Illuminating the Mandelbrot Set

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1 Introduction

The Mandelbrot set $\mathcal{M}$ is one of the most interesting and beautiful objects in all of mathematics. It is also one of the most intricate planar sets. Contrary to the fact that it is named after Benoit Mandelbrot, the father of fractal geometry, the boundary of this set can be thought of as the antithesis of a fractal. Most definitions of a fractal set specify that the set is "self-similar;" that is, whenever you magnify a portion of this set, you always see a small copy of the original set. As we shall discuss in this paper, every tiny area of the boundary of the Mandelbrot set has its own "identity." That is, using some tools from geometry and complex analysis, we can read off exactly where this boundary point is and, more importantly, we can tell exactly what the corresponding dynamical behavior is. So there is no self-similarity along the boundary of the Mandelbrot set.

2 Preliminaries

Amazingly, the Mandelbrot set arises when the simple quadratic function $P_c(z) = z^2 + c$ is iterated. The Mandelbrot set is a summary of this iterated behavior in the parameter plane, i.e., the c-plane, where c is a complex number. To be precise, the Mandelbrot set (which we denote by $\mathcal{M}$) consists of those c values for which the orbit of 0, i.e., the sequence $0, P_c(0), P_c(P_c(0)) = P_c^2(0), P_c^3(0), \ldots$ does not tend to infinity.

There are two reasons for singling out the orbit of 0. The first is the following important fact from complex dynamics: If P_c possesses an attracting cycle, then the orbit of 0, the critical point, must converge to that cycle. Recall that a cycle is an orbit $z_0, P_c(z_0), \ldots P_c^n(z_0) = z_0$ that returns to itself after exactly n iterations. Such a cycle is called attracting if all sufficiently nearby orbits tend to the cycle. This occurs when the derivative of P_c^n at z_0 has magnitude strictly less than one. The cycle is called repelling if all nearby points have orbits that move away from the cycle. This occurs when the derivative of P_c^n at z_0 has magnitude greater than one. And the cycle is neutral if the derivative at z_0 has magnitude equal to one.

Since 0 must tend to any attracting cycle of P_c, it follows that P_c can have at most one attracting cycle since 0 is the only critical point of P_c. Also, such a c-value must lie in $\mathcal{M}$ since the orbit of 0 is bounded. In fact, the c-values for which P_c has an attracting cycle comprise all of the visible interior regions in the Mandelbrot set. By visible, we mean that nobody has ever found experimentally or otherwise a component of the interior that does not have this property. (One of the main conjectures concerning $\mathcal{M}$ is that its interior consists of *only* c-values for which there is an attracting cycle.)

The second important fact regarding the orbit of 0 involves the filled Julia set of P_c. By definition, the filled Julia set of P_c is the set of all points in the complex plane whose orbits are bounded under iteration of P_c. We denote the filled Julia set of P_c by K_c. The fundamental dichotomy says that K_c assumes one of two possible shapes depending upon the fate of the orbit of 0. If the orbit of 0 tends to ∞, then K_c is a Cantor set. On the other hand, if the orbit of 0 is bounded, then K_c is connected. See [1], [7] for a proof of this. Therefore, for the function $z^2 + c$, there are only two types of filled Julia sets: those that consist of a single component, and those that are totally disconnected (and hence consist of uncountably many point components). Thus, the Mandelbrot set may also be defined as the set of c-values for which the filled Julia set is connected.

The Mandelbrot set is displayed in Figure 1. If the orbit of 0 does not tend to ∞ for a given c-value, then the corresponding parameter c lies in $\mathcal{M}$ and we color this point black. If the orbit does escape to ∞, then c is not in $\mathcal{M}$ and we color c according to how quickly the orbit of 0 reaches the exterior of a large disk surrounding the origin (with red points escaping fastest, followed in order by orange, yellow, green, blue, and violet).

In complex dynamics, the object of central interest in the dynamical plane is the *Julia set*, which we denote by J_c. By definition, the Julia set is the boundary of the filled Julia set. Given a point in the Julia set, any open neighborhood of this point, no matter how small, contains some points whose orbits tend to ∞ and other points whose orbits remain bounded. In fact, using Montel's Theorem from complex analysis, this open neighborhood is eventually mapped over the entire complex plane, minus at most one point. So the family of iterates of P_c on the Julia set is very chaotic.

The large black regions (called *hyperbolic components*) visible in the Mandelbrot set are regions for which P_c has an attracting cycle of some given period. For example, any c-value drawn from the large central cardioid has an attracting fixed point. For c in the large open disk just to the left of this cardioid, P_c has an attracting 2-cycle. We therefore call this the period 2-bulb. And, for c in the northernmost and southernmost bulbs off the main cardioid, P_c has an attracting cycle of period 3, so these are the period 3-bulbs.

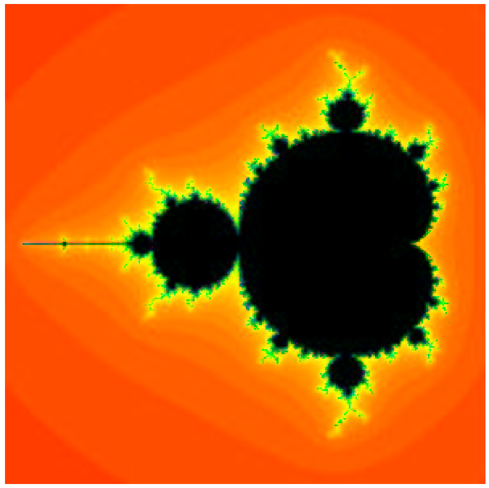

Figure 1: The Mandelbrot set. Colored points are c-values for which the orbits of 0 escape to ∞; black points are c-values for which this does not happen. So the Mandelbrot set is the black region in this image.

As c moves from one hyperbolic component to another, the map undergoes a bifurcation. The simplest part of this bifurcation is the fact that we move from having an attracting cycle of some period when we are in one hyperbolic component to having an attracting cycle of some other period in the subsequent hyperbolic component. But, in fact, much more happens: the topology of the Julia sets changes dramatically. For example, if we move from the main cardioid to the period-2 bulb, the Julia set, which is just a simple closed curve when c is in the main cardioid, becomes a "basilica" when c is in the period 2-bulb. See Figure 2. What happens is a repelling 2-cycle that lies in J_c when c is in the cardioid suddenly merges with the attracting fixed point and thereby makes it neutral when the parameter reaches the boundary of the cardioid. So two points in J_c become identified to one point. Meanwhile, infinitely many pairs of preimages of this point also become identified. This is what accounts for the infinitely many "pinch-points" visible in the basilica. As another example, as we move from the main cardioid to the period 3-bulbs, a period 3-cycle becomes identified and the Julia set transforms into the "Douady rabbit." See Figure 2.

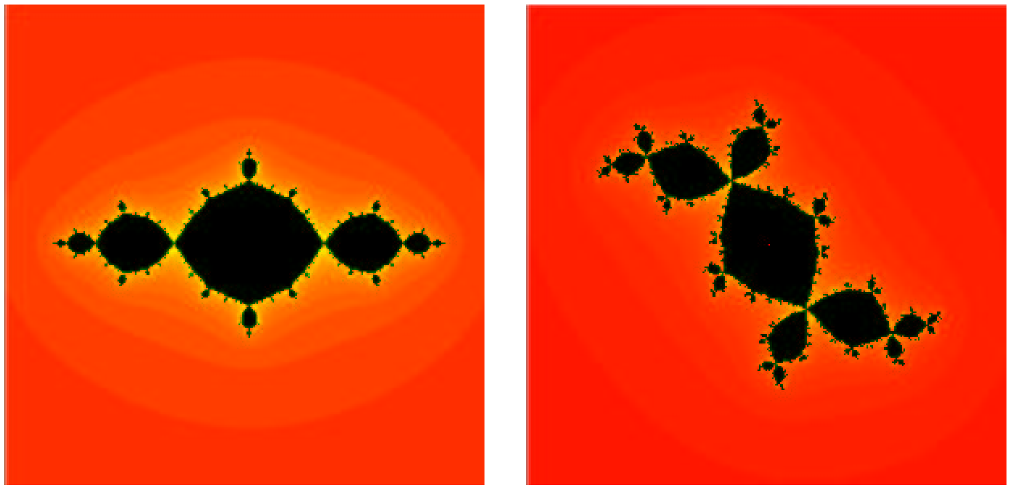

Figure 2: The Julia sets for $z^2 - 1$ (the basilica) and $z^2 - 0.12 + .75i$ (the Douady rabbit). The filled Julia sets are the black regions, so the Julia sets here are the boundaries between the black and colored regions.

3 Periods of the Bulbs

A natural question is how do we understand the arrangement of the bulbs in $\mathcal{M}$. Amazingly, as mentioned earlier, if we zoom in to any portion of the boundary of the Mandelbrot set, it turns out that this zoom is very different from any other zoom that is non-symmetric with respect to $c \mapsto \overline{c}$. More importantly, with a keen eye for geometry, one can deduce exactly where in the boundary of $\mathcal{M}$ this zoom is, and, even more importantly, what the corresponding dynamical behavior for parameters drawn from the associated bulb is. It turns out that there are several different geometric and dynamical ways to understand the structure of these bulbs. We will first look at this geometrically, but the real way to understand this uses techniques from complex analysis that we will describe later.

For simplicity, let's concentrate for the remainder of this paper on the bulbs attached to the main cardioid. The maps corresponding to a parameter drawn from one of these bulbs all have an attracting cycle of some given period. How do we know what this period is? One way is easy: look at the bulb. There is an antenna attached to this bulb. This antenna has a junction point from which a certain number of spokes emanate. The number of these spokes tells us exactly what the period is. For example, in Figure 3, we display two bulbs having periods 5 and 7. Note that this is the exact number of spokes hanging off the junction point in the main antenna attached to each bulb.

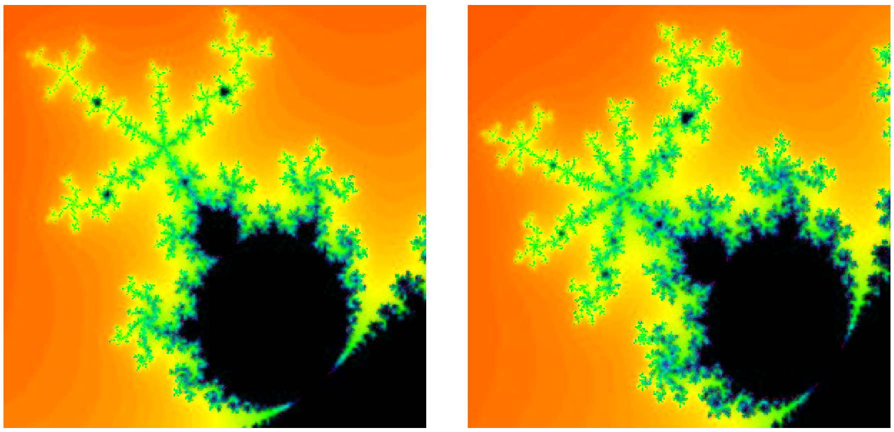

Figure 3: Period 5 and 7 bulbs hanging off the main cardioid.

There is another way to read off the periods of these bulbs. Choose a parameter from the interior of a period n bulb and plot the corresponding filled Julia set. There is a central disk in these filled Julia sets that surrounds the origin. Then there are exactly $n-1$ smaller disks that join this main disk at certain junction points. For example, in Figure 2, we see that the rabbit has two "ears" attached to the central disk and the period of this bulb is $2+1=3$. Similarly, the basilica has just 1 "ear" attached and the period here is $1+1=2$. In Figure 4, we display Julia sets from the above period 5 and period 7 bulbs, and we see the same phenomenon.

Now let us turn to the arrangement of the bulbs around the main cardioid. To do this, we assign a fraction p/q to each of these bulbs. Here q is the period of the bulb, so the question is: what is p? There are several geometric and dynamical ways to determine p. Look at the period five bulb in Figure 3. We call the spoke of the antenna that extends down to the bulb from the junction point the *principal spoke*. Note that the "shortest" spoke (that is not the principal spoke) is located $2/5$ of a turn in the counter-clockwise direction from the principal spoke. So this bulb is then the $2/5$-bulb. In that

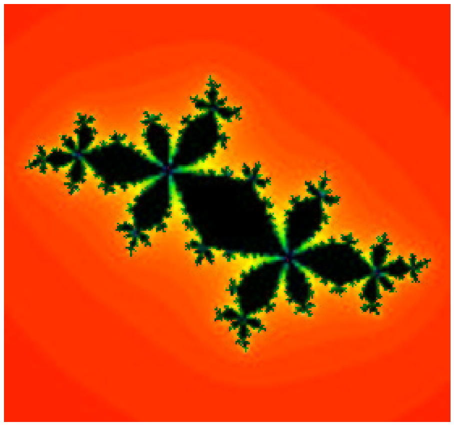 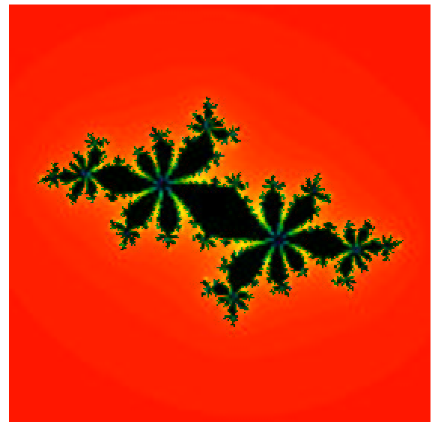

Figure 4: Julia sets drawn from the above period 5 and 7 bulbs hanging off the main cardioid. Note that there are 4 and 6 "ears" hanging off the central disks of these filled Julia sets.

same figure, we also see that the period 7-bulb is, in fact, the 3/7-bulb.

A second way to see this is to turn to the filled Julia set. In Figure 4, each of the filled Julia sets has a main component that surrounds the origin together with $q-1$ ears attached at one point. Note where the "smallest" ear is located; it is exactly p/q of a turn in the counterclockwise direction from main component.

And then there is a third way to read off p/q. Simply plot the points on the attracting cycle of period q in the Fatou set. What you see is that this cycle moves around the ears and the main component, rotating roughly speaking by p/q of a turn at each stage. So there is a very nice connection between the geometry of the Mandelbrot set and Julia sets and the dynamics of P_c.

4 External Rays

In the previous section, we referred to the smallest "spokes" of the antennas and "ears" of the filled Julia sets and we just looked at these sets to determine which is the smallest. So the question is: how do we really tell which of these is the "smallest?" The main tool for explaining the size of portions of the

Mandelbrot and filled Julia sets is the theory of external rays developed by Douady and Hubbard [6]. In this section we summarize the part of their work relevant to this paper.

Given c in one of the bulbs, it is known that there is an analytic homeomorphism ϕ_c taking the basin of ∞ which we denote by U_c to the exterior of the unit circle (with ∞ mapped to ∞). Moreover, ϕ_c conjugates the map P_c in U_c to the squaring map $Q_0(z) = z^2$ outside the unit circle. That is, on U_c we have $\phi_c(P_c(z)) = (\phi_c(z))^2$. If we specify that $\phi_c'(\infty) > 0$, then ϕ_c is the unique such analytic conjugacy.

An *external ray in* U_c is then the preimage under ϕ_c of a straight ray of the form $re^{2\pi i\theta}$ where $r > 1$. The number θ is called the *external angle* of the ray. We always specify external angles mod 1. Since the squaring map sends straight rays to straight rays, it follows from the fact that ϕ_c is a conjugacy that P_c maps external rays to external rays. Moreover, the action of P_c on these rays is the same as the doubling map given by $\theta \to 2\theta$ mod 1.

In case P_c admits an attracting cycle, it is known that all of the external rays land at a point in J_c. By this we mean that

$$\lim_{r \to 1} \phi_c^{-1}(re^{2\pi i\theta})$$

exists for each θ and this limit point lies in J_c. The limit point is called the landing point of the external ray with angle θ. Moreover, in this case, each point in J_c is the landing point for at least one external ray.

If c is drawn from a bulb attached to the main cardioid, then it is known that there is a repelling fixed point z_c lying in J_c and having the property that this fixed point is the sole intersection point of the boundaries of the immediate basins of the corresponding attracting cycle. What happens is, when c lies in the main cardioid, the corresponding map has a single attracting fixed point with just one immediate attracting basin. But, as c meets the boundary of the main cardioid where a period k bulb is attached, this fixed point becomes neutral and merges with a repelling periodic cycle of period k which had been in J_c while c was in the main cardioid. At this point, the basin of attraction suddenly becomes a set on k distinct disks. Then, as c enters the period k bulb, the fixed point z_c becomes repelling, while there are now k distinct immediate basins, each containing one of the points on the new attracting cycle. All of this happens continuously as c varies. Moreover, the boundaries of each of the immediate basins meet at exactly one point, namely z_c. Then it can be shown that there are exactly k external rays that now land on z_c, and each of these rays separate a particular pair of the immediate basins of the attracting cycle.

Since external rays cannot cross in U_c, this means that, if θ_1 and θ_2 are the external angles corresponding to two external rays that meet at z_c and separate a single basin of the cycle from all of the others, then any external ray of angle θ with $\theta_1 < \theta < \theta_2$ must in fact accumulate on the boundary of that immediate basin. So this is how we measure the size of the "ears" in the dynamical plane: the size is just the length of the interval of external angles that accumulate on the boundary of that ear, i.e., in the above case, the size is just $\theta_2 - \theta_1$. An explicit method to compute these external angles is given in [4] and [5].

If c does not lie in the Mandelbrot set, then the conjugacy ϕ_c is still defined, but only in a neighborhood of ∞. We may still pull back this conjugacy, but not to the entire complement of K_c, for the conjugacy cannot be extended to the preimage of c, as c has only one preimage under P_c, namely 0. However, we may still define the conjugacy in a neighborhood U_c of ∞ that contains c. Hence, for each c in the complement of $\mathcal{M}$, there is defined $\Phi(c) = \phi_c(c)$. Then the remarkable theorem of Douady and Hubbard asserts that the map

$$\Phi : \mathbf{C} - \mathcal{M} \rightarrow \{z \mid |z| > 1\}$$

is an analytic homeomorphism which is the exterior Riemann map for $\mathcal{M}$.

With this homeomorphism in hand, we may define the external rays of $\mathcal{M}$ as in the case of K_c. The preimage of the straight ray $\theta =$ constant under Φ is called the *external ray with angle θ* to $\mathcal{M}$. It is known that all rays with rational external angles land on $\mathcal{M}$, with landing defined exactly as in the case of K_c. At this time, it is not yet known whether all irrational rays actually land on $\mathcal{M}$.

Recall that a component of the interior of $\mathcal{M}$ is called a *hyperbolic component* if P_c admits an attracting cycle for all c in that component. The period of the attracting cycle is necessarily constant over each hyperbolic component and so is called the period of the component.

It is known that hyperbolic components have smooth boundaries. At a dense set of c-values along these boundaries there is attached a hyperbolic component whose period is a multiple of the original period. These satellite components are attached to the original component at a c-value called the *root point* of the satellite component.

We can now state a fundamental result of Douady and Hubbard regarding the external rays of $\mathcal{M}$.

Theorem. *Suppose B is a satellite hyperbolic component of $\mathcal{M}$ with period k. Then there are exactly two external rays that land at the root point of this component and each of them has external angle that has period k under the angle doubling map $\theta \mapsto 2\theta \bmod 1$.*

This is the way that we will specify the size of certain regions of $\mathcal{M}$. Given two rays θ_1 and θ_2 that land at the same point in $\mathcal{M}$, we know that none of the rays between these two can cross the given rays. As a consequence, all of these rays must approach $\mathcal{M}$ in the region cut off by the θ_i. It is therefore natural to measure the "size" of this region by determining the length of the arc between θ_1 and θ_2. The methods described in [4] and [5] also apply here to compute the external rays landing at root points in $\mathcal{M}$.

5 Farey Addition

One curious fact that relates to the Farey tree involves the size of the bulbs hanging off the main cardioid. To begin, we think of the root point of the main cardioid as being the cusp at $c = 1/4$. Then we call the main cardioid the 0/1-bulb. The root point of any other bulb is just the point where this bulb is attached to the main cardioid. Now which is the largest bulb between the root points of the 0/1 and 1/2-bulbs (in, say, the upper portion of $\mathcal{M}$)? It is clearly the 1/3-bulb. And note that 1/3 is obtained from the previous two fractions by *Farey addition*, i.e., adding the numerators and adding the denominators

$$\frac{0}{1} \text{``}+\text{''} \frac{1}{2} = \frac{1}{3}.$$

Similarly, the largest bulb between the 1/3 and 1/2-bulbs is the 2/5-bulb, again given by Farey addition.

$$\frac{1}{3} \text{``}+\text{''} \frac{1}{2} = \frac{2}{5}.$$

And the largest bulb between the 2/5 and 1/2-bulb is the 3/7-bulb while the largest bulb between the 2/5 and 1/3-bulbs is the 3/8-bulb and so on along the "Farey tree."

Then it follows that these bulbs are arranged around the boundary of the main cardioid in the exact order of the rational numbers in the unit interval.

Actually, techniques from calculus can be used to prove this fairly easily. For more details, see [2], [3], [4], and [5]. An online, interactive discussion of this (with plenty of animations) called the Mandelbrot Set Explorer is available at `http://math.bu.edu/DYSYS/explorer`.

Using similar techniques from geometry, one can identify the other sub-bulbs in the Mandelbrot set. Unfortunately, there are many other points in the Mandelbrot set that this approach does not apply to; indeed, despite the simplicity of the function $z^2 + c$, there are still many c-values in $\mathcal{M}$ for which we have no idea what is happening in the corresponding Julia set and what is the nearby structure in the Mandelbrot set. For example, along the boundary of the main cardioid we have only looked at the parameters corresponding to "rational" root points as discussed above. But there are uncountably many other points along the boundary of the cardioid. These correspond to "irrational" points. We understand the behavior of P_c at the so-called "highly" irrational points, but the parameters at the "not-so-irrational" points have behavior that is still not understood. This is one of the major open problems in this area of mathematics. For a basic introduction to complex dynamics, see [1]. A more advanced survey of this field is John Milnor's book [7].

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