{W_0}$ .

Now suppose that there is an additional component of  $\overline{\mathbb{C}} - \overline{B}$  that is disjoint from  $W_0$ . Call this component  $W_1$ . Note that  $-W_1$

is also a component of  $\overline{\mathbb{C}} - \overline{B}$  and that  $\pm W_1$  are disjoint. We have that  $F_\lambda(W_1)$  does not meet  $W_0$  since all preimages of points in  $W_0$  lie in  $W_0$ . Also, from the above, we have that  $H(\pm W_1)$  lies in  $W_0$ .

We claim that there are no critical points in  $W_1$ . For, if  $c_\lambda \in W_1$ , then we must have  $-c_\lambda \in -W_1$  and so  $F_\lambda$  maps both  $\pm W_1$  onto an open set  $Q$  in two-to-one fashion. Now  $Q$  lies in  $\overline{\mathbb{C}} - \overline{B}$  and hence  $Q$  must be some connected component of this set, say  $Q = W_k$ . Then we have  $k \neq 0$  and that all four preimages of any point in  $W_k$  lie in  $\pm W_1$ . But, since  $F_\lambda(H(z)) = F_\lambda(z)$ , there must also be preimages of these points in  $H(\pm W_1) \subset W_0$ , as we saw above. Thus we have more than four preimages for these points, so this cannot happen. We conclude that there can be no critical points in  $W_1$ .

Thus we have that any additional component of  $\overline{\mathbb{C}} - \overline{B}$  cannot contain either a critical point or a prepole of  $F_\lambda$ . Now we know that the set of components  $\cup W_j$  excluding  $W_0$  is mapped onto itself by  $F_\lambda$ . But then either one of these domains must be periodic under  $F_\lambda$  or else we have no periodic domains in  $\cup W_j$ . The former is impossible, since such a periodic domain would necessarily have a critical point belonging to it, while the latter is impossible by the Sullivan No Wandering Domains Theorem. See [9].

We conclude that there are no other  $W_j$  to start with in  $\overline{\mathbb{C}} - \overline{B}$ , and so  $\overline{\mathbb{C}} - \overline{B} = W_0$ , an open, connected, simply connected set as claimed. □

As a remark, the fact that there is only one component to the complement of  $\overline{B}$  does not preclude the existence of quadratic-like filled Julia sets with infinitely many pinch points. These often reside as subsets of  $W_0$  as depicted in Figure 3.

**Corollary.** *Suppose  $z_0 \in \partial B \cap \partial T$ . Then  $z_0$  is a critical point of  $F_\lambda$ .*

**Proof:** Suppose that  $z_0$  is not a critical point of  $F_\lambda$ . Then  $F_\lambda(z_0) = w_0$  has four distinct preimages:  $\pm z_0$  and  $\pm z_1$  with  $z_0 \neq \pm z_1$ . Let  $\pm U_i$  be open neighborhoods of  $\pm z_i$  and suppose that the  $\pm U_i$  are disjoint and that  $F_\lambda(\pm U_i) = W$  where  $W$  is an open neighborhood of  $w_0$ .

Since  $z_0 \in \partial B$ , we may find a pair of external rays  $\alpha_0$  and  $\beta_0$  that land at distinct points in  $U_0 \cap \partial B$ . Let  $\gamma(\alpha_0, \beta_0)$  denote the

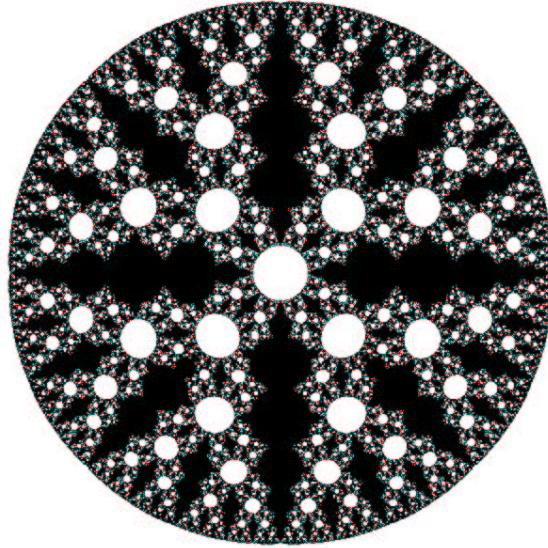


FIGURE 3. The Julia set of  $F_\lambda$  when  $\lambda = 0.01$ . For this  $\lambda$ -value,  $F_\lambda$  admits an attracting cycle of period 2. Note the black regions lying inside  $W_0$  that resemble the Julia set of  $z^2 - 1$ ; these are the basins of attraction of the two-cycle for  $F_\lambda$ .

union of the external rays contained between  $\alpha_0$  and  $\beta_0$  (where we assume that the angle between these two rays is smaller than  $\pi/2$ ). We may choose  $\alpha_0$  and  $\beta_0$  so that the closure of  $\gamma(\alpha_0, \beta_0)$  contains  $z_0$ , i.e., that these external rays land on either “side” of  $z_0$ . The set  $-\gamma(\alpha_0, \beta_0)$  lies in  $B$  and has similar properties near  $-z_0$ .

Now  $z_0$  lies in  $\partial T \cap \partial B$ . Hence  $\pm z_1$  also lies in  $\partial T \cap \partial B$  since  $H(\pm z_0) = \pm z_1$  and  $H$  maps  $\partial T$  to  $\partial B$  and  $\partial B$  to  $\partial T$ . Thus we may find two other external rays  $\alpha_1$  and  $\beta_1$  in  $B$  such that  $\alpha_1$  and  $\beta_1$  terminate in  $U_1$  on either side of  $z_1$  and, moreover, have the property that  $\gamma(\alpha_1, \beta_1)$  is disjoint from  $\pm\gamma(\alpha_0, \beta_0)$ . As above,  $-\gamma(\alpha_1, \beta_1)$  lies in  $B$  and terminates in  $-U_1$ .

Since  $\pm\gamma(\alpha_0, \beta_0)$  and  $\pm\gamma(\alpha_1, \beta_1)$  are disjoint, it follows that the images of these sets are distinct. (Since  $F_\lambda$  is even we know that  $F_\lambda(\gamma(\alpha_i, \beta_i)) = F_\lambda(-\gamma(\alpha_i, \beta_i))$  and hence if the images of these four sets were not distinct then there would exist points in  $B$  with four

preimages in  $B$  and this cannot happen since  $F_\lambda$  is  $2 - 1$  on  $B$ .) Also, these images accumulate near  $w_0$ . Thus we have two disjoint intervals of rays in  $B$  that accumulate on  $\partial B$ , and both contain  $w_0$  in their closure. It follows that there must be a subset  $R$  of  $\partial B$  that is disjoint from the boundary of  $W_0$  and  $R$  is separated from  $W_0$  by these sets of rays. (See Figure 4.)

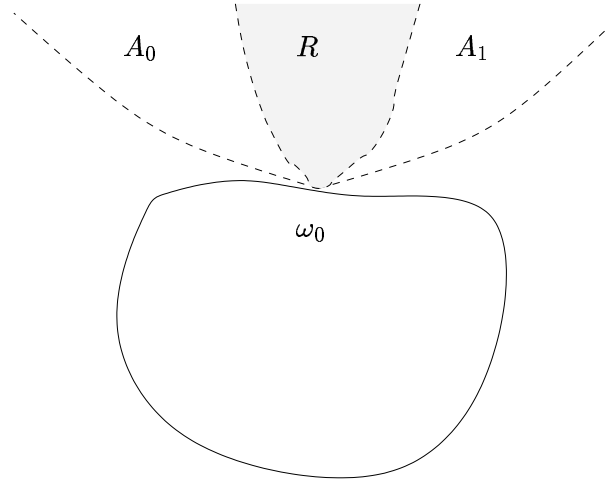


FIGURE 4. The region  $R$  where  $F_\lambda(\pm\gamma(\alpha_i, \beta_i)) = A_i$

The set  $R$  cannot bound an open component of  $\overline{\mathbb{C}} - \overline{B}$ , as we saw above. Hence  $R$ , which is separate from  $W_0$ , must have empty interior. But there must be preimages of this region on the boundary of  $W_0$ , and so there must be points in  $W_0$  that map arbitrarily close to these points. This is impossible.

We conclude that  $w_0$  could not have had four preimages and so  $z_0$  must have been a critical point.  $\square$

## 6. PROOF OF THE SIERPINSKI CURVE CRITERION

In this section we investigate the general case where the critical orbit escapes through the trap door into  $B$ . Here we complete the proof that, when this occurs, the Julia set is a Sierpinski curve.

In this case  $F_\lambda$  is a rational map of degree  $d \geq 2$  whose postcritical closure is disjoint from its Julia set. Hence  $F_\lambda$  is dynamically hyperbolic. Since  $B$  is a simply connected Fatou component for  $F_\lambda$ , we have that  $\partial B$  is locally connected and that the Julia set of  $F_\lambda$  is connected and locally connected [9]. Since  $B$  is simply connected we know by the Riemann mapping theorem that there is a conformal isomorphism  $\psi : \mathbb{D} \rightarrow B$  where  $\mathbb{D}$  is the open unit disk. The following result is well known. See [9].

**Theorem (Caratheodory).** *A conformal isomorphism  $\psi : \mathbb{D} \rightarrow U \subset \overline{\mathbb{C}}$  extends to a continuous map from the closed disk  $\overline{\mathbb{D}}$  onto  $\overline{U}$  if and only if the boundary  $\partial U$  is locally connected.*

This tells us that the Riemann map  $\psi : \mathbb{D} \rightarrow B$  extends to a continuous map  $\hat{\psi} : \overline{\mathbb{D}} \rightarrow \overline{B}$ . In particular, we have a continuous map from  $\partial \mathbb{D}$  to  $\partial B$ . Therefore we know that all external rays  $R_t$  (with  $t \in \mathbb{R}/\mathbb{Z}$ ) in  $B$  land on a single point in  $\partial B$ .

This allows us to prove:

**Theorem.**  *$\partial B$  is a simple closed curve.*

Combining this result with the techniques described in Section 3 allows us to conclude that  $J(F_\lambda)$  is a Sierpinski curve when the critical orbit escapes through the trap door.

**Proof:**  $\partial B$  is a simple closed curve if and only if exactly one external ray lands at each point in  $\partial B$ . Assume this is not the case. Suppose that there exists  $p \in \partial B$  such that two external rays  $R_{t_1}$  and  $R_{t_2}$  land on  $p$ . Since these rays together with the point  $p$  form a Jordan curve and  $W_0$  is connected and simply connected, we have that  $W_0$  lies entirely within one of the two open components created by this Jordan curve. Without loss of generality, assume that  $W_0$  is such that  $W_0 \cap \gamma(t_1, t_2) = \emptyset$  (so  $W_0$  is “outside” the sector  $\gamma(t_1, t_2)$  between  $R_{t_1}$  and  $R_{t_2}$ ).

We claim that there exist positive integers  $q$  and  $n$  such that the region

$$\gamma\left(\frac{q}{n}, \frac{q+1}{n}\right) \subset \gamma(t_1, t_2)$$

and neither of  $R_{q/n}$  nor  $R_{(q+1)/n}$  land on  $\partial W_0$ . If this is not possible then all rays with angle  $s$  such that  $R_s \subset \gamma(t_1, t_2)$  land at  $p$ . This gives a contradiction because the set of  $s \in \mathbb{R}/\mathbb{Z}$  such that  $\gamma(s) = p$

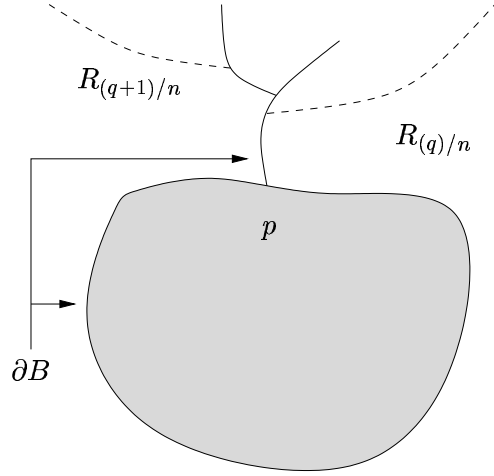


FIGURE 5. A possible landing pattern.

has measure zero [9]. Therefore, if we have two rays landing at  $p$ , then we can find  $q$  and  $n$  such that

$$\gamma\left(\frac{q}{n}, \frac{q+1}{n}\right) \subset \gamma(t_1, t_2)$$

and neither of  $R_{q/n}$  nor  $R_{(q+1)/n}$  land on  $\partial W_0$ . (See Figure 5.)

Assume that we have such  $q$  and  $n$ . As above, let  $\gamma(q/n, (q+1)/n)$  denote the union of the external rays contained between  $q/n$  and  $(q+1)/n$ . After  $n$  iterations  $\gamma(q/n, (q+1)/n)$  is mapped over all of  $\partial B$ . In particular, if  $R_\theta$  is an external ray landing on  $\partial W_0$  we know that there is a ray  $R_\phi \in \gamma(q/n, (q+1)/n)$  such that  $F_\lambda^n(R_\phi) = R_\theta$ . Since  $R_\phi \in \gamma(q/n, (q+1)/n)$  we know that the landing point  $\gamma(\phi)$  is not on  $\partial W_0$ . Hence there exists a neighborhood  $N_\phi$  of  $\gamma(\phi)$  such that  $N_\phi \cap W_0$  is empty. However, since  $F_\lambda^n(N_\phi)$  is on the boundary of  $W_0$  we know that  $F_\lambda^n(N_\phi) \cap W_0$  is not empty. This is a contradiction since points not in  $W_0$  never enter  $W_0$ . Hence, we can never have two rays landing at the same point on  $\partial B$ , implying that  $\partial B$  is a simple closed curve.

□



Although the above was written for  $\lambda$  such that the critical points escape, the results also hold for the Misiurewicz case. In this case  $F_\lambda$  is subhyperbolic and all of the theorems above hold with minor adjustments (i.e., all of the proofs depending on hyperbolicity still go through when hyperbolicity is replaced by subhyperbolicity). Therefore we know that  $\partial B$  is a simple closed curve for the Misiurewicz case as well.

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