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THE MCMULLEN DOMAIN: SATELLITE MANDELBROT SETS AND SIERPINSKI HOLES

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ABSTRACT. In this paper we describe some features of the parameter planes for the families of rational maps givenb by $F_\lambda(z) = z^n + \lambda/z^n$ where $n \geq 3, \lambda \in \mathbb{C}$. We assume $n \geq 3$ since, in this case, there is a McMullen domain surrounding the origin in the λ-plane. This is a region where the corresponding Julia sets are Cantor sets of concentric simple closed curves. We prove here that the McMullen domain in parameter plane is surrounded by infinitely many simple closed curves $\mathcal{S}^k$ for $k = 1, 2, \ldots$ having the property that:

(1) Each curve $\mathcal{S}^k$ surrounds the McMullen domain as well as $\mathcal{S}^{k+1}$, and the $\mathcal{S}^k$ accumulate on the boundary of the McMullen domain as $k \to \infty$;
(2) The curve $\mathcal{S}^k$ meets the centers of τ_k^n Sierpinski holes, each with escape time $k+2$ where
$$\tau_k^n = (n-2)n^{k-1} + 1.$$
(3) The curve $\mathcal{S}^k$ also passes through τ_k^n parameter values which are centers of the main cardioids of baby Mandelbrot sets with base period k.

1. INTRODUCTION

Our goal in this paper is to describe some of the interesting features of the parameter plane for the families of rational maps given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

where λ is a complex parameter and $n \geq 3$ is a positive integer. The reason why $n \neq 1, 2$ will be made clear below. We regard these maps as singular perturbations of the simple map $z \mapsto z^n$. We call these maps singular perturbations since, when $\lambda = 0$, F_0 has degree n, but when $\lambda \neq 0$, the degree of F_λ jumps to $2n$. As we discuss below, the corresponding Julia sets for these maps become much more complex (and interesting) when λ is non-zero.

The reason for the interest in such singular perturbations stems from Newton's method: when Newton's method is applied to a family of maps that, at a particular parameter value, has a multiple root, a similar jump in degree and complexity arises as the parameters are varied.

While each of the rational maps F_λ with $\lambda \neq 0$ has degree $2n$, in fact, just as in the case of the well-studied quadratic polynomial family, this family forms a natural one-parameter family of maps since there is essentially only one "free" critical orbit for each map. This occurs since each of the $2n$ non-zero finite critical points of F_λ

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have orbits that behave symmetrically. Hence the λ-plane is the natural parameter plane to record the behavior of maps in these families.

As another similarity with the quadratic family, the point at ∞ is a superattracting fixed point for each F_λ when $n > 1$, and so it may be the case that the critical orbits all enter the immediate basin of attraction of this fixed point. This may happen in a number of different manners. To describe these different scenarios, we denote the immediate basin of attraction of ∞ by B_λ. Since 0 is a pole, there is a neighborhood of 0 that is mapped into B_λ. Now either this neighborhood is itself contained in B_λ, or else 0 lies in a disjoint preimage of B_λ which we denote by T_λ. In the latter case, we note that F_λ maps T_λ in n to 1 fashion onto B_λ while $F_\lambda \mid B_\lambda$ is also n to 1. Hence the only preimages of B_λ are B_λ itself and T_λ.

In the quadratic family there is only one way for the critical orbit to escape to ∞: either the critical point and its entire subsequent orbit lies in the basin of ∞ or else no point on this orbit lies in the basin. In contrast, there are three distinctly different manners in which the critical orbit of F_λ may escape to ∞, and this in turn determines three different topological structures for the Julia sets corresponding to these escape parameters. The following Theorem describes this trichotomy (see [6]):

Theorem (The Escape Trichotomy).

1. *If one and hence all of the critical values of F_λ lie in B_λ, then the Julia set of F_λ is a Cantor set;*
2. *If one and hence all of the critical values lie in T_λ, then the Julia set is a Cantor set of simple closed curves;*
3. *If the critical values all lie in preimages of T_λ under F_λ^j for some $j > 0$, then the Julia set is a Sierpinski curve.*

A *Sierpinski curve* is a planar set that is characterized by the following five properties: it is a compact, connected, locally connected and nowhere dense set with two or more complementary domains that are all bounded by simple closed curves that are pairwise disjoint. It is known from work of Whyburn [16] that any two Sierpinski curves are homeomorphic. In fact, they are homeomorphic to the well-known Sierpinski carpet fractal. From the point of view of topology, a Sierpinski curve is a universal set in the sense that it contains a homeomorphic copy of any planar, compact, connected, one-dimensional set. The first example of a Sierpinski curve Julia set was given by Milnor and Tan Lei [13].

Case 2 of the Escape Trichotomy was first observed by McMullen [10], who showed that this phenomenon occurs in each of these families provided that $n \neq 1, 2$ and λ is sufficiently small.

In Figure 1 we display the parameter plane for the family $F_\lambda(z) = z^3 + \lambda/z^3$. The black points in this picture correspond to parameter values for which the critical orbit does not escape to ∞. Again in analogy with the quadratic polynomial family, for these parameters the Julia set is known to be a connected set [1]. The white regions in this picture represent λ-values for which the critical orbit tends to ∞. The exterior region corresponds to parameter values for which the Julia set is a Cantor set; we call this set the *Cantor set locus.* The small white region in the center of the picture corresponds to parameter values for which the Julia set is a Cantor set of simple closed curves. We call this region the *McMullen domain.* It is known [3] that there is a unique such domain for each $n \geq 3$ and that this region

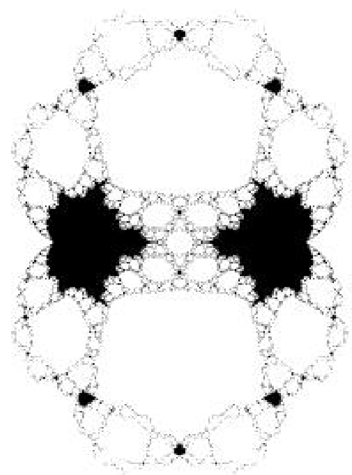

FIGURE 1. The parameter plane for the family $z^3 + \lambda/z^3$.

is a simply connected open set that is bounded by a simple closed curve. All other white regions in this picture correspond to parameters for which the Julia set is a Sierpinski curve. These are called *Sierpinski holes.* Hence the Julia set of F_λ is a connected set for all λ-values except those in the Cantor set locus and the McMullen domain. So we call this set of parameters the *connectedness locus.*

It is known that there are infinitely many disjoint Sierpinski holes for each of these families [1]. As we show below, there is a unique parameter in each Sierpinski hole for which the orbit of the critical point lands on 0 at some iteration, say iteration $k - 1 > 0$, and therefore on ∞ at iteration k. We call this λ-value the *center of the Sierpinski hole* and k the *escape time* of the hole. All other parameters in this Sierpinski hole have the property that the critical point has orbit that lands in B_λ at the escape time iterate. By Whyburn's result, the Julia sets corresponding to any two parameters drawn from a Sierpinski hole are homeomorphic. However, as shown in [6], there exist Sierpinski holes corresponding to each escape time $k \geq 3$, and these have the property that if λ_1 and λ_2 lie in Sierpinski holes with different escape times, then F_{λ_1} and F_{λ_2} are not topologically conjugate on their Julia sets. Consequently, even though the topology of these sets is the same, the dynamical behavior on these sets is quite different.

In Figure 1 we see that, in the case $n = 3$, there appear to be two large copies of a Mandelbrot set that straddle the positive and negative real axes. These are called the *principal Mandelbrot sets* for F_λ. For the λ-value in the center of the main cardioid of the Mandelbrot set on the right, two of the critical points are fixed. These critical points are symmetrically located under $z \mapsto -z$. At the center of the Mandelbrot set on the left, a pair of symmetric critical points are interchanged by F_λ. Hence these critical points lie on a superstable cycle of period 2. We shall often encounter this kind of discrepancy when n is odd in the sequel, so we will say that a Mandelbrot set has *base period* k if the parameter at the center of its main cardioid has a critical point that is mapped to either itself or to its negative (which is also a critical point) by F_λ^k, and k is the minimal such iteration. By symmetry, it follows that the orbit of this critical point is periodic with period either k or $2k$. The reason for this somewhat strange choice of terminology is that it will simplify the statement of our main result.

It is known [2] that there are $n-1$ such principal Mandelbrot sets of base period 1 in the degree $2n$ case and that each of these sets is homeomorphic to the standard quadratic Mandelbrot set. Moreover, for each λ in one of these principal Mandelbrot sets, there is an invariant set within the Julia set of F_λ on which either F_λ or F_λ^2 is topologically conjugate to the corresponding quadratic polynomial on its Julia set. Also apparent in this image are two large Sierpinski holes along the positive and negative imaginary axis. These holes have escape time 3. In fact, the centers of these Sierpinski holes and the centers of the principal Mandelbrot sets lie along an actual circle in the parameter plane that we call the dividing circle.

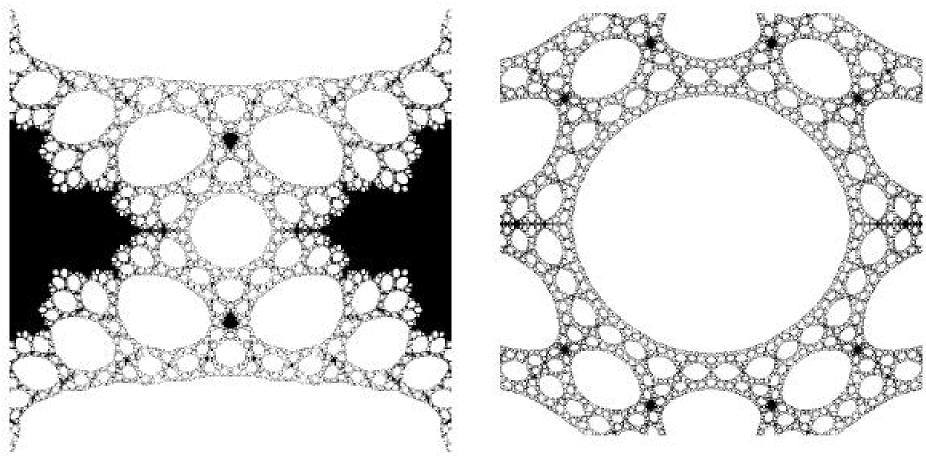

FIGURE 2. Magnifications of the parameter plane for the family $z^3 + \lambda/z^3$ around the McMullen domain.

In Figure 2, we have displayed several magnifications of the region around the McMullen domain in the case $n = 3$. In the first image, note that there appear to be four large Sierpinski holes that surround the McMullen domain. These Sierpinski holes are known to have escape time 4. Between the two upper and the two lower Sierpinski holes there appear to be small copies of a Mandelbrot set, while between the two left and two right holes we see the period two bulb of a principal Mandelbrot set and the remainder of the "tail" of this set. In fact, the centers of the main cardioids of the two small Mandelbrot sets are parameters for which two symmetric critical points are interchanged by F_λ^2, so we again say that these Mandelbrot sets have base period 2. Thus, there appears to be another curve encircling the McMullen domain that contains four parameter values lying at the centers of Sierpinski holes with escape time 4 and four other parameters for which the critical orbit is periodic of period 2 (under either F_λ or F_λ^2).

Inside the closed curve passing through these four Sierpinski holes appears to be another simple closed curve meeting ten Sierpinski holes. Each of these holes has escape time 5. Also, each pair of these holes apparently has either a small copy of a Mandelbrot set or a copy of such a set inside the principal Mandelbrot set between them. Each of these Mandelbrot sets has base period 3. Examining the further magnification in Figure 2, there is another closed curve meeting 28 Sierpinski holes with escape time 6 and, inside that curve, an even smaller curve meeting 82 Sierpinski holes with escape time 7. It appears that, in this case, the k^{th} curve meets exactly $3^k + 1$ Sierpinski holes with escape time $k + 3$ as well as the same number of Mandelbrot sets. We call these curves the *rings around the McMullen domain.*

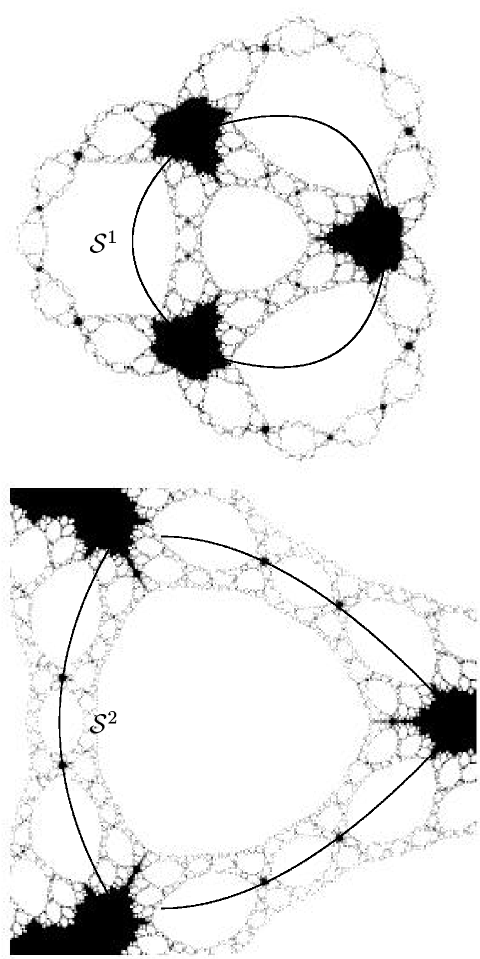

FIGURE 3. The curves $\mathcal{S}^1$ and $\mathcal{S}^2$ in parameter plane for $n = 4$.

In Figure 3 we display the parameter plane for F_λ when $n = 4$ as well as a magnification of the region around the McMullen domain. Now there appears to be a closed curve $\mathcal{S}^1$ that meets the three principal Mandelbrot sets caught between three large Sierpinski holes, each of which has escape time 3. This is the dividing circle. Inside this curve is another closed curve $\mathcal{S}^2$ meeting 9 Sierpinski holes, each with escape time 4. Between these holes we again see either a smaller copy of a

Mandelbrot set or a portion of the principal Mandelbrot set. Further inside there is another curve curve containing 33 holes with escape time 5.

The following result concerning the existence of these special closed curves in parameter space was proved in [4]:

Theorem. (Rings Around the McMullen Domain). *For each $n \geq 3$, the McMullen domain in the parameter space for the family $z^n + \lambda/z^n$ is surrounded by infinitely many disjoint simple closed curves (rings) $\mathcal{S}^d$ for $d = 1, 2, \ldots$ having the property that:*

(1) *Each curve $\mathcal{S}^d$ surrounds the McMullen domain as well as $\mathcal{S}^{d+1}$, and the $\mathcal{S}^d$ accumulate on the boundary of the McMullen domain as $d \to \infty$;*
(2) *The curve $\mathcal{S}^d$ contains τ_d^n parameter values for which the critical orbits all land on ∞ after $d + 2$ iterations where*
$$\tau_d^n = (n-2)n^{d-1} + 1.$$
(3) *The curve $\mathcal{S}^d$ also passes through τ_d^n superstable parameter values where one of the critical points of F_λ is periodic of period d (when n is even) or is mapped to either itself or interchanged with its negative by F_λ^d (when n is odd).*

Our goal in this paper is to extend this result in two ways. We first prove that, with one exception, each of the superstable parameters lying on these curves is actually the center of a small copy of a Mandelbrot set:

Theorem (Satellite Mandelbrot Sets). *For $n \geq 3$ and $d \neq 2$, each of the τ_d^n superstable parameter values lying on the curve $\mathcal{S}^d$ lie at the center of the main cardioid of a Mandelbrot set with base period d. When $d = 2$, exactly $n - 1$ of these parameters lie at the center of the period 2 bulb of the principal Mandelbrot sets that have base period 1, while the remaining superstable parameters along $\mathcal{S}^2$ lie at the centers of Mandelbrot sets with base period 2.*

Our second result applies to the Sierpinski holes. Roesch [15] has shown that, in the case $n = 2$, each Sierpinski hole is homeomorphic to a disk and that there is a natural uniformization of this disk taking the center of the hole to the origin of the unit disk in $\mathbb{C}$. This uniformization is similar to the uniformization of the exterior of the Mandelbrot set in [7]. Here we make a minor modification to the methods in Roesch's paper in order to extend this result to the case of any $n \geq 3$:

Theorem (Structure of Sierpinski Holes). *Each Sierpinski hole in parameter space is a simply connected open set in which there is a unique parameter for which the critical orbits all land on ∞ at a particular iteration of F_λ.*

As a consequence of this result, each of the rings $\mathcal{S}^d$ also passes through the centers of τ_d^n Sierpinski holes in parameter space with escape time $d + 2$, and these Sierpinski holes and baby Mandelbrot sets are arranged in alternate fashion around $\mathcal{S}^d$.

More generally, this result allows us to determine the precise number of Sierpinski holes lying in the parameter plane for each value of n:

Corollary. *For each $n \geq 3$, there are exactly $(2n)^{k-3}(n-1)$ Sierpinski holes with escape time $k \geq 3$ in the parameter plane.*

2. Prior Results

For the remainder of this paper we shall restrict attention to the family of rational maps given by

$$F_\lambda(z) = z^n + \lambda/z^n$$

where $n \geq 3$ since it is known that there is no McMullen domain when $n = 1, 2$. Our goal in this section is to summarize some of the known results concerning this family that are relevant to our main results.

2.1. **Elementary Mapping Properties.** In the dynamical plane, the object of principal interest is the *Julia set* of F_λ, which we denote by $J(F_\lambda)$. The Julia set is the set of points at which the family of iterates of F_λ, $\{F_\lambda^n\}$, fails to be a normal family in the sense of Montel. It is known that $J(F_\lambda)$ is also the closure of the set of repelling periodic points for F_λ as well as the boundary of the set of points whose orbits escape to ∞ under iteration of F_λ. See [12].

The point at ∞ is a superattracting fixed point for F_λ and we denote the immediate basin of ∞ by B_λ. It is well known that F_λ is conjugate to $z \mapsto z^n$ in a neighborhood of ∞ in B_λ. There is also a pole of order n for F_λ at the origin, so there is a neighborhood of 0 that is mapped into B_λ by F_λ. If this neighborhood is disjoint from B_λ, then we denote the preimage of B_λ that contains 0 by T_λ. So F_λ maps both B_λ and T_λ in n-to-1 fashion over B_λ. We call T_λ the *trap door* since any orbit that eventually enters the immediate basin of ∞ must "fall through" T_λ enroute to B_λ.

The map F_λ has $2n$ other critical points given by $\lambda^{1/2n}$. We call these points the *free critical points* for F_λ. There are, however, only two critical values for each F_λ, and these are given by $\pm 2\sqrt{\lambda}$. We denote a free critical point by c_λ and a critical value by v_λ. The map also has $2n$ prepoles given by $(-\lambda)^{1/2n}$. Note that all of the critical points and prepoles lie on the circle of radius $|\lambda|^{1/2n}$ centered at the origin. We call this circle the *critical circle* and denote it by C_λ. A straightforward computation shows that C_λ is mapped in $2n$-to-1 fashion onto the straight line segment connecting the two critical values and passing through the origin. We call this line segment the *critical segment.* One also checks easily that any other circle centered at the origin is mapped by F_λ in n-to-1 fashion onto an ellipse surrounding the critical segment whose foci lie at the critical values. In particular, F_λ maps the open disk outside the critical circle as an n to 1 covering onto the exterior of the critical segment in $\overline{\mathbb{C}}$. F_λ maps the interior of the critical circle in similar fashion to the complement of the critical segment.

We call the straight ray connecting the origin to ∞ and passing through one of the critical points (resp., prepoles) a *critical point ray* (resp., *prepole ray*). Each of the $2n$ critical point rays is mapped 2-to-1 onto one of the two straight line segments of the form tv_λ, where $t \geq 1$ and v_λ is the image of the critical point on this ray. So the image of a critical point ray is a straight ray connecting v_λ to ∞. Each of the prepole rays is mapped 1-to-1 onto the straight line $it\sqrt{\lambda}$, where t is now any real number. Note that the image of each prepole ray is a straight line that is perpendicular to the line tv_λ, $t \in \mathbb{R}$, that contains the critical segment as well as the images of all of the critical point rays.

Let S_λ be an open sector bounded by two prepole rays corresponding to adjacent prepoles on C_λ, so S_λ is a sector in the plane having angle $2\pi/2n$. We call S_λ a *critical point sector* since it contains at its "center" a unique critical point of F_λ.

Similarly, let P_λ be the open sector bounded by two critical point rays corresponding to adjacent critical points on C_λ. We call P_λ a *prepole sector*. The following result follows immediately from the above:

Proposition (Mapping Properties of F_λ).

(1) *F_λ maps each critical point sector* 2-*to*-1 *onto the open half plane that is bounded by the image of the prepole lines and contains the critical value that is the image of the unique critical point in the sector;*
(2) *F_λ maps each prepole sector* 1-*to*-1 *onto the entire plane minus the two half lines $\pm tv_\lambda$ where $t \geq 1$;*
(3) *F_λ maps the region in both the interior and the exterior of the critical circle onto the complement of the critical segment as an n-to-1 covering map.*

For more details, we refer to [1].

2.2. **Symmetries.** We now turn to the symmetry properties of F_λ in both the dynamical and parameter planes. Let $\beta = \exp(\pi i/n)$ so that β is a primitive $2n^{\text{th}}$ root of unity. Then, for each j, we have $F_\lambda(\beta^j z) = (-1)^j F_\lambda(z)$. Hence, if n is even, we have $F_\lambda^2(\beta^j z) = F_\lambda(z)$. Therefore the orbits of z and $\beta^j z$ land on the same orbit after two iterations and so have the same eventual behavior for each j. If n is odd, the orbits of $F_\lambda(z)$ and $F_\lambda(\beta^j z)$ are either the same or else they become the negatives of each other after the first iteration. In either case it follows that the orbits of $F_\lambda(\beta^j z)$ behave symmetrically under $z \mapsto -z$ for each j. Hence the Julia set of F_λ is symmetric under $z \mapsto \beta z$ and we say that $J(F_\lambda)$ has $2n$-fold symmetry. In particular, each of the free critical points eventually maps onto the same orbit (in case n is even) or onto one of two symmetric orbits (in case n is odd). Thus these orbits all have the same ultimate behavior. This is why the λ-plane is a natural parameter plane for each of these families.

Let $H_\lambda(z)$ be one of the n involutions given by $H_\lambda(z) = \lambda^{1/n}/z$. Then $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, so that the Julia set is also preserved by each of these involutions. Note that each H_λ maps the critical circle to itself and maps circles centered at the origin that lie outside the critical circle to similar circles that lie inside the critical circle. It follows that two such circles, one inside and one outside the critical circle, are mapped n-to-1 onto the same ellipse by F_λ.

The parameter plane (see Figures 2 and 3) for F_λ also possesses several symmetries. First of all, we have

$$\overline{F_\lambda(z)} = F_{\overline{\lambda}}(\overline{z})$$

so that F_λ and $F_{\overline{\lambda}}$ are conjugate via the map $z \mapsto \overline{z}$. Therefore the parameter plane is preserved by complex conjugation.

We also have $(n-1)$-fold symmetry in the parameter plane for F_λ. To see this, let $\omega = \exp(2\pi i/(n-1)$, so ω is a primitive $(n-1)^{\text{st}}$ root of unity. Then, if n is even, we compute that

$$F_{\lambda\omega}(\omega^{n/2} z) = \omega^{n/2}(F_\lambda(z)).$$

As a consequence, for each $\lambda \in \mathbb{C}$, the maps F_λ and $F_{\lambda\omega}$ are conjugate under the linear map $z \mapsto \omega^{n/2} z$. In particular, since when $\lambda \in \mathbb{R}^+$, the positive real axis is preserved by F_λ, it follows that the straight ray $\omega^{n/2} \cdot \mathbb{R}^+$ is preserved by $F_{\lambda\omega}$.

When n is odd, the situation is a little different. We now have

$$F_{\lambda\omega}(\omega^{n/2} z) = -\omega^{n/2}(F_\lambda(z)).$$

Since $F_\lambda(-z) = -F_\lambda(z)$, we therefore have that $F^2_{\lambda\omega}$ is conjugate to F^2_λ via the map $z \mapsto \omega^{n/2}z$. This means that the dynamics of F_λ and $F_{\lambda\omega}$ are "essentially" the same, though subtly different. For example, if F_λ has a fixed point, then under the conjugacy, this fixed point and its negative are mapped to a 2-cycle for $F_{\lambda\omega}$. The line $\omega^{n/2} \cdot \mathbb{R}^+$ is no longer invariant under $F_{\lambda\omega}$ when $\lambda \in \mathbb{R}^+$ but rather it is interchanged with $-\omega^{n/2} \cdot \mathbb{R}^+$. In any event, it follows that the behavior of F_λ for $\lambda \in \omega^j \cdot \mathbb{R}^+$ is essentially the same as that of maps with corresponding positive real parameters. We therefore call the rays $\omega^j \cdot \mathbb{R}^+$ the symmetry axes in parameter space.

To summarize the symmetry properties of F_λ, we have:

Proposition (Symmetries in the dynamical and parameter plane). *The dynamical plane for F_λ is symmetric under the map $z \mapsto \beta z$ where β is a primitive $2n^{\text{th}}$ root of unity. The parameter plane is symmetric under both $z \mapsto \overline{z}$ and $z \mapsto \omega z$ where ω is a primitive $(n-1)^{\text{st}}$ root of unity.*

2.3. Rings in Parameter Plane.

For each n, let λ^* be the unique real solution to the equation

$$|v_\lambda| = 2|\sqrt{\lambda}| = |\lambda|^{1/2n} = |c_\lambda|.$$

A computation shows that

$$\lambda^* = \left(\frac{1}{4}\right)^{\frac{n}{n-1}}.$$

The circle of radius λ^* plays an important role in the parameter plane, for if λ lies on this circle, it follows that both of the critical values lie on the critical circle for F_λ. We call the circle of radius λ^* in parameter plane plane the *dividing circle.* We will be primarily concerned in later sections with values of the parameter that lie on or inside the dividing circle. In a subsequent paper we shall describe the very different structure of the parameter plane in the exterior of the dividing circle.

The dividing circle plays the role of the ring $\mathcal{S}^1$ in parameter plane. If λ lies on this circle, then both v_λ and c_λ lie on the same circle (the critical circle) in the dynamical plane. As λ winds once around the dividing circle in the counterclockwise direction beginning on the real axis, the critical points and prepoles of F_λ (given by $\lambda^{1/2n}$ and $(-\lambda)^{1/2n}$ respectively) each wind $1/2n$ of a turn around the critical circle, while the critical values $(\pm 2\sqrt{\lambda})$ each wind one half of a turn around the critical circle, all monotonically in the counterclockwise direction. Hence there are $n-1$ special parameter values on the dividing circle for which a critical point of the corresponding map equals a critical value, so for these special λ-values we have a superattracting fixed point or 2-cycle for F_λ. There are also $n-1$ other parameters on this circle for which the critical value is a prepole, so these are centers of a Sierpinski hole with escape time 3.

For the rest of this paper, we shall restrict attention to parameter values that lie inside the dividing circle in parameter space. For these parameters, we have that both critical values lie strictly inside the critical circle, so the entire critical segment lies inside this circle as well. Since F_λ maps the region outside the critical circle as an n-to-1 covering onto the complement of the critical segment, it follows that there is a simple closed curve that is mapped by F_λ n-to-1 to the critical circle. Denote this curve by γ_1. Then F_λ maps the exterior of γ_1 as an n-to-1 covering onto the exterior of the critical circle, so there is another simple closed curve γ_2

lying outside γ_1 that is mapped n-to-1 to γ_1. Continuing in this manner, we find a succession of simple closed curves γ_j having the property that γ_j lies outside γ_{j-1} and F_λ maps γ_j as an n-to-1 covering onto γ_{j-1} and hence as an n^j-to-1 covering onto the critical circle.

We may parametrize the γ_j as follows. We restrict to the case where $\lambda \notin \mathbb{R}^+$ so $0 < \arg\lambda < 2\pi$. First consider the critical circle. Let $c_0 = c_0(\lambda)$ denote the critical point on the critical circle satisfying $\arg c_0 = (\arg\lambda)/2n$. Then parametrize the critical circle by $\gamma_0(\theta)$ where $\gamma_0(0) = c_0$ and $\gamma_0(\theta)$ rotates once around the critical circle in the usual manner in the counterclockwise direction as θ increases from 0 to 2π. Note that this parametrization depends analytically on λ since c_0 does. Then parametrize γ_1 by requiring that $F_\lambda(\gamma_1(\theta)) = \gamma_0(\theta)$. So $\gamma_1(\theta)$ is periodic with period $2n\pi$. Inductively, define $\gamma_j(\theta)$ in similar fashion by requiring that $F_\lambda(\gamma_j(\theta)) = \gamma_{j-1}(\theta)$ and so $\gamma_j(\theta)$ is periodic of period $2^j\pi$.

As proved in [4], there is then a unique parameter λ_θ^j for which the second image of c_0 is given by $\gamma_j(\theta)$. This then gives a parametrization of the portion of the ring $\mathcal{S}^j$ in parameter plane lying off $\mathbb{R}^+$. In particular, the centers of the baby Mandelbrot sets and the Sierpinski holes in our main results will be such parameters.

2.4. **Parameters on the Real Axis.** Since there is a principal Mandelbrot set $\mathcal{M}$ straddling $\mathbb{R}^+$, it is well known from real dynamics that there is a sequence of parameters $\lambda_d \in \mathbb{R}^+$ for $d = 1, 2, \ldots$ having the property that the critical point $c_0 \in \mathbb{R}^+$ has period d and the orbit of c_0 is monotonic along $\mathbb{R}^+$ in the sense that, for $\lambda = \lambda_d$,

$$0 < F_\lambda(c_0) < c_0 = F_\lambda^d(c_0) < F_\lambda^{d-1}(c_0) < F_\lambda^{d-2}(c_0) < \cdots < F_\lambda^2(c_0).$$

Because of this monotonicity, these periodic orbits cannot have arisen from a bifurcation from some cycle of lower period (except in the case $d = 2$, where this does in fact occur). Consequently each λ_d (except λ_2) lies at the center of the main cardioid of a baby Mandelbrot set lying along the positive real axis. For λ_1, this Mandelbrot set is the principal Mandelbrot set $\mathcal{M}$ alluded to earlier. For each $d > 2$, λ_d lies at the center of a small copy of a Mandelbrot set contained in the "spine" of $\mathcal{M}$ and we have $\lambda_{d+1} < \lambda_d$ for each d. The parameter λ_2 lies at the center of the period 2 bulb of $\mathcal{M}$. Also, as $d \to \infty$, λ_d approaches the tip of the tail of $\mathcal{M}$ which is known to lie on the boundary of the McMullen domain. Hence $\mathcal{M}$ extends all the way from the Cantor set locus to the McMullen domain along $\mathbb{R}^+$.

As above, let ω be the primitive $(n-1)^{\text{st}}$ root of unity given by $\exp(2\pi i/(n-1))$. By the symmetry described above, there is a similar sequence of parameter values $\omega\lambda_d$ lying on the line $\omega \cdot \mathbb{R}^+$ in the parameter plane. For these parameter values, the ray $\omega^{n/2} \cdot \mathbb{R}^+$ is invariant if n is even, whereas, if n is odd, F_λ interchanges $\pm\omega^{n/2} \cdot \mathbb{R}^+$. In this latter case, it follows that $F_{\omega\lambda_d}$ has a superattracting cycle of period $2d$ when d is odd, or a pair of superattracting cycles of period d when d is even.

3. Baby Mandelbrot Sets

In this section we prove the existence of $(n-2)n^{d-1} + 1$ homeomorphic copies of the Mandelbrot set with base period $d \geq 2$ arranged around the ring $\mathcal{S}^d$ surrounding the McMullen domain.

3.1. **Polynomial-like Maps.** The main tool for proving the existence of these sets is the Douady-Hubbard theory of polynomial-like maps. To define these maps, let U' and U be open simply connected subsets of $\mathbb{C}$ with $\overline{U}' \subset U$. An analytic map $G : U' \to U$ is called a *polynomial-like map* (of degree two) if G is proper of degree two (so each point in U has exactly two preimages in U' when counted with multiplicity). It follows that such a polynomial-like map has a unique critical point in U'.

Now suppose we have a family of polynomial-like maps given by $G_\mu : U'_\mu \to U$. Usually the range of such a family also depends on the parameter, but this will not be the case for us. We assume that:

(1) G_μ depends analytically on μ and the parameters μ lie in a closed disk W;
(2) The set U'_μ depends continuously on μ;
(3) For μ in ∂W, v_μ lies in $U - U'_\mu$, where c_μ is the critical point of G_μ in U'_μ and $v_\mu = G_\mu(c_\mu)$;
(4) As μ rotates once around the boundary of W, the winding number of $v_\mu - c_\mu$ is ± 1.

Under these conditions, Douady and Hubbard [8] have shown that there is a homeomorphic copy of the Mandelbrot set lying in W. In addition, if μ lies in this Mandelbrot set, then the set of points in U'_μ whose orbits under G_μ never leave U'_μ is homeomorphic to the filled Julia set of the quadratic polynomial corresponding to G_μ under the above homeomorphism.

In the case of the families F_λ when n is odd, we sometimes need a slight modification of this result. Recall that, in this case, we have $F_\lambda(-z) = -F_\lambda(z)$. So suppose that the functions G_μ above also have this property. Let $E_\mu(z) = -G_\mu(z)$. Then $E_\mu^2 = G_\mu^2$ so that orbits of E_μ and G_μ agree after every second iteration. In addition to the four assumptions above, suppose also that, for each μ, the open sets U_μ have the property that

(1) $G_\mu(U_\mu)$ is disjoint from U_μ;
(2) but the family of maps E_μ is a polynomial-like family for $\mu \in W$.

Then, as above, there is a Mandelbrot set in W for the family E_μ. Hence there is also a Mandelbrot set for the family G_μ in W. The only difference here is that the actual periods of the bulbs in the Mandelbrot set for G_μ are twice the periods of the corrresponding bulbs for E_μ.

3.2. **Existence of Baby Mandelbrot Sets.** In this section we prove that for each $d \geq 2$ there are $(n-2)n^{d-1} + 1$ baby Mandelbrot sets with base period d whose centers lie at the superstable parameters on the ring $\mathcal{S}^d$ (with the special exception in the case $d = 2$ noted earlier).

Recall that $\omega = \exp(2\pi i/(n-1))$. As we showed earlier, the parameter plane is symmetric under the rotation $\lambda \mapsto \omega\lambda$. From the results of Section 2.4, we know that, for each $d \geq 3$, there is a unique parameter value $\lambda_d \in \mathbb{R}^+$ that lies at the center of a baby Mandelbrot set that has base period d and whose center lies in $\mathcal{S}^d \cap \mathbb{R}^+$. For $d = 1$, the parameter value λ_1 lies at the center of the principal Mandelbrot set along $\mathbb{R}^+$; for $d = 2$, the parameter value λ_2 lies at the center of the period 2 bulb of this Mandelbrot set. The parameters $\omega^j\lambda_d$ have similar properties on the other symmetry axes, except that the periods may be $2d$ in certain cases when n is odd. This gives $n-1$ baby Mandelbrot sets of base period d whose centers lie at the intersection of $\mathcal{S}^d$ with one of the symmetry axes, so we need only prove the existence of

$$(n-2)\cdot n^{d-1}-n+2=(n-2)(n^{d-1}-1)$$

additional such sets. By the symmetry in the parameter plane, it suffices to restrict attention to the sector in parameter plane given by $0<\arg\lambda<2\pi/(n-1)$. Since this sector represents $1/(n-1)$ of the parameter plane, we shall therefore show that there exists

$$\kappa(d)=(n-2)(n^{d-1}-1)/(n-1)=(n-2)(n^{d-2}+n^{d-3}+\ldots+n+1)$$

additional such baby Mandelbrot sets with centers on $\mathcal{S}^d$ in the open sector given by $0<\arg\lambda<2\pi/(n-1)$.

For each λ in this sector, there is a unique critical point $c_0=c_0(\lambda)$ that satisfies $\arg c_0=(\arg\lambda)/2n$. We denote the remaining critical points by $c_j=c_j(\lambda)$ where the index j increases as the c_j move around the critical circle in the counterclockwise direction. Let $v_\lambda=F_\lambda(c_{2j})$ so that $-v_\lambda=F_\lambda(c_{2j+1})$. We similarly define $p_0=p_0(\lambda)$ to be the prepole satisfying $\arg p_0=(\pi+\arg\lambda)/2n$ and index the remaining p_j as above. So, along the critical circle, we have the following arrangement of critical points and prepoles:

$$0<\arg c_0<\arg p_0<\cdots<\arg c_{2n-1}<\arg p_{2n-1}<2\pi.$$

Of special importance in the sequel are the points given by $F_\lambda^2(c_j)=F_\lambda(\pm v_\lambda)$. We compute that

$$F_\lambda(\pm v_\lambda)=(\pm 2)^n\lambda^{n/2}+\frac{1}{(\pm 2)^n\lambda^{\frac{n}{2}-1}},$$

so this point is independent of which critical value we choose if n is even, while, if n is odd, there are two symmetrically located points. So, to specify this point uniquely, we define the function G by

$$G(\lambda)=F_\lambda^2(c_0)=2^n\lambda^{n/2}+\frac{1}{2^n\lambda^{\frac{n}{2}-1}}.$$

Since we have restricted attention to the open sector in the parameter plane given by $0<\arg\lambda<2\pi/(n-1)$, we always use the square root of λ that lies in the region $0<\arg\sqrt{\lambda}<\pi/(n-1)$ to compute G. Note that G is defined in the parameter plane but takes values in the dynamical plane.

Let

$$\nu=\left(\frac{n-2}{n}\right)^{\frac{1}{n-1}}\left(\frac{1}{4}\right)^{\frac{n}{n-1}}\in\mathbb{R}^+.$$

A straightforward computation shows that G has $n-1$ critical points that are given by $\omega^j\nu$ for $j=0,\ldots,n-2$. Moreover, G maps the interval $[0,\nu]\subset\mathbb{R}^+$ univalently onto $[G(\nu),\infty]\subset\mathbb{R}^+$ and the segment $\omega\cdot[0,\nu]$ univalently onto the ray $\omega^{n/2}\cdot[G(\nu),\infty]$. Note that this latter ray lies on the line $\arg z=n\pi/(n-1)$.

Recall that $\lambda^*=4^{n/(1-n)}$ is the radius of the dividing circle in parameter plane. Then we have

$$\nu=\left(\frac{n-2}{n}\right)^{\frac{1}{n-1}}\left(\frac{1}{4}\right)^{\frac{n}{n-1}}<\left(\frac{1}{4}\right)^{\frac{n}{n-1}}=\lambda^*.$$

Consider the restriction of G to the arc along the dividing circle lying in the sector $0<\arg\lambda<2\pi/(n-1)$. Let η denote this arc. If $\lambda\in\eta$, then both $c_0(\lambda)$ and $v_\lambda=F_\lambda(c_0)$ lie on the critical circle for the corresponding F_λ and we have

$$0<\arg c_0(\lambda)<\arg v_\lambda=(\arg\lambda)/2<\pi/(n-1).$$

Since v_λ lies on the critical circle, it follows that $G(\lambda) = F_\lambda(v_\lambda)$ lies on the critical segment for F_λ. By the above, since the critical segment connects $\pm v_\lambda$, it is constrained to lie in the region $0 < \arg\lambda < \pi/(n-1)$ and the negative of this sector given by $\pi < \arg\lambda < n\pi/(n-1)$. The only exception to this occurs at the special parameter value $\hat{\lambda} = 4^{n/(1-n)}\exp(i\pi/(n-1))$. This is the unique parameter on η for which $F_{\hat{\lambda}}^2(c_0) = 0$, i.e., $\hat{\lambda}$ is the center of a Sierpinski hole with escape time 3 that meets this portion of the dividing circle. Hence G maps η strictly outside the sector $n\pi/(n-1) \leq \arg\lambda \leq 2\pi$ except at $\hat{\lambda}$ which is mapped to the origin.

We next claim that G is univalent on η. Indeed, since $G(\lambda)$ lies on the critical segment for F_λ and the argument of this segment, $\arg\lambda/2$, increases monotonically as $\arg\lambda$ increases from 0 to $2\pi/(n-1)$, it follows that no two parameters on η are mapped to the same point by G.

Let U denote the open sector in dynamical plane given by $n\pi/(n-1) < \arg z < 2\pi$. Also let X denote the portion of the sector in parameter plane given by $0 < \arg\lambda < 2\pi/(n-1)$ that lies on or inside the dividing circle. The set X is bounded by the segments $[0,\lambda^*]$ and $\omega\cdot[0,\lambda^*]$ together with the arc η. We have shown that G maps the boundary of X onto a curve in $\overline{\mathbb{C}}$. The map is univalent on the boundary of X with the following exceptions:

(1) On the portion of ∂X in $\mathbb{R}^+$, G maps the interval $[\nu, \lambda^*]$ one-to-one to $[G(\nu), G(\lambda^*)]$ and, since ν is a critical point of G, there is a second interval of the form $[a,\nu]$ with $a > 0$ that is also mapped to the same interval;

(2) On the portion of ∂X in $\omega\cdot\mathbb{R}^+$, there are a symmetric pair of intervals mapped to the same interval in $\omega^{n/2}\cdot\mathbb{R}^+$.

By the above, we have that the image of the boundary of X therefore lies either along the boundary of the closed sector $\overline{U}$ or else outside of it. If $\lambda \in X$ is very close to the origin, then

$$G(\lambda) \approx \frac{1}{2^n\lambda^{\frac{n}{2}-1}},$$

so the argument of $G(\lambda)$ lies between $n\pi/(n-1)$ and 2π. Since there are no critical points of G in the interior of X, it follows that G maps $\overline{X}$ univalently (except on the two intervals above) over a region that completely contains the sector $\overline{U}$. Hence there is a curve ξ in $\overline{X}$ that is mapped onto the union of the two segments $[0, G(\nu)]$ and $\omega^{n/2}\cdot[0, G(\nu)]$ on the boundary of $\overline{U}$. Note that ξ meets the dividing circle only at the parameter value $\hat{\lambda}$ that is mapped to 0 by G, and ξ meets $\mathbb{R}^+$ at ν and $\omega\cdot\mathbb{R}^+$ at $\omega\nu$.

We can now define the subset W of parameter space that will be used in the polynomial-like map construction. Let W be the open region in X bounded by the segments $[0,\nu]$ and $\omega\cdot[0,\nu]$ together with the curve ξ. See Figure 4. By the above, G maps the boundary of W univalently onto the union of $\mathbb{R}^+$, $\omega\cdot\mathbb{R}^+$, the origin, and ∞, i.e., the boundary of $\overline{U}$. We have shown:

Proposition 1. *G maps the closure of W univalently onto the closed sector $\overline{U}$ in $\overline{\mathbb{C}}$. Moreover, as λ travels once around the boundary of W, $G(\lambda)$ travels once around the boundary of this sector in $\overline{\mathbb{C}}$.*

Remark. Notice that the set W does not depend on λ or on which of the many Mandelbrot sets that will be produced. This is not a problem as it is the domains of the polynomial-like family that will depend on λ as will how they are mapped around the plane under various iterations of F_λ.

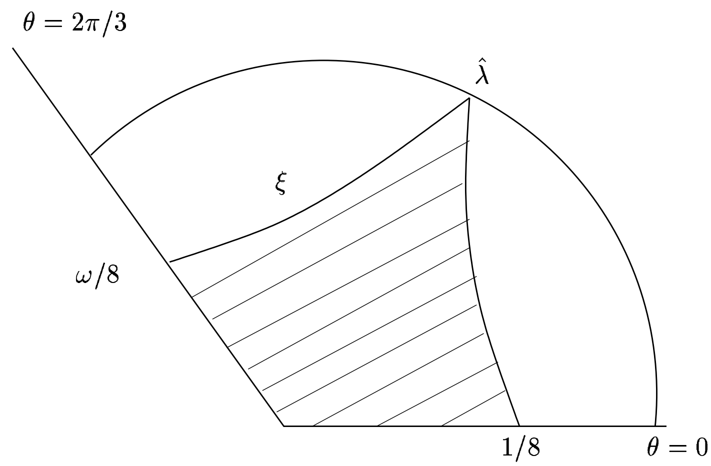

FIGURE 4. The region W in the parameter plane when $n = 4$. In this case, $\nu = 1/8$.

We now turn to the construction of the sets V'_λ that will serve as the domains of the functions in the polynomial-like family. We make the further restriction for the rest of this section that $\lambda \in W$. Since $0 < \arg \lambda < 2\pi/(n-1)$, we have that the $n-2$ critical points of F_λ given by $c_{n+2}(\lambda), \ldots, c_{2n-1}(\lambda)$ always lie in U. This follows since

$$\arg c_{n+2} = \frac{\arg \lambda}{2n} + \frac{(n+2)\pi}{n} > \frac{(n+2)\pi}{n} > \frac{n\pi}{n-1}$$

while

$$\begin{aligned} \arg c_{2n-1} &= \frac{\arg \lambda}{2n} + \frac{(2n-1)\pi}{n} \\ &< \frac{2\pi}{2n} + \frac{(2n-1)\pi}{n} = 2\pi. \end{aligned}$$

A similar computation shows that there are $n-1$ prepoles in U for each $\lambda \in W$, namely $p_{n+1}, \ldots, p_{2n-1}$. Hence there are $n-2$ critical point sectors that lie in the open sector U for each $\lambda \in W$.

Now fix $d \geq 2$. We shall construct $\kappa(d)$ different sets V'_λ inside U that are each mapped onto U properly of degree two by F^d_λ. The difference between these $\kappa(d)$ sets will be determined by how the orbits of the sets $F^j_\lambda(V'_\lambda)$ move through the plane for $0 \leq j < d$.

Let $P_0(\lambda)$ denote the closed inner prepole sector containing $p_0(\lambda)$, i.e., the sector bounded by the rays through c_0 and c_1 and contained on or inside the critical circle. We have $0 < \arg z < \pi/(n-1)$ for any $z \in P_0(\lambda)$ and $\lambda \in W$ since $0 < \arg c_0 < \arg c_1 < \pi/(n-1)$ for these λ-values. Recall that U is the open sector given by

$$\frac{n\pi}{n-1} < \arg z < 2\pi.$$

Proposition 2. *For each $\lambda \in W$, there is a preimage of U under F_λ that is a simply connected open set in $P_0(\lambda)$ and F_λ maps this preimage univalently onto U.*

The closure of this preimage meets the boundary of $P_0(\lambda)$ only at the origin and p_0.

Proof: The image of the entire set $P_0(\lambda)$ under F_λ is a half plane bounded by the straight line passing through both critical values and extending to ∞ in both directions. This line lies outside of U for each λ since $0 < \arg v_\lambda < \pi/(n-1)$ and $\pi < \arg(-v_\lambda) < n\pi/(n-1)$. The interior of $P_0(\lambda)$ is mapped univalently onto one of the two open half planes bounded by this straight line. A computation shows that the image of the prepole line in $P_0(\lambda)$ is a straight line that is perpendicular to the straight line passing through the critical values and that lies in the right half plane. Hence it follows that the image half plane contains U. Since the boundary of U meets the straight line boundary of this half plane only at the origin and ∞, it follows that the closure of the preimage of U in $P_0(\lambda)$ meets the boundary of $P_0(\lambda)$ only at the origin and p_0.

□

Let V_λ^1 denote the union of the preimage of U in $P_0(\lambda)$ together with the negative of this set. By Proposition 1, $G(\lambda) = F_\lambda(v_\lambda)$ lies in U for each $\lambda \in W$, so v_λ lies in the preimage of U in $P_0(\lambda)$ under F_λ. Similarly, $-v_\lambda$ lies in the other component of V_λ^1, so V_λ^1 contains both critical values. When n is even, since $F_\lambda(-z) = F_\lambda(z)$, the negative of the preimage of U is also a preimage of U. When n is odd, this set is mapped by F_λ to $-U$. In this case, the negative of the preimage of U is contained in the inner prepole sector $P_n(\lambda) = -P_0(\lambda)$ containing the prepole $p_n(\lambda)$. As in the previous Proposition, this preimage is a simply connected open set and its closure meets the boundary of $P_n(\lambda)$ only at the origin and p_n. Note that, unlike U, these two preimage sets depend on λ. Also, since $0 < \arg c_0 < \arg c_1 < \pi/n - 1$, we have that the sets $P_0(\lambda)$ and $P_n(\lambda)$ do not meet either of $\pm U$, and so V_λ^1 is disjoint from $\pm U$. See Figure 5. We have shown:

Proposition 3. *For each $\lambda \in W$, the set V_λ^1 is contained in the union of the prepole sectors $P_0(\lambda)$ and $P_n(\lambda)$ and is disjoint from $\pm U$.*

Moreover, we also have:

Proposition 4. *Suppose $\lambda \in W$. Then the entire critical segment (excluding the origin) lies in V_λ^1.*

Proof: Let tv_λ be a point on the critical segment with $0 < |t| < 1$. We shall show that $F_\lambda(tv_\lambda)$ lies in U for each such t (when n is even) or in $\pm U$ (when n is odd).

First suppose that n is even. Since $v_\lambda = 2\sqrt{\lambda}$, we have that

$$F_\lambda(tv_\lambda) = (2t)^n \lambda^{\frac{n}{2}} + \frac{1}{(2t)^n \lambda^{\frac{n}{2}-1}}.$$

Note that the term

$$\frac{1}{(2t)^n \lambda^{\frac{n}{2}-1}}$$

in this sum always lies in U. Indeed, we have

$$0 < \arg \lambda^{\frac{n}{2}-1} < \frac{(n-2)\pi}{n-1}$$

so that

$$\frac{n\pi}{n-1} < \arg \frac{1}{\lambda^{\frac{n}{2}-1}} < 2\pi.$$

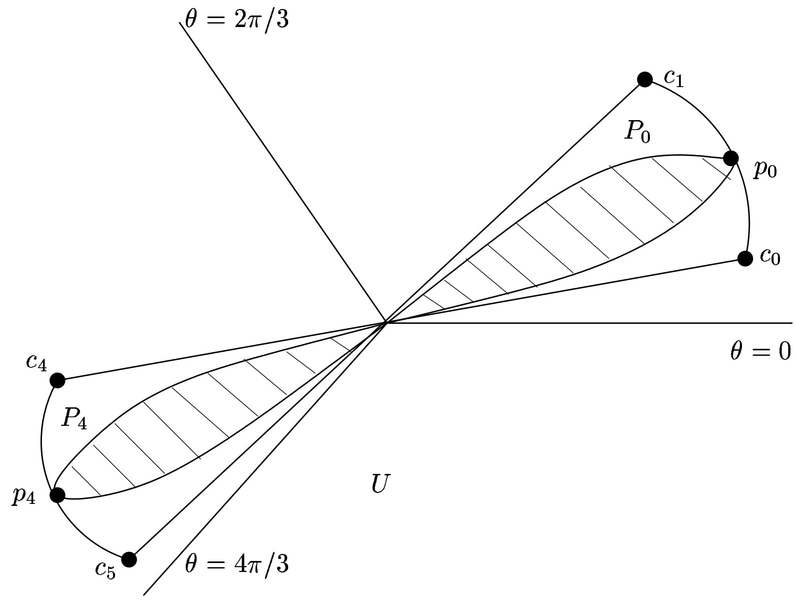

FIGURE 5. The region V_λ^1 in the dynamical plane when $n = 4$.

Since n is even, it therefore follows that

$$\frac{1}{(2t)^n \lambda^{\frac{n}{2}-1}}$$

lies in U for each $t \neq 0$.

Consider the closed sector bounded by the ray passing through the origin and $1/2^n\lambda^{\frac{n}{2}-1}$ and also the ray passing through the origin and $F_\lambda(v_\lambda)$. Call this sector Z. Note that $Z \subset U$ since both $1/2^n\lambda^{\frac{n}{2}-1}$ and $G(\lambda) = F_\lambda(v_\lambda)$ lie in U. Also note that, for each $t \neq 0$, the point $1/2^n t^n \lambda^{\frac{n}{2}-1}$ lies on the boundary of Z.

We now proceed geometrically. When $t = 1$, we think of the term $2^n\lambda^{n/2}$ as a vector in the plane extending from $1/2^n\lambda^{\frac{n}{2}-1}$ to $F_\lambda(v_\lambda)$ since we have $1/2^n\lambda^{\frac{n}{2}-1} + 2^n\lambda^{n/2} = F_\lambda(v_\lambda)$. For $|t| \neq 1$, the vector $2^n t^n \lambda^{n/2}$ is shorter than $2^n\lambda^{n/2}$ but points in the same direction as $2^n\lambda^{n/2}$. Meanwhile, the point $1/2^n t^n \lambda^{\frac{n}{2}-1}$ lies on the boundary of Z but further away from the origin than $1/2^n\lambda^{\frac{n}{2}-1}$. Hence the sum of $2^n t^n \lambda^{n/2}$ and $1/2^n t^n \lambda^{\frac{n}{2}-1}$ lies in Z and therefore also in U. So we have $F_\lambda(tv_\lambda) \in U$ for each t with $0 < |t| \leq 1$ and n even.

When n is odd, we have that the above proof works for $0 < t < 1$. For $-1 < t < 0$, by the $z \mapsto -z$ symmetry, we then have that $F_\lambda(tv_\lambda)$ lies in $-U$.

□

Now consider the entire preimage of V_λ^1 under F_λ, not just the preimages lying in $\pm P_0(\lambda)$. We denote this preimage by V_λ^2. Recall that each critical point sector is mapped two-to-one onto a half plane bounded by the straight line given by itv_λ with $t \in \mathbb{R}$, i.e., the line that is the image of the prepole rays and hence perpendicular to the critical segment. Observe that the interiors of the inner prepole sectors $P_0(\lambda)$

and $P_n(\lambda)$ are each contained in one of these half planes. This follows since

$$0 < \arg c_0 < \arg c_1 < \frac{\pi}{n-1} \leq \frac{\pi}{2} < \arg iv_\lambda < \frac{(n+1)\pi}{2(n-1)} \leq \pi$$

for $\lambda \in W$. Therefore each of the two open sets comprising V_λ^1 is completely contained in one of these half planes. It then follows that V_λ^2 consists of a collection of $2n$ disjoint open sets, one lying in each critical point sector. Each of these open sets contains the entire portion of the critical circle that lies in this critical point sector since, by Proposition 4, the critical segment minus the origin lies in the image of these sets. Each of these open sets in V_λ^2 is also mapped two-to-one onto one of the two components of V_λ^1. Note also that the open sets in V_λ^2 do not meet V_λ^1. This follows since the image of V_λ^1 is contained in $\pm U$ whereas the images of the open sets in V_λ^2, namely the components of V_λ^1, lie in the complement of these sectors by Proposition 3. Since the boundary of V_λ^1 meets the boundary of the image of each critical point sector only at the origin, it follows that the boundary of each component of V_λ^2 meets the boundary of the critical point sector in which it lies only at the two prepoles lying on the boundary rays of the sector. V_λ^2 therefore is a collection of $2n$ symmetrically arranged open sets whose closures meet only at the prepoles. See Figure 6. To summarize, we have:

Proposition 5. *Each of the $2n$ open sets comprising V_λ^2:*

(1) *lies in a distinct critical point sector;*
(2) *is disjoint from V_λ^1 and hence disjoint from the critical segment as well;*
(3) *meets the critical circle in the open arc between adjacent prepoles;*
(4) *is mapped two-to-one onto one of the two components of V_λ^1.*

Recall that γ_1 is the preimage of the critical circle lying outside C_λ and that $H_\lambda(\gamma_1)$ is the corresponding preimage of the critical circle lying inside C_λ. The set V_λ^2 does not meet γ_1 or $H_\lambda(\gamma_1)$ since the image of this set does not meet the critical circle. (The boundary of V_λ^2 does meet γ_1 and $H_\lambda(\gamma_1)$, but only at the preimages of the prepoles p_0 or p_n.)

3.3. **The Case $d = 2$.** We can now prove the Satellite Mandelbrot Sets Theorem in the case $d = 2$. We need to prove the existence of $\kappa(2) = n - 2$ baby Mandelbrot sets inside W with centers on $\mathcal{S}^2$. Consider the $n - 2$ critical point sectors that contain the critical points c_j where $j = n+2, \ldots, 2n-1$. We claim that the closures of each of these sectors is contained in $U \cup \{0\}$ for each $\lambda \in W$. Indeed, the union of these sectors is contained in the sector bounded by the prepole rays through p_{n+1} and p_{2n-1}. We have, since $n \geq 3$,

$$\arg p_{n+1} = \frac{\arg \lambda + \pi}{2n} + \frac{(n+1)\pi}{n} > \frac{(2n+3)\pi}{2n} \geq \frac{n\pi}{n-1}$$

and

$$\arg p_{2n-1} = \frac{\arg \lambda + \pi}{2n} + \frac{(2n-1)\pi}{n} < \frac{(4n^2-5n+3)\pi}{2n(n-1)} \leq 2\pi.$$

Now fix any j with $n + 2 \leq j \leq 2n - 1$. Define V_λ' to be the component of V_λ^2 that contains c_j. We shall show that F_λ^2 is a polynomial-like family that takes V_λ' to U for each $\lambda \in W$ when n is even, or to one of $\pm U$ when n is odd.

Suppose first that n is even. By construction, we then have that F_λ^2 takes V_λ' onto U in two-to-one fashion. We claim that $\overline{V}_\lambda'$ is strictly contained inside U. This

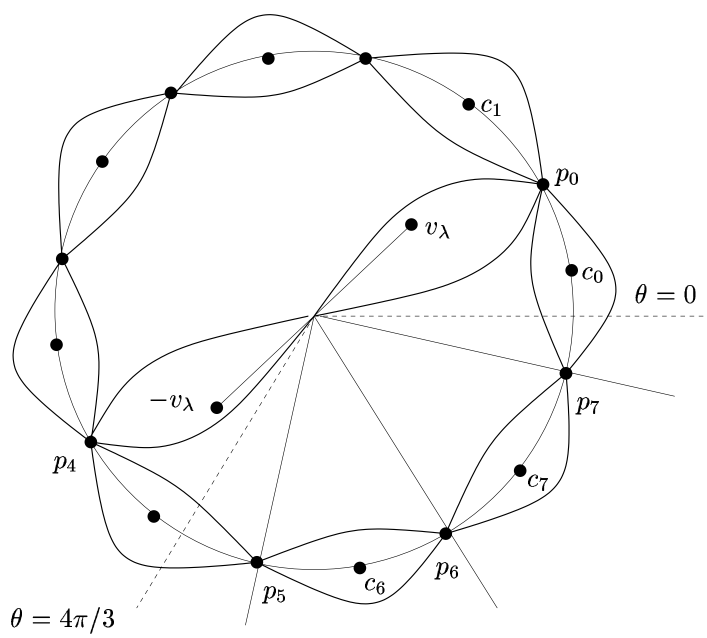

FIGURE 6. The region V_λ^2 in the dynamical plane together with the critical point sectors containing c_7 and c_8 when $n = 4$.

follows since the closure of the critical point sector containing V'_λ is contained in $U \cup \{0\}$ and $\overline{V}'_\lambda$ is bounded away from 0 in this sector. If $\lambda \in \partial W$, then $F_\lambda^2(c_j)$ lies on the boundary of U and makes one transit around this boundary as λ moves around the boundary of W as shown in Proposition 1. Hence $F_\lambda^2(c_j)$ winds once around V'_λ in the region $U - V'_\lambda$ for these λ-values. It follows that F_λ^2 is a polynomial-like family on W. This produces a baby Mandelbrot set with base period 2. By construction, the center of this Mandelbrot set is a parameter for which the critical point c_j is periodic with period 2, so this parameter lies on the curve $\mathcal{S}^2$. Moreover, since the centers of each of these Mandelbrot sets correspond to different superstable cycles, we have produced $n+1$ such sets, each having centers corresponding to a different critical point among the points $c_{n+2}, \ldots, c_{2n-1}$.

When n is odd, the image of V'_λ under F_λ^2 lies in either U or $-U$. In the former case, the previous arguments apply to produce a baby Mandelbrot set of base period 2 in W; in the latter case, by our modification of the Douady-Hubbard theory of polynomial-like maps in Section 3.1, we apply the result to $-F_\lambda^2$ to produce a baby Mandelbrot set of base period 2. This proves the Satellite Mandelbrot Set Theorem in the case $d = 2$.

3.4. **The Case $d \geq 3$.** For the general case $d \geq 3$, we again first restrict attention to the case where n is even. Let U_{out} denote the portion of the open sector U that lies on or outside the critical circle.

Proposition 6. *Suppose n is even. There is a closed subset Q_λ contained in U_{out} that has the property that F_λ maps Q_λ as a covering onto the set of points satisfying*

$$n\pi + \frac{n\pi}{n-1} \leq \arg z \leq 2n\pi.$$

Remark. We interpret this result geometrically as follows: Since

$$2n\pi - n\pi - \frac{n\pi}{n-1} = \left(\frac{n-2}{2}\right) \cdot 2\pi + \left(\frac{n-2}{n-1}\right)\pi,$$

it follows that F_λ wraps the region Q_λ a total of $(n-2)/2$ full revolutions in the clockwise direction around $\mathbb{C}$ starting at $\mathbb{R}^+$, followed by an additional rotation of $(n-2)\pi/(n-1)$ radians. In particular, the image of Q_λ covers the sector

$$2\pi - \left(\frac{n-2}{n-1}\right)\pi = \frac{n\pi}{n-1} \leq \arg z \leq 2\pi$$

exactly $n/2$ times and its complement $(n/2)-1$ times.

Proof: The boundary of U_{out} consists of three curves: the portion of the critical circle lying in $\overline{U}$ together with the rays in $\mathbb{R}^+$ and $\arg z = n\pi/(n-1)$ that extend from the critical circle to ∞. We claim that each of these curves is mapped to the complement of U by any F_λ with $\lambda \in W$, and this image meets $\overline{U}$ only at the origin.

To see this, note first that the critical circle portion of the boundary is wrapped by F_λ around the the critical segment and this segment lies outside of $\overline{U}$ (except at 0) for each $\lambda \in W$. The ray $\mathbb{R}^+$ is mapped to a curve of the form $t^n + \lambda/t^n$ with $t \geq 0$, so this image resides in the sector $0 < \arg z < 2\pi/(n-1)$ which lies in the upper half plane and so is disjoint from $\overline{U}$ since $\arg\lambda > 0$. Finally, if $\arg z = n\pi/(n-1)$, then

$$\arg z^n = \frac{n^2\pi}{n-1} = n\pi + \frac{n\pi}{n-1}.$$

So z^n lies along the straight line with argument $n\pi/(n-1)$ since n is even. Also,

$$-\frac{n\pi}{n-1} < \arg\frac{\lambda}{z^n} < \frac{(2-n)\pi}{n-1}$$

or, adding 2π to both sides,

$$\frac{(n-2)\pi}{n-1} < \arg\frac{\lambda}{z^n} < \frac{n\pi}{n-1}.$$

Hence the sum of these two terms must satisfy

$$\frac{(n-2)\pi}{n-1} < \arg\left(z^n + \frac{\lambda}{z^n}\right) < \frac{n\pi}{n-1},$$

and so the image of this boundary curve lies in a sector to the left of $\overline{U}$. Consequently, the image of each of the boundary curves lies outside $\overline{U}$ (except at 0) as claimed.

Now there are no critical points in the interior of U_{out}, so F_λ is a covering map on this set. Consider the arc of a circle ν with very large radius that lies in U_{out}. If $z \in \nu$, then

$$\frac{n\pi}{n-1} \leq \arg z \leq 2\pi.$$

Then, using the above, since $F_\lambda(z) \approx z^n$ for $|z|$ large, we have that $F_\lambda(\nu)$ is a nearly circular arc that extends from above $\arg z = 2n\pi$ to below

$$\arg z = \frac{n^2\pi}{n-1} = n\pi + \frac{n\pi}{n-1}.$$

So we define Q_λ to be the preimage of the region

$$Y = \{z \in \mathbb{C} \,|\, n\pi + \frac{n\pi}{n-1} \leq \arg z \leq 2n\pi\}$$

under F_λ. So Q_λ is a closed subset of U_{out} and F_λ takes Q_λ onto Y as a covering. This completes the proof.

□

Recall that F_λ is an n-to-1 covering map in the exterior of the critical circle that takes this region to the exterior of the critical segment. As we observed in Section 2, since the critical segment lies strictly inside the critical circle, it follows that there is a collection of simple closed curves γ_j for $j = 1, 2, \ldots$ lying outside the critical circle and having the property that F_λ maps γ_1 to the critical circle as an n-to-1 covering and also maps γ_{j+1} as an n-to-1 covering onto γ_j for each $j \geq 1$. So F_λ^j takes γ_j onto the critical circle as a n^j-fold covering. These simple closed curves are all disjoint and converge outward to the boundary of the basin of ∞ as $j \to \infty$ and γ_j is contained in the region inside γ_{j+1} for each j.

To construct the baby Mandelbrot sets along $\mathcal{S}^d$ for $d \geq 3$, we proceed by induction. We continue to assume that n is even. First consider the case $d = 3$. Recall that V_λ^2 consists of $2n$ disjoint open sets whose boundaries meet only at the prepoles along the critical circle. By Proposition 5, the closure of V_λ^2 contains the entire critical circle and is disjoint from the critical segment.

Since F_λ is an n-to-1 covering map outside the critical circle, and each component of V_λ^2 does not meet the critical segment, each component of V_λ^2 therefore has exactly n preimages outside the critical circle. We denote the union of these $2n^2$ preimages by V_λ^3. Each component of V_λ^3 is disjoint from V_λ^2 since, again invoking Proposition 5, the sets $F_\lambda(V_\lambda^3) = V_\lambda^2$ and $F_\lambda(V_\lambda^2) = V_\lambda^1$ are disjoint. Hence the entire set V_λ^3 lies strictly outside the critical circle. Also, each component of V_λ^3 meets the curve γ_1 since the image of this set meets the critical circle. Finally, the closure of V_λ^3 completely contains γ_1 since $F_\lambda(\gamma_1)$ is the critical circle, which lies in the closure of V_λ^2.

We claim that there are exactly $(n-2)(n+1)$ components of V_λ^3 that lie in the set $Q_\lambda \subset U_{\text{out}}$ given by Proposition 6. To see this, recall that $n-2$ of the $2n$ components of V_λ^2 are strictly contained inside U. By Proposition 6, $F_\lambda \,|\, Q_\lambda$ covers each component of V_λ^2 that does not lie in U a total of $(n-2)/2$ times, while $F_\lambda \,|\, Q_\lambda$ covers the $n-2$ components of V_λ^2 one additional time. Therefore there are

$$2n(n-2)/2 + n - 2 = (n-2)(n+1)$$

components of V_λ^3 that lie in Q_λ, as claimed. As in the case $d = 2$, each of these preimages is strictly contained inside U_{out} and is mapped by F_λ onto a component of V_λ^2 and then by F_λ^3 in two-to-one fashion onto U. Note that each of these components varies continuously with λ for $\lambda \in W$. Hence we have a family of polynomial-like maps defined on each of these preimages. Since the closures of each of these components is contained inside U, we may invoke a similar argument as in the case $d = 2$ to show that the critical value of F_λ^3 moves once around the boundary of U for parameters in the boundary of W. It follows that there is a baby

Mandelbrot set in W corresponding to each of these $(n+2)(n-1)$ polynomial-like families.

For $d > 3$, we continue in this fashion. There are a total of $2n^2$ components of V_λ^3 and $(n-2)(n+1)$ of them lie inside U. As above, F_λ maps Q_λ over the region Y as a covering, so there are

$$2n^2(n-2)/2 + (n-2)(n+1) = (n-2)(n^2+n+1)$$

components of V_λ^4 lying in U and F_λ^4 is a polynomial-like family on each of them. Continuing in this fashion shows that there are $(n-2)(n^{d-1} + \ldots + n + 1)$ similar components of V_λ^d in U on which F_λ^d is polynomial-like. This concludes the proof when n is even.

We finally turn to the case where n is odd. Because of the different symmetry present in this case, we need to modify the above proof as well as the consequences of Proposition 6 somewhat. Note first that, since n is odd, F_λ now maps the ray $\arg z = n\pi/(n-1)$ that bounds U into a different sector, namely the sector

$$|\arg z| < \frac{\pi}{n-1}.$$

This follows since $\arg z^n = \pi/(n-1)$ whereas

$$-\frac{\pi}{n-1} < \arg\frac{\lambda}{z^n} < \frac{\pi}{n-1}$$

so the sum of these terms lies in $|\arg z| < \pi/(n-1)$. As a consequence, the image of this boundary curve of U_{out} lies outside of the region $-U$, not U, as before. As in the case where n was even, the image of the portion of the boundary of U_{out} in $\mathbb{R}^+$ is mapped to the upper half plane. Note that this curve may meet a portion of $-U$, but this will not affect the definition of Q_λ below. And, again as before, the image of the critical circle misses both $\pm U$.

We then define Q_λ as before to be the preimage in U_{out} of the set

$$Y = \{z \in \mathbb{C} \,|\, n\pi + \frac{n\pi}{n-1} \le \arg z \le 2n\pi\}.$$

By the above, the straight line boundaries of U_{out} are mapped outside of the region Y. As when n was even, F_λ wraps Q_λ a total of $(n-2)/2$ revolutions around $\mathbb{C}$ in the clockwise direction, starting at $2n\pi$, and then followed by an additional rotation of $(n-2)\pi/(n-1)$ radians. Since n is odd, this now means that F_λ wraps Q_λ around the entire plane in the clockwise direction beginning at $2n\pi$ a total of $(n-1)/2$ full revolutions, minus a rotation of $\pi/(n-1)$ radians. Since F_λ maps the boundaries of U_{out} beyond the boundaries of Y, we again have that F_λ is a covering map taking the subset Q_λ onto Y.

Now recall that there are $n-2$ components of V_λ^2 that are strictly contained in U. By symmetry, there are the same number of components in $-U$. Since V_λ^2 has $2n$ components, that leaves four components of V_λ^2, two of which intersect each of the sectors between $\pm U$ (though they need not be completely contained in these regions). In particular, only 2 meet the sector $0 < \arg z < \pi/(n-1)$ that lies between U and $-U$ in the right half plane. Therefore we have that $F_\lambda(Q_\lambda)$ covers $2n(n-1)/2 - 2 = (n-2)(n+1)$ of these components. Hence there are the same number of components of V_λ^3 lying in U_{out}. So we have $(n-2)(n+1)$ different families of polynomial-like maps F_λ^3 defined on these different components.

The proof in the case $d > 3$ now proceeds exactly as before, with these modifications. Similarly the proof of the existence of the baby Mandelbrot sets also goes through as above, except that, since n is odd, F_λ^j may take a component of V_λ^j onto $-U$, but the base periods of these Mandelbrot sets are still j since we use $-F_\lambda^j$ as in the modification of the polynomial-like mapping theorem in Section 3.1. This completes the proof of the Satellite Mandelbrot Sets Theorem.

4. Sierpinski Holes

In this section we investigate the structure of all of the Sierpinski holes in the parameter plane of F_λ, not just those that lie along the rings $\mathcal{S}^d$. Let S be such a region. We shall construct a natural analytic homeomorphism Φ taking S to complement of the unit disk $\overline{\mathbb{C}} - \overline{\mathbb{D}}$. Φ will be a natural generalization of the uniformizing map of the exterior of the Mandelbrot set constructed by Douady and Hubbard in [7]. Φ will also be a straightforward modification of a similar map constructed by Roesch in [15] for the case $n = 2$. For this reason we call the map Φ the *Roesch map*.

Let $\lambda \in S$. Let $v_\lambda = 2\sqrt{\lambda}$ be one of the critical values of F_λ where we assume v_λ varies analytically with λ over all of S. This is possible since $\lambda \neq 0$ in S. Let V_λ be the component of the complement of the Julia set of F_λ that contains v_λ. Since there are no critical points in the forward orbit of V_λ until this orbit meets the trap door, there exists $d > 0$ such that F_λ^d is an analytic homeomorphism taking V_λ onto T_λ. Then H_λ is an analytic homeomorphism taking T_λ to B_λ. To specify H_λ, we fix a choice of one of the n involutions H_λ so that H_λ varies analytically as λ ranges over S. See Section 2.2. Finally there is another analytic homeomorphism, ϕ_λ (the Böttcher coordinate), taking B_λ onto $\overline{\mathbb{C}} - \overline{\mathbb{D}}$ and satisfying $\phi_\lambda(F_\lambda(z)) = (\phi_\lambda(z))^n$. So the composition $\phi_\lambda \circ H_\lambda \circ F_\lambda^d$ is an analytic homeomorphism taking V_λ to $\overline{\mathbb{C}} - \overline{\mathbb{D}}$.

We now define the Roesch map $\Phi : S \to \overline{\mathbb{C}} - \overline{\mathbb{D}}$ by

$$\Phi(\lambda) = \phi_\lambda(H_\lambda(F_\lambda^d(v_\lambda))).$$

Clearly, Φ is analytic and maps S into $\overline{\mathbb{C}} - \overline{\mathbb{D}}$. We need to show that Φ is one-to-one and onto. This will prove that each Sierpinski hole is simply connected. This will also prove that there is a unique λ in S that is mapped to ∞ with degree one by Φ, so there is a unique λ-value lying at the center of the Sierpinski hole, i.e., a unique parameter for which the critical points eventually land on ∞. Therefore we can give an exact count of the number of Sierpinski holes in the parameter plane. See below.

Recall that $\beta = \exp(\pi i/n)$ so that $\beta^{2n} = 1$ and $F_\lambda(\beta z) = -F_\lambda(z)$. Hence the Julia set of F_λ is symmetric under the map $z \mapsto \beta z$.

We now fix $\lambda \in S$. Let $B_\epsilon(v_\lambda)$ denote the ball of radius ϵ about v_λ. We choose ϵ small enough so that $B_{3\epsilon}(v_\lambda) \subset V_\lambda$. For any $w \in B_\epsilon(v_\lambda)$, we define a diffeomorphism $\tau_w : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ satisfying:

(1) τ_w is the identity map everywhere except on the $2n$ disjoint disks given by $\beta^j \cdot B_{3\epsilon}(v_\lambda)$ for $j = 1, \ldots, 2n$.
(2) Inside $B_\epsilon(v_\lambda)$, $\tau_w(z) = z + w - v_\lambda$, so τ_w just translates points in this ball by $w - v_\lambda$.
(3) In the annulus $B_{3\epsilon}(v_\lambda) - B_\epsilon(v_\lambda)$, τ_w is a C^∞ diffeomorphism mapping this annulus to the annulus $B_{3\epsilon}(v_\lambda) - B_\epsilon(w)$ and extending smoothly to the boundaries of $B_{3\epsilon}(v_\lambda) - B_\epsilon(v_\lambda)$.
(4) $\tau_w(\beta^j z) = \beta^j \tau_w(z)$, so this defines τ_w on all the symmetric copies of $B_{3\epsilon}(v_\lambda)$.
(5) τ_{v_λ} is the identity map everywhere and τ_w varies smoothly with w.

So, for each w, τ_w is a "bump function" that holomorphically moves v_λ to w and the symmetric images of v_λ to the corresponding symmetric images of w, and τ_w does nothing outside the balls of radius 3ϵ around these points.

Let $G_w = \tau_w \circ F_\lambda$. The function G_w is not holomorphic, but it is smooth. Note that G_w has a superattracting fixed point of order n at ∞ and a pole of order n at the origin. In fact, for each $w \in B_\epsilon(v_\lambda)$, the basin of ∞ for G_w is just B_λ and the trap door is T_λ since $G_w = F_\lambda$ on these sets. Indeed, all of the preimages of B_λ and T_λ under F_λ^{-i} and G_w^{-i} are the same sets; only the orbits of points within these sets have changed. Also, G_w has critical points (that is, points where G_w is not locally one-to-one) at the points c_λ, but the critical values are now given by $\pm w$ rather than $\pm v_\lambda$, and G_w has the same $2n$-fold symmetry as F_λ, i.e., $G_w(\beta^j z) = (-1)^j G_w(z)$.

We may define an ellipse field μ_w on $\overline{\mathbb{C}}$ that is preserved by G_w as follows. Let μ_w be the standard complex structure defined everywhere except on all of the preimages of the $2n$ open sets $\beta^j \cdot B_{3\epsilon}(v_\lambda)$ under G_w^{-k} for $k \geq 1$. On these preimages, let μ_w be the pullback by G_w^{-k} of the standard complex structure on the sets $\beta^j \cdot B_{3\epsilon}(v_\lambda)$. So, for example, μ_w is the standard complex structure on $B_{3\epsilon}(v_\lambda)$ but is given on $G_w^{-1}(B_{3\epsilon}(v_\lambda))$ by applying $F_\lambda^{-1} \circ \tau_w^{-1}$ to the standard complex structure on $B_{3\epsilon}(v_\lambda)$. Note that, by construction, G_w preserves the ellipse field μ_w and that μ_w varies smoothly with w.

Since for each j, the ellipse field μ_w on $F_\lambda^{-1}(\beta^j \cdot B_{3\epsilon}(v_\lambda))$ is obtained by pulling back the standard complex structure by a composition of a diffeomorphism and a holomorphic map, it follows μ_w has bounded dilatation on this subset. On subsequent preimages, μ_w is obtained by pulling back this structure by just a holomorphic map, so it follows that μ_w has bounded dilatation on all of $\overline{\mathbb{C}}$. Also, since these pullback maps respect the $z \mapsto \beta z$ symmetry, it follows that μ_w also has this $2n$-fold symmetry.

Since μ_w has bounded dilatation, we may apply the Measurable Riemann Mapping Theorem to straighten μ_w. This gives a quasiconformal homeomorphism $h_w : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that converts μ_w to the standard complex structure almost everywhere. We may normalize h_w so that $h_w(\infty) = \infty$ and $h_w(0) = 0$. Then the map given by $R_w = h_w \circ G_w \circ h_w^{-1}$ takes the standard complex structure to itself, so R_w is a holomorphic map which is, in fact, a rational map of degree $2n$. We may further normalize h_w by requiring that $h_w(\beta c_\lambda) = \beta h_w(c_\lambda)$ since we have only specified two conditions on h_w so far. This requirement actually implies much more about the symmetry of h_w.

Lemma. *$h_w(\beta z) = \beta h_w(z)$ for all $z \in \overline{\mathbb{C}}$.*

Proof: Suppose $h_w(\beta z) \neq \beta h_w(z)$ for some z. Then the function

$$g_w(z) = \beta^{-1} h_w(\beta z)$$

also straightens the ellipse field μ_w since $z \mapsto \beta z$ preserves μ_w, h_w converts μ_w to the standard complex structure, and then $z \mapsto \beta^{-1} z$ preserves the standard structure. But we have $g_w(\infty) = \infty$, $g_w(0) = 0$, and $g_w(c_\lambda) = \beta^{-1} h_w(\beta c_\lambda) = h_w(c_\lambda)$. So h_w and g_w agree at $0, \infty$, and c_λ. Therefore $h_w(z) = g_w(z)$ for all $z \in \overline{\mathbb{C}}$.

□

The map R_w has a pole of order n at the origin and a superattracting fixed point of order n at ∞ since G_w has these properties. Similarly, since $h_w(\beta z) = \beta h_w(z)$, it follows that $R_w(\beta^j z) = (-1)^j R_w(z)$. Therefore R_w actually assumes the form $F_\alpha(z) = z^n + \alpha/z^n$ for some $\alpha = \alpha(w)$ (see below).

It follows that h_w gives a quasiconformal conjugacy between the map G_w and F_α. Note that the basin of ∞ for F_α is given by $B_\alpha = h_w(B_\lambda)$ and the trap door for F_α is given by $T_\alpha = h_w(T_\lambda)$ since B_λ is the basin of ∞ and T_λ is the trap door for G_w. Moreover, the critical points for F_α are given by $h_w(c_\lambda)$ while the critical values of F_α are given by $\pm h_w(w)$. Finally, since the construction of μ_w and h_w depends continuously on w, it follows that $\alpha(w)$ is a continuous function of w and that $\alpha(w)$ lies in S for each w.

Let H_α be the involution that fixes the pair of critical points for F_α that correspond under h_w to those that H_λ fixes for F_λ. Then, on T_λ, we have $H_\alpha \circ h_w = h_w \circ H_\lambda$.

Now we compute $\Phi(\alpha)$:

$$\begin{aligned}\Phi(\alpha) &= \phi_\alpha(H_\alpha(F_\alpha^d(h_w(w))))\\ &= \phi_\alpha(H_\alpha(h_w(G_w^d(w))))\\ &= \phi_\alpha(H_\alpha(h_w(F_\lambda^d(w))))\\ &= \phi_\alpha(h_w(H_\lambda(F_\lambda^d(w)))).\end{aligned}$$

In B_α, we have

$$F_\alpha = h_w \circ F_\lambda \circ h_w^{-1}$$

so

$$\phi_\lambda(h_w^{-1}(F_\alpha(z))) = \phi_\lambda(F_\lambda(h_w^{-1}(z))) = \left(\phi_\lambda(h_w^{-1}(z))\right)^n.$$

Therefore

$$\phi_\alpha = \phi_\lambda \circ h_w^{-1}$$

and so

$$\Phi(\alpha) = \phi_\lambda(H_\lambda(F_\lambda^d(w))).$$

Since $\phi_\lambda \circ H_\lambda \circ F_\lambda^n$ is an analytic homeomorphism in a neighborhood of v_λ, it therefore follows that Φ is one-to-one in a neighborhood of λ.

Now we show that Φ is onto. Suppose that λ_0 belongs to ∂S. Then we claim that $v_{\lambda_0} \notin B_{\lambda_0}$. If $v_{\lambda_0} \in B_{\lambda_0}$, this would imply that λ_0 lies in the Cantor set locus in parameter space, which is an open set. Hence all nearby parameter values would lie in this locus, which cannot happen since $\lambda_0 \in \partial S$. Consequently, we have a Böttcher coordinate ϕ_{λ_0} defined for F_{λ_0} on B_{λ_0} and the functions ϕ_λ vary continuously with λ at each $\lambda_0 \in S \cup \partial S$.

If the map Φ does not take S onto the set $\overline{\mathbb{C}} - \overline{\mathbb{D}}$, then there is a sequence $\lambda_j \in S$ that accumulates on some point in ∂S, say λ_0, but $\Phi(\lambda_j)$ accumulates on some point, say z_0, in the open set $\overline{\mathbb{C}} - \overline{\mathbb{D}}$. By definition of Φ, we have

$$H_{\lambda_0}(F_{\lambda_0}^d(v_{\lambda_0})) = \phi_{\lambda_0}^{-1}(z_0).$$

Hence $H_{\lambda_0}(F_{\lambda_0}^d(v_{\lambda_0}))$ lies in the basin of ∞, so $\lambda_0 \in S$. This contradiction shows that the sequence $\Phi(\lambda_j)$ cannot accumulate on a point in the interior of $\overline{\mathbb{C}} - \overline{\mathbb{D}}$. This shows that Φ is onto.

□

Proposition. *Suppose R is a rational map of degree $2n$ with a single pole of order n at the origin. Suppose also that $R(\beta^j z) = (-1)^j R(z)$ where β is a primitive $2n^{th}$ root of unity. Then R assumes the form*

$$R(z) = z^n + \frac{\lambda}{z^n}$$

for some $\lambda \neq 0$.

Proof: We may write $R(z) = P(z)/z^n$ where
$$P(z) = a_{2n}z^{2n} + \ldots a_1 z + a_0.$$
We may first conjugate R by the linear map $z \mapsto a_{2n}^{1/(n-1)}z$ so that we may assume that the coefficient $a_{2n} = 1$. Then the equation $R(\beta z) = -R(z)$ implies that
$$a_{2n-1}\beta^{2n-1}z^{2n-1} + \ldots a_2\beta^2 z^2 + a_1\beta z = a_{2n-1}z^{2n-1} + \ldots a_2 z^2 + a_1 z$$
for all $z \in \mathbb{C}$. Therefore $a_j = 0$ for $j = 1, \ldots, 2n-1$.

□

Recall that a Sierpinski hole S has escape time k if the orbit of each critical point of F_λ lands in B_λ at the k^{th} iteration for some and hence all $\lambda \in S$.

Proposition. *There are exactly $2^{k-3}n^{k-3}(n-1)$ Sierpinski holes with escape time k in the parameter plane for F_λ for each $k \geq 3$.*

Proof: Define the functions $P_2(\lambda)$ and $Q_2(\lambda)$ by
$$\frac{P_2(\lambda)}{Q_2(\lambda)} = F_\lambda^2(c_\lambda) = 2^n\lambda^{\frac{n}{2}} + \frac{1}{2^n\lambda^{\frac{n}{2}-1}} = \frac{2^{2n}\lambda^{n-1}+1}{2^n\lambda^{\frac{n}{2}-1}}.$$
So P_2 is a polynomial in λ of degree $n-1$ with $n-1$ distinct nonzero roots. Q_2 is not a polynomial, but the only root of Q_2 is 0. Note that P_2 and Q_2 have no common factors. Consequently, there are $n-1$ distinct λ-values for which $F_\lambda^2(c_\lambda) = 0$. Since there is a unique λ-value that lands on 0 in each Sierpinski hole, it then follows that there are exactly $n-1$ Sierpinski holes with escape time 3.

Now define
$$\frac{P_3(\lambda)}{Q_3(\lambda)} = F_\lambda^3(c_\lambda) = \frac{P_2^{2n} + \lambda Q_2^{2n}}{(P_2Q_2)^n}.$$
We have that P_2^{2n} is a polynomial in λ of degree $2n(n-1)$. Also λQ_2^{2n} is a polynomial of degree n^2-2n+1, so the degree of λQ_2^{2n} is less than the degree of P_2^{2n}. Hence the numerator P_3 is a polynomial of degree $2n(n-1)$. In the denominator, Q_3 is the product of the polynomial P_2^n of degree $n(n-1)$ and the term
$$\lambda^{((n/2)-1)n},$$
so the degree of Q_3 is
$$n(n-1) + n\left(\frac{n}{2}\right).$$
Therefore Q_3 has smaller degree than P_3.

Note that P_3 and Q_3 have no common roots. This follows since the roots of P_2 and Q_2 are distinct, as we saw earlier. Hence any root of the denominator $P_2^nQ_2^n$ is a root of only one of P_2 or Q_2. Therefore only one of the two terms in the numerator vanishes at this root, so P_3 is nonzero at this point. Therefore we have that there are $2n(n-1)$ λ-values counted with multiplicity for which $F_\lambda^3(c_\lambda) = 0$. However, since there is a unique λ-value that lands on 0 in each Sierpinski hole, it follows that none of these roots are multiple roots. Therefore there are exactly $2n(n-1)$ Sierpinski holes with escape time 4.

For the general case, we proceed by induction. For $k \geq 3$ let
$$\frac{P_k(\lambda)}{Q_k(\lambda)} = F_\lambda^k(c_\lambda).$$
Assume that

(1) P_k is a polynomial in λ of degree $(2n)^{k-2}(n-1)$;

(2) Q_k is the product of a polynomial in λ of degree $(2^{k-2}-1)n^{k-2}(n-1)$ and the expression

$$\lambda^{((n/2)-1)n^{k-2}};$$

(3) P_k and Q_k have no common roots.

Then $P_{k+1} = P_k^{2n} + \lambda Q_k^{2n}$. Now P_k^{2n} is a polynomial in λ of degree

$$(2n)^{k-1}(n-1) = 2^{k-1}n^k - 2^{k-1}n^{k-1}.$$

Also, Q_k^{2n} is the product of a polynomial in λ of degree

$$2(2^{k-2}-1)n^{k-1}(n-1)$$

and

$$\lambda^{(n-2)n^{k-1}}$$

which is also a polynomial. So Q_k^{2n} is a polynomial of degree

$$2(2^{k-2}-1)n^{k-1}(n-1) + (n-2)n^{k-1} = (2^{k-1}-1)n^k - 2^{k-1}n^{k-1}.$$

Hence λQ_k^{2n} has degree

$$(2^{k-1}-1)n^k - 2^{k-1}n^{k-1} + 1$$

which is smaller than the degree of P_k^{2n}. Hence the numerator P_{k+1} is a polynomial of degree $2^{k-1}n^{k-1}(n-1)$.

In the denominator we have $Q_{k+1} = P_k^n Q_k^n$ so Q_{k+1} is the product of polynomials of degree $(2n)^{k-2}(n)(n-1)$ and degree $(2^{k-2}-1)n^{k-1}(n-1)$ as well as the expression

$$\lambda^{((n/2)-1)n^{k-1}}.$$

The product of the two polynomials has degree

$$(2n)^{k-2}(n)(n-1) + (2^{k-2}-1)n^{k-1}(n-1) = (2^{k-1}-1)n^{k-1}(n-1)$$

so Q_{k+1} is the product of a polynomial of degree $(2^{k-1}-1)n^{k-1}(n-1)$ and the term

$$\lambda^{((n/2)-1)n^{k-1}}.$$

So the degree of Q_{k+1} is

$$(2^{k-1}-1)n^{k-1}(n-1) + n^{k-1}\left(\frac{n}{2}-1\right) = 2^{k-1}n^{k-1}(n-1) - \frac{n^k}{2}.$$

Therefore the degree of P_{k+1} is larger than Q_{k+1}. By induction, P_k and Q_k have no common roots. Hence any root of the denominator is a root of only one of the two terms in the numerator, and so the numerator cannot vanish at such a point. Hence there are $2^{k-1}n^{k-1}(n-1)$ distinct roots of the equation $F_\lambda^{k+1}(c_\lambda) = 0$. As above, each of these roots is a center of a Sierpinski hole and we have shown that there is a unique such center in each hole. This proves that there are exactly $2^{k-1}n^{k-1}(n-1)$ Sierpinski holes with escape time $k+2$, or $2^{k-3}n^{k-3}(n-1)$ Sierpinski holes with escape time k

$\square$

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