Mandelpinski Necklaces in the Parameter Planes of Rational Maps*

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Abstract

In this paper we give a survey of some recent results involving "Mandelpinski necklaces" that occur in the family of complex rational maps of the form $z^n + \lambda/z^d$ where $\lambda \in \mathbb{C}$ and $n, d \geq 2$. A Mandelpinski necklace is a simple closed curve in the parameter plane for these maps that passes alternately through a certain number of baby Mandelbrot sets and Sierpinski holes. At the end of the paper we describe the very special case that occurs when $n = d = 2$.

For the family of maps

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

a "Mandelpinski necklace" is a simple closed curve the the parameter plane that passes alternately through a certain number of centers of baby Mandelbrot sets and Sierpinski holes. The center of a baby Mandelbrot set is the parameter that lies at the "center" of the main cardioid of this set, and hence is a parameter for which one of the critical orbits is periodic. A Sierpinski hole is a disk in the parameter plane containing parameters for which the corresponding Julia sets are Sierpinski curves, i.e., sets homeomorphic to the well known Sierpinski carpet fractal. The center of such a hole is a parameter for which the critical orbits all eventually map to ∞. The main result that we shall focus on in this paper is the following: In the parameter plane for the maps $z^n + \lambda/z^d$, there are infinitely many disjoint simple closed curves $\mathcal{S}^k$ for $k = 1, 2, 3, \ldots$ surrounding the McMullen domain, with the $\mathcal{S}^k$ converging down to the boundary of the McMullen domain (when n and d are not both equal to 2). The curve $\mathcal{S}^1$ passes through exactly $n - 1$ centers of baby Mandelbrot sets and Sierpinski holes. The curve $\mathcal{S}^k$ for $k > 1$ passes through exactly $dn^{k-2}(n-1) - n^{k-1} + 1$ centers of baby Mandelbrot sets and Sierpinski holes.

1 Introduction

For simplicity, we shall concentrate for most of this paper on the family of complex rational maps given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

where $\lambda \neq 0$ is a complex parameter and $n \geq 3$. The reason for this simplification is that this family has $2n$ "free" critical points. However, like the well-studied quadratic family $z^2 + c$, because of certain symmetries, there is really only one free critical orbit since all of the critical orbits behave symmetrically. Moreover, there are certain symmetries in the dynamical plane that are present when $n = d$ but not so when $n \neq d$. For complete results in the case where $n \neq d$, see [12], [13].

As another similarity with the quadratic family, the point at ∞ is a superattracting fixed point for each λ. Hence we have an immediate basin of attraction at ∞ which we denote by B_λ. Also, 0 is a pole of order n, and so there is an open set containing 0 that is mapped onto B_λ. If this open set is disjoint from B_λ, we call this set the "trap door" and denote it by T_λ. Note that F_λ maps both T_λ and B_λ n-to-1 over B_λ.

As usual in complex dynamics, we are interested in the Julia set for F_λ, which we denote by $J(F_\lambda)$. As in the quadratic case, the Julia set has several equivalent definitions. First, $J(F_\lambda)$ is the boundary of the set of points whose orbits tend to ∞. Second, $J(F_\lambda)$ is the closure of the set of repelling periodic points. And third, $J(F_\lambda)$ is the set on which the map F_λ is chaotic.

The following result was proved in [8].

Theorem (The Escape Trichotomy). *For the family of functions*

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

with $n \geq 2$

1. *If the critical values lie in B_λ, then the Julia set is a Cantor set.*
2. *If the critical values lie in T_λ, then the Julia set is a Cantor set of simple closed curves.*
3. *If the critical values lie in any other preimage of T_λ, then the Julia set is a Sierpinski curve.*

A *Sierpinski curve* is a planar set that is characterized by the following five properties: it is a compact, connected, locally connected and nowhere dense set whose complementary domains (of which there must be at least two) are bounded by simple closed curves that are pairwise disjoint. It is known from work of Whyburn [18] that any two Sierpinski curves are homeomorphic. In fact, they are homeomorphic to the well-known Sierpinski carpet fractal. From the point of view of topology, a Sierpinski curve is a universal set in the sense that it contains a homeomorphic copy of any planar, compact, connected, one-dimensional set. The first example of a Sierpinski curve Julia set was given by Milnor and Tan Lei [17].

Case 2 of the Escape Trichotomy was first observed by McMullen [14], who showed that this phenomenon occurs in each family provided that $n \neq 2$ and λ is sufficiently small. As we describe later, when $n = 2$, the critical values of F_λ cannot lie in T_λ.

In Figure 1 we display the parameter plane for the family $F_\lambda(z) = z^3 + \lambda/z^3$. The external red region in this set corresponds to parameter values for which the Julia set is a Cantor set; we call this set the *Cantor set locus.* The small red region in the center is a disk surrounding the origin that contains parameter values for which the Julia set is a Cantor set of simple closed curves. We call this region the *McMullen domain.* All of the other red disks contain parameters for which the Julia set is a Sierpinski curve. These disks are called *Sierpinski holes.* In each such hole, there is a unique parameter

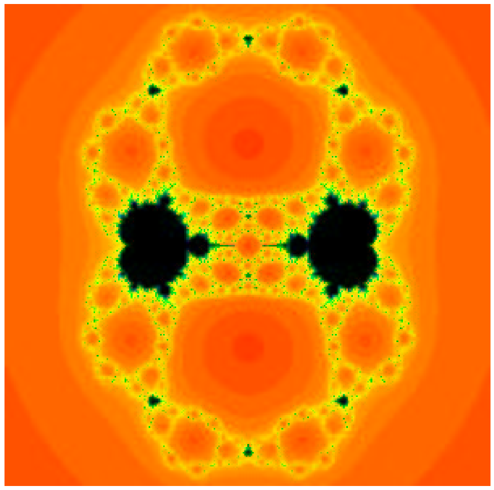


Figure 1: The parameter plane for the family $z^3 + \lambda/z^3$.

for which the orbit of some critical point lands on 0 at some iteration and therefore on ∞ at the next iteration, say at iteration $k > 2$. We then call this parameter the center of the Sierpinski hole and k the *escape time* of the hole.

Our goal in this paper is to investigate further properties of the parameter plane for these maps and, in particular, the structure of the parameter plane in a neighborhood of the McMullen domain. It is known [3] that there is a unique McMullen domain in the parameter plane for each $n \geq 3$, and this region is an open disk surrounding the origin that is bounded by a simple closed curve.

In Figure 2, we have displayed several magnifications of the region around the McMullen domain in the case $n = 3$. In the first image, note that there are four large Sierpinski holes symmetrically placed around the McMullen domain. These Sierpinski holes all have escape time 4. Between the two upper and the two lower Sierpinski holes there appear to be small copies of a Mandelbrot set, while between the two left and two right holes we see the

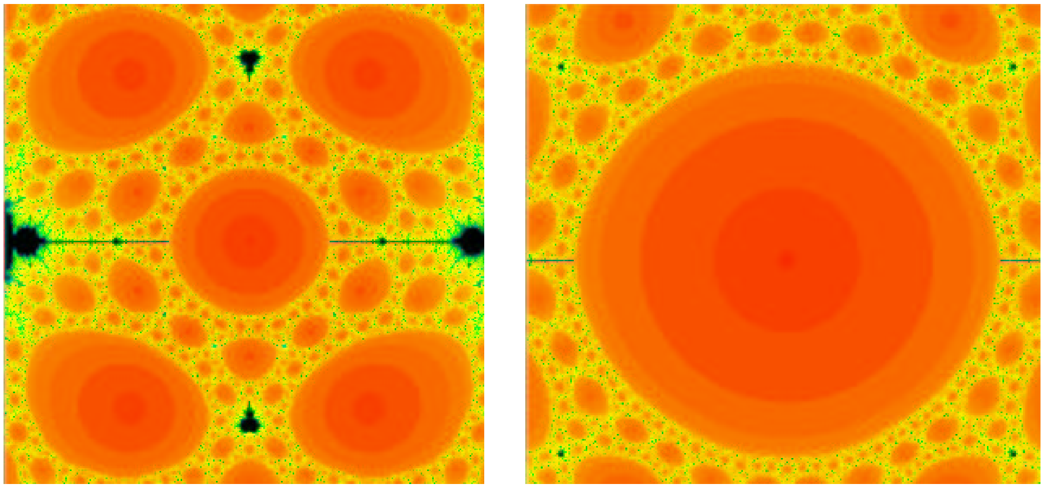


Figure 2: Magnifications of the parameter plane for the family $z^3 + \lambda/z^3$ around the McMullen domain.

period two bulb of a principal Mandelbrot set and the remainder of the "tail" of this set. Indeed, one may draw a simple closed curve that encircles the McMullen domain and passes through the centers of each of these Sierpinski holes, the centers of the main cardioids of the two smaller Mandelbrot sets, and the centers of the two period two bulbs of the principal Mandelbrot sets. That is, on this simple closed curve, we find four parameter values for which the map has a a superstable periodic point and four other values for which F_λ^4 maps the critical points to ∞, and these parameter values alternate between the superstable and the centers of Sierpinski holes as the parameter winds around the closed curve.

Inside these four Sierpinski holes appear to be another simple closed curve containing ten Sierpinski holes. Each of these holes has escape time 5. Also, each pair of these holes apparently has either a small copy of a Mandelbrot set or a portion of a principal Mandelbrot set (the two largest Mandelbrot sets displayed in Figure 1) between them. Examining the further magnification in Figure 2, we see a smaller closed curve containing 28 Sierpinski holes with escape time 6 and, inside that curve, an even smaller curve containing 82 Sierpinski holes with escape time 7. It appears that the k^{th} curve meets exactly $3^k + 1$ Sierpinski holes with escape time $k + 3$ as well as the same number of (portions of) Mandelbrot sets (though these are so small that they are not quite visible). These are the curves that we call Mandelpinski necklaces.

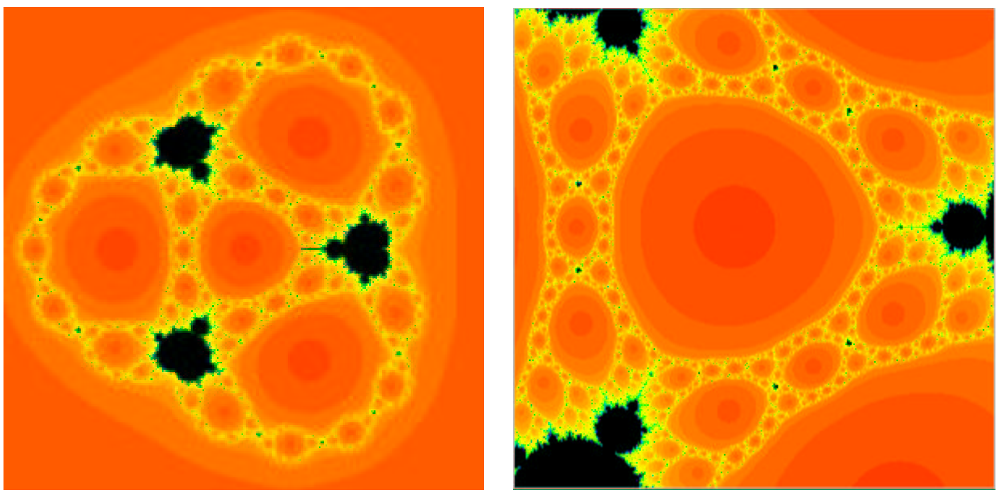


Figure 3: The parameter plane for the family $z^4 + \lambda/z^4$ and a magnification around the McMullen domain.

Actually, the formula in the general case is a little more complicated than that. In Figure 3 we display the parameter plane for the case $n = 4$ as well as a magnification of the McMullen domain. Here we see three principal Mandelbrot sets arranged between three large Sierpinski holes, each of which has escape time 3. Inside these sets is a curve containing 9 Sierpinski holes, each with escape time 4, and inside another curve containing 33 holes of escape time 5. Further magnification shows that there are $2 \cdot 4^{k-1} + 1$ holes with escape time $k + 2$ in case $n = 4$.

Our main goal in this paper is to make these observations rigorous. We shall prove:

Theorem. (Mandelpinski Necklace Theorem). *For each $n \geq 3$, the McMullen domain for the family $z^n + \lambda/z^n$ is surrounded by infinitely many simple closed curves (or rings) $\mathcal{S}^k$ for $k = 1, 2, \ldots$ having the property that:*

1. *Each ring $\mathcal{S}^k$ surrounds the McMullen domain as well as $\mathcal{S}^{k+1}$, and the $\mathcal{S}^k$ accumulate on the boundary of the McMullen domain as $k \to \infty$;*

2. *The ring $\mathcal{S}^k$ meets the centers of τ_k^n Sierpinski holes, each with escape time $k + 2$ where*
$$\tau_k^n = (n-2)n^{k-1} + 1.$$

3. *The ring $\mathcal{S}^k$ also passes through τ_k^n superstable parameter values where a critical point is periodic of period k or $2k$.*

Using techniques from complex dynamics, it has been shown [4] that these superstable parameter values each lie at the center of the main cardioid of a Mandelbrot set when $k \neq 2$, while the Sierpinski holes surrounding the centers are all simply connected sets. When $k = 2$, $\mathcal{S}^2$ passes through exactly $n - 1$ centers of period 2 bulbs of the largest Mandelbrot sets and also the centers of $\tau_2^n - (n-1)$ centers of smaller baby Mandelbrot sets. As a remark, the case where $n = 2$ is very different and quite special. We shall describe the result in this case at the end of this paper.

2 Elementary Mapping Properties

Besides 0 and ∞, F_λ has $2n$ other critical points given by $\lambda^{1/2n}$. We call these points the *free critical points* for F_λ. There are, however, only two critical values, and these are given by $\pm 2\sqrt{\lambda}$. We denote a free critical point by c_λ

and a critical value by v_λ. The map also has $2n$ prepoles given by $(-\lambda)^{1/2n}$. Note that all of the critical points and prepoles lie on the circle of radius $|\lambda|^{1/2n}$ centered at the origin. We call this circle the *critical circle* and denote it by C_λ.

The map F_λ has some very special properties when restricted to circles centered at the origin. The following is a straightforward computation (see [3]):

Proposition.

1. *F_λ takes the critical circle $2n$-to-one onto the line interval connecting the two critical values $\pm 2\sqrt{\lambda}$;*

2. *F_λ takes any other circle centered at the origin to an ellipse whose foci are the critical values.*

We call the image of the critical circle the *critical segment.* We call the straight line connecting the origin to ∞ and passing through one of the critical points (resp., prepoles) a *critical point ray* (resp., *prepole ray*). Similar straightforward computations show that each of the critical point rays is mapped in two-to-one fashion onto one of the two straight line segments of the form tv_λ, where $t \geq 1$ and v_λ is the image of the critical point on this ray. So the image of a critical point ray is a straight ray connecting either v_λ or $-v_\lambda$ to ∞. Thus the critical segment together with these two rays forms a straight line through the origin.

Similarly, each of the $2n$ prepole rays is mapped in one-to-one fashion onto the straight line given by $it\sqrt{\lambda}$, where t is now any real number. Note that the image of the prepole rays is the line that is perpendicular to the line tv_λ for $t \in \mathbb{R}$, i.e., the line that contains the critical segment and the images of all of the critical point rays.

Let U_λ be a sector bounded by two prepole rays corresponding to adjacent prepoles on C_λ, i.e., U_λ is a sector in the plane with angle $2\pi/2n$. We call U_λ a *critical point sector* since it contains at its "center" a unique critical point of F_λ. Similarly, let V_λ be the sector bounded by two critical point rays corresponding to adjacent critical points on C_λ. We call V_λ a *prepole sector.* The next result follows immediately from the above:

Proposition (Mapping Properties of F_λ).

1. *F_λ maps the interior of each critical point sector in two-to-one fashion onto the open half plane that is bounded by the image of the prepole rays and contains the critical value that is the image of the unique critical point in the sector;*

2. *F_λ maps the interior of each prepole sector in one-to-one fashion onto the entire plane minus the two half lines $\pm t v_\lambda$ where $t \geq 1$;*

3. *F_λ maps the region in either the interior or the exterior of the critical circle onto the complement of the critical segment as an n-to-one covering map of $\mathbb{C}$.*

We now turn to the symmetry properties of F_λ in both the dynamical and parameter planes. Let ν be the primitive $2n^{\text{th}}$ root of unity given by $\exp(\pi i/n)$. Then, for each j, we have $F_\lambda(\nu^j z) = (-1)^j F_\lambda(z)$. Hence, if n is even, we have $F_\lambda^2(\nu^j z) = F_\lambda^2(z)$ for each j. Therefore the points z and $\nu^j z$ land on the same orbit after two iterations and so their orbits have the same eventual behavior for each j. If n is odd, the orbits of $F_\lambda(z)$ and $F_\lambda(\nu^j z)$ are either the same or else they are the negatives of each other after the first iteration. In either case it follows that the orbits of $\nu^j z$ behave symmetrically under $z \mapsto -z$ for each j. Hence the Julia set of F_λ is always symmetric under $z \mapsto \nu z$. In particular, each of the free critical points eventually maps onto the same orbit (in case n is even) or onto one of two symmetric orbits (in case n is odd). Thus these orbits all have the same behavior and so the λ-plane is a natural parameter plane for each of these families. Note also that, if n is even and the orbit of some critical point eventually lands on some other critical point at iteration $j \geq 1$, then in fact one of the critical points of F_λ must be periodic of period j. If n is odd, then there are two possibilities: either one of the critical points has period j or else it has period $2j$.

Let $H_\lambda(z)$ be one of the n involutions given by $H_\lambda(z) = \lambda^{1/n}/z$. Then we have $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, so that the Julia set is also preserved by each of these involutions. Note that each H_λ maps the critical circle to itself and also fixes a pair of critical points of the form $\pm\sqrt{\lambda^{1/n}}$. H_λ also maps circles centered at the origin outside the critical circle to similar circles inside the critical circle and vice versa. It follows that two such circles, one inside and one outside the critical circle, are mapped onto the same ellipse by F_λ.

The parameter plane (see Figures 1 and 3) for F_λ also possesses several symmetries. First of all, we have

$$\overline{F_\lambda(z)} = F_{\overline{\lambda}}(\overline{z})$$

so that F_λ and $F_{\overline{\lambda}}$ are conjugate via the map $z \mapsto \overline{z}$. Therefore the parameter plane is symmetric under the map $\lambda \mapsto \overline{\lambda}$.

We also have $(n-1)$-fold symmetry in the parameter plane for F_λ. To see this, let ω be the primitive $(n-1)^{\text{st}}$ root of unity given by $\exp(2\pi i/(n-1))$. Then, if n is even, we compute that

$$F_{\lambda\omega}(\omega^{n/2}z) = \omega^{n/2}(F_\lambda(z)).$$

As a consequence, for each $\lambda \in \mathbb{C}$, the maps F_λ and $F_{\lambda\omega}$ are conjugate under the linear map $z \mapsto \omega^{n/2}z$. In particular, since, when λ is real, the real line is preserved by F_λ, it follows that the straight line passing through 0 and $\omega^{n/2}$ is preserved by $F_{\lambda\omega}$.

When n is odd, the situation is a little different. We now have

$$F_{\lambda\omega}(\omega^{n/2}z) = -\omega^{n/2}(F_\lambda(z)).$$

Since $F_\lambda(-z) = -F_\lambda(z)$ when n is odd, we therefore have that $F_{\lambda\omega}^2$ is conjugate to F_λ^2 via the map $z \mapsto \omega^{n/2}z$. This means that the dynamics of F_λ and $F_{\lambda\omega}$ are "essentially" the same, though subtly different. For example, if F_λ has a fixed point, then under the conjugacy, this fixed point and its negative are mapped to a 2-cycle for $F_{\lambda\omega}$. Since the real line is invariant when λ is real, it follows that the straight lines passing through the origin and $\pm\omega^{n/2}$ are interchanged by $F_{\lambda\omega}$ and hence invariant under $F_{\lambda\omega}^2$.

To summarize the symmetry properties of F_λ, we have:

Proposition (Symmetries in the dynamical and parameter plane). *The dynamical plane for F_λ is symmetric under the map $z \mapsto \nu z$ where $\nu = \exp(\pi i/n)$. The parameter plane is symmetric under both $z \mapsto \overline{z}$ and $z \mapsto \omega z$ where $\omega = \exp(2\pi i/(n-1))$.*

The following result shows that the McMullen domain and all of the Sierpinski holes are located inside the unit circle in parameter space.

Proposition (Location of the Cantor set locus.) *Suppose $|\lambda| \geq 1$. Then v_λ lies in B_λ so that λ lies in the Cantor set locus.*

Proof: Suppose $|z| \geq 2|\lambda|^{1/2} \geq 2$. Then, since $|z| \geq |\lambda|^{1/2}$, we have

$$|F_\lambda(z)| \geq |z|^n - \frac{|\lambda|}{|z|^n} \geq |z|^n - |\lambda|^{1-\frac{n}{2}} \geq |z|^n - 1 \geq |z|^{n-1} > |z|$$

since $n > 2$. Hence $|F_\lambda^j(z)| \to \infty$ so the region $|z| \geq 2|\lambda|^{1/2}$ lies in B_λ. In particular, $v_\lambda \in B_\lambda$.

□

For each n, let $\lambda^* = \lambda_n^*$ be the unique real solution to the equation

$$|v_\lambda| = 2|\sqrt{\lambda}| = |\lambda|^{1/2n} = |c_\lambda|.$$

Using this equation, we compute easily that

$$\lambda^* = \left(\frac{1}{4}\right)^{\frac{n}{n-1}}.$$

The circle of radius λ^* plays an important role in the parameter plane, for if λ lies on this circle, it follows that both of the critical values lie on the critical circle for F_λ. If λ lies inside this circle, then F_λ maps the critical circle strictly inside itself. We call the circle of radius λ^* in parameter plane the *dividing circle.* We denote by $\mathcal{O} = \mathcal{O}_n$ the open set of parameters inside the dividing circle. We will be primarily concerned in later sections with values of the parameter that lie in $\mathcal{O}$. In particular, we shall show that all of the rings around the McMullen domain $\mathcal{S}^k$ with $k > 1$ lie in this region while the ring $\mathcal{S}^1$ is the dividing circle itself.

3 Some Special Cases

In this section we discuss the dynamics of several special cases of F_λ that will help define the rings around the McMullen domain later.

First suppose that λ lies on the dividing circle, i.e., $|\lambda| = \lambda^*$. In this case, all of the critical points, critical values, and prepoles of F_λ lie on the same circle (the critical circle) in dynamical plane, namely the circle

$$|z| = \left(\frac{1}{2}\right)^{\frac{1}{n-1}}.$$

As λ winds once around the dividing circle in the counterclockwise direction beginning on the real axis, the critical points and prepoles of F_λ wind $1/2n$ of a turn around the critical circle, while the critical values wind one-half of a turn around the critical circle, all in the counterclockwise direction. Hence there are exactly $n-1$ special parameter values on the dividing circle for which a critical point of the corresponding map equals a critical value, so for these special λ-values we have a superattracting fixed or period two point for F_λ. Equivalently, one computes that these $n-1$ parameters are given by

$$\lambda = \left(\frac{1}{4}\right)^{\frac{n}{n-1}}.$$

There are $n-1$ other parameters on this circle for which the critical value is a prepole, and these are given by

$$\lambda = \left(\frac{-1}{4}\right)^{\frac{n}{n-1}}.$$

This proves the case $k = 1$ of the Mandelpinski Necklace Theorem.

Theorem. *The ring $\mathcal{S}^1$ is the dividing circle in parameter plane. It contains $n-1$ superstable parameters and the same number of centers of Sierpinski holes.*

See Figure 4.

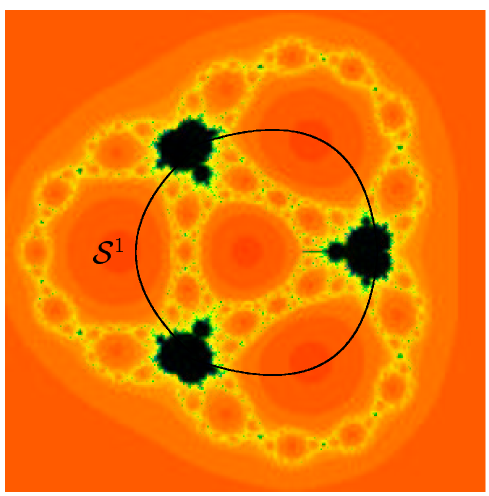


Figure 4: The ring $\mathcal{S}^1$ in the parameter plane for $n = 4$.

We next restrict attention to values of λ lying in $\mathbb{R}^+$. The graph of F_λ shows that, in this case, F_λ maps $\mathbb{R}^+$ to itself and that there is a unique critical point lying in $\mathbb{R}^+$. We denote this critical point by $c_0 = c_0(\lambda)$. See Figure 5.

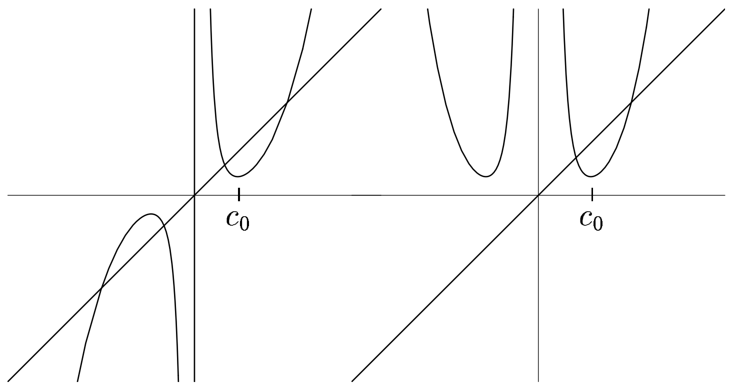


Figure 5: The graphs of $x^3 + 0.01/x^3$ and $x^4 + 0.01/x^4$.

It is known [2] that there is a Mandelbrot set (a principal Mandelbrot set) whose central spine lies along an interval $[\lambda_-, \lambda_+]$ contained in $\mathbb{R}^+$. Moreover, if $\lambda > \lambda_+$, then λ lies in the Cantor set locus, whereas if $0 < \lambda < \lambda_-$, then λ lies in the McMullen domain. The graph of $F_\lambda \,|\, \mathbb{R}^+$ shows that F_λ undergoes a saddle-node bifurcation at λ_+ and that the critical point c_λ maps onto the repelling fixed point in $\partial B_\lambda \cap \mathbb{R}^+$ after two iterations when $\lambda = \lambda_-$. Since each F_λ is conjugate on the real line to a real quadratic polynomial of the form $Q_c(x) = x^2 + c$, standard facts from quadratic dynamics yield the following:

Proposition (Superstable parameters for $\lambda \in \mathbb{R}^+$.) *There is a decreasing sequence of parameters in $\mathbb{R}^+$ $\lambda_1 > \lambda_2 \ldots$ converging to λ_- such that, for $\lambda = \lambda_k$, the critical point c_0 is periodic with period k and the critical orbit in $\mathbb{R}^+$ has the special form when $k \geq 2$:*

$$0 < v_\lambda = F_\lambda(c_0) < c_0 = F_\lambda^k(c_0) < F_\lambda^{k-1}(c_0) < \ldots < F_\lambda^3(c_0) < F_\lambda^2(c_0).$$

In particular, λ_k is a superstable parameter value of period k and the orbit *of* $F^2_{\lambda_k}(c_0)$ *is monotonically decreasing for* $k-1$ *iterations along* $\mathbb{R}^+$*.*

Portions of the graphs of F_{λ_k} for $k = 4$ and $k = 5$ when $n = 4$ are displayed in Figure 6. Note that the parameter λ_1 necessarily lies on the dividing circle $\mathcal{S}^1$. We shall show below that each λ_k lies on $\mathcal{S}^k$.

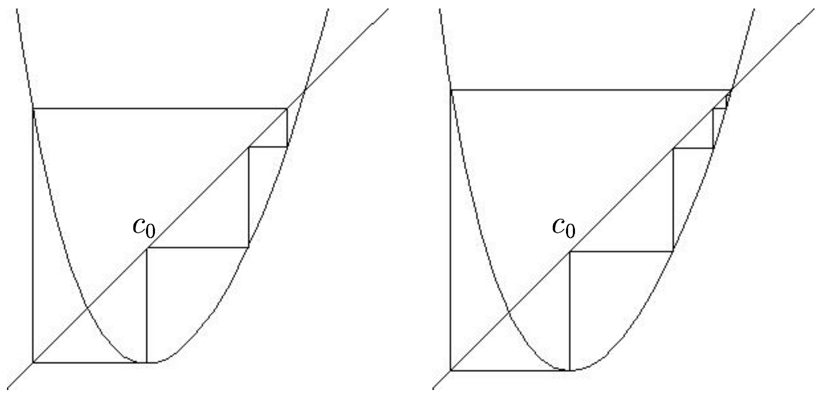


Figure 6: The graphs of F_λ for $\lambda = \lambda_4$ and $\lambda = \lambda_5$ when $n = 4$.

Because of the $(n-1)$-fold symmetry in the parameter plane, we have a similar sequence of superstable parameter values along the ray $\lambda = \omega \cdot \mathbb{R}^+$ in parameter plane. To be more precise, first suppose that n is even. Suppose that $\lambda = a\omega$ with $a > 0$ and, as before, $\omega = \exp(2\pi i/(n-1))$. Then, using the results in Section 2, we have that, if $t > 0$,

$$F_\lambda(\omega^{\frac{n}{2}} t) = \omega^{\frac{n}{2}} F_a(t)$$

so that F_λ on the line $\omega^{n/2} \cdot \mathbb{R}^+$ is conjugate to F_a on $\mathbb{R}^+$.

Now F_λ has critical points at

$$\begin{aligned} c_0 &= (a\omega)^{\frac{1}{2n}} \\ c_1 &= \nu(a\omega)^{\frac{1}{2n}} \\ c_{n+1} &= \nu^{n+1}(a\omega)^{\frac{1}{2n}} = -\nu(a\omega)^{\frac{1}{2n}} = -c_1. \end{aligned}$$

Note that the critical point c_{n+1} lies on the line $\omega^{n/2} \cdot \mathbb{R}^+$. This follows since

$$\begin{aligned} -\nu(a\omega)^{\frac{1}{2n}} &= -(a)^{\frac{1}{2n}} \left(\exp\left(\frac{\pi i}{n}\right) \exp\left(\frac{\pi i}{n(n-1)}\right)\right) \\ &= -(a)^{\frac{1}{2n}} \exp\left(\frac{\pi i}{n-1}\right) \\ &= -a^{\frac{1}{2n}}\omega^{\frac{1}{2}} = a^{\frac{1}{2n}}\omega^{\frac{n}{2}}. \end{aligned}$$

Therefore the above Proposition goes over to the case where $\lambda = a\omega$ with $a = \lambda_k \in \mathbb{R}^+$ provided we now use the critical point c_{n+1} lying on the line $\omega^{n/2} \cdot \mathbb{R}^+$. We note that the symmetric critical point c_1 lies on the line $\omega^{1/2} \cdot \mathbb{R}^+$ and maps onto the critical value on the line $\omega^{n/2} \cdot \mathbb{R}^+$ after one iteration.

The case where n is odd is similar modulo the $z \mapsto -z$ symmetry described earlier. The difference is that the superattracting cycles now have period $2k$ and alternate back and forth between $\omega \cdot \mathbb{R}^+$ and $-\omega \cdot \mathbb{R}^+$. We have:

Proposition (Superstable parameters for $\lambda \in \omega \cdot \mathbb{R}^+$). *Let $\lambda_1 > \lambda_2 \dots$ be the decreasing sequence in $\mathbb{R}^+$ in the previous Proposition. Suppose n is even. For $\lambda = \lambda_k\omega$, the critical point c_{n+1} is periodic with period k and the critical orbit along the line $\omega^{n/2} \cdot \mathbb{R}^+$ has the special form when $k \geq 2$*

$$F_\lambda(c_{n+1}) < c_{n+1} = F_\lambda^k(c_{n+1}) < F_\lambda^{k-1}(c_{n+1}) < \dots < F_\lambda^3(c_{n+1}) < F_\lambda^2(c_{n+1}).$$

In particular, $\lambda = \lambda_k\omega$ is a superstable parameter value of period k and the orbit of $F_\lambda^2(c_{n+1})$ is monotonically decreasing for $k-1$ iterations along $\omega^{n/2} \cdot \mathbb{R}^+$. When n is odd, replace F_λ with F_λ^2. The cycle corresponding to $\lambda = \lambda_k\omega$ now has period $2k$.

4 Rings in Dynamical Plane

In this section we prove the existence of infinitely many rings γ_λ^k for $k = 0, 1, \dots$ in the dynamical plane. Each ring γ_λ^k is a smooth, simple closed curve that is mapped n^k-to-1 onto the critical circle by F_λ^k. We shall use these rings in the next section to construct the rings $\mathcal{S}^k$ in the parameter plane.

We begin by defining γ_λ^0 to be the critical circle. Recall that, if $\lambda \in \mathcal{O}$, then F_λ maps γ_λ^0 strictly inside itself. Since all of the critical points of F_λ lie on γ_λ^0, it follows that F_λ takes the exterior of γ_λ^0 as an n-to-1 covering onto the plane minus the critical segment and hence over the entire exterior of γ_λ^0. Thus there is a preimage γ_λ^1 lying outside of γ_λ^0 and mapped n-to-1 onto γ_λ^0 by F_λ. Since F_λ is a covering map, it follows that γ_λ^1 must be a single simple closed curve. Then F_λ maps the exterior of γ_λ^1 as an n-to-1 covering onto the exterior of γ_λ^0, so there is a preimage of γ_λ^1 lying in this region and mapped n-to-1 to γ_λ^1. Call this simple closed curve γ_λ^2. Continuing inductively, we find a collection of simple closed curves γ_λ^k for $k \geq 1$ having the properties that:

1. γ_λ^{k+1} lies in the exterior of γ_λ^k;
2. F_λ takes γ_λ^{k+1} as an n-to-1 covering onto γ_λ^k;
3. so F_λ takes γ_λ^{k+1} as an n^{k+1}-to-1 covering of the critical circle;
4. the γ_λ^{k+1} converge outward to the boundary of B_λ as $k \to \infty$.

We now construct a parameterization of each of the γ_λ^k. In order for this parametrization to be well-defined, we need to restrict attention to parameters in the region $\mathcal{O}' = \mathcal{O} - (-\lambda^*, 0]$, so that $-\pi < \operatorname{Arg} \lambda < \pi$. We therefore assume that λ lies in $\mathcal{O}'$ for the remainder of this paper.

For $\lambda \in \mathcal{O}'$, there is a unique critical point of F_λ lying in the region $|\operatorname{Arg} z| < \pi/2n$. Call this critical point $c_0 = c_0(\lambda)$. Note that $c_0 \in \mathbb{R}^+$ if $\lambda \in \mathbb{R}^+$. We index the remaining critical points by c_j with the argument of c_j increasing as j increases.

To parametrize the critical circle γ_λ^0, we set $\gamma_\lambda^0(0) = c_0(\lambda)$. By the Mapping Properties Proposition, for each $\theta \in \mathbb{R}$, we then let $\gamma_\lambda^0(\theta)$ be the natural continuation of this parametrization of the circle in the counterclockwise direction. So $\gamma_\lambda^0(\theta)$ is 2π-periodic in θ and depends analytically on λ for $\lambda \in \mathcal{O}'$.

To parametrize $\gamma_\lambda^1(\theta)$, consider the portion of the critical point sector containing $c_0(\lambda)$ that lies outside the critical circle. There is a unique point in this region mapped to c_0 by F_λ; call this point $\gamma_\lambda^1(0)$. Then define $\gamma_\lambda^1(\theta)$ by requiring that

$$F_\lambda(\gamma_\lambda^1(\theta)) = \gamma_\lambda^0(\theta)$$

and that $\gamma_\lambda^1(\theta)$ varies continuously with θ. Note that $\gamma_\lambda^1(\theta)$ is $2n\pi$ periodic since F_λ is n-to-1 on γ_λ^1. We then proceed inductively to define $\gamma_\lambda^k(\theta)$ by first specifying that, in the outside portion of the critical point sector containing c_0, $\gamma_\lambda^k(0)$ is the unique point that is mapped by F_λ to $\gamma_\lambda^{k-1}(0)$ and then using F_λ to complete this parameterization. As above, for each k, $\gamma_\lambda^k(\theta)$ is $2n^k\pi$ periodic in θ and depends analytically on λ.

To prove the existence of the rings in the parameter plane, we need to be more specific about the location of the rings in the dynamical plane. Let V_+ be the portion of the prepole sector lying on and outside the critical circle and also between the two critical point rays through c_0 and c_1. That is,

$$V_+ = \left\{ z \,\middle|\, |z| \geq |\lambda|^{1/2n},\ \frac{\operatorname{Arg}\lambda}{2n} \leq \operatorname{Arg} z \leq \frac{\operatorname{Arg}\lambda}{2n} + \frac{\pi}{n} \right\}.$$

Let $V_- = \nu^{-1} \cdot V_+$. So V_- is the portion of the prepole sector bounded by the critical lines through c_0 and c_{-1} and lying on or outside the critical circle. Let $V_\lambda = V_+ \cup V_-$. See Figure 7.

Since $|\operatorname{Arg}\lambda| < \pi$ and $n \geq 3$, we have for $z \in V_\lambda$

$$|\operatorname{Arg} z| \leq \left| \frac{\operatorname{Arg}\lambda}{2n} \right| + \frac{\pi}{n} < \frac{3\pi}{2n} \leq \frac{\pi}{2}.$$

So for each $\lambda \in \mathcal{O}'$, the region V_λ is contained in the half plane $\operatorname{Re} z > 0$.

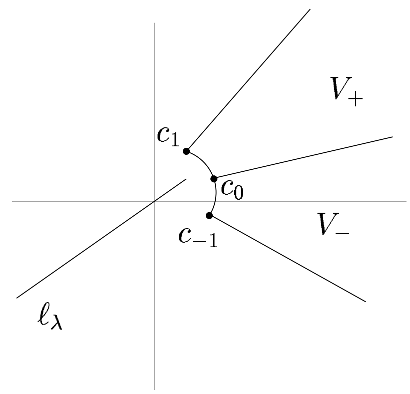


Figure 7: The region $V_\lambda = V_+ \cup V_-$.

Now F_λ maps the portion of boundary of V_+ lying along the critical circle one-to-one to the critical segment since the endpoints of this arc are adjacent critical points along C_λ that are mapped to distinct critical values. Also, F_λ maps the portion of the critical point line containing c_0 lying on the boundary of V_+ one-to-one onto the ray $tv_\lambda = 2t\sqrt{\lambda}$ with $t \geq 1$ and $\text{Arg}\sqrt{\lambda} > 0$, while F_λ maps the other boundary ray containing c_1 to the negative of this ray. Hence the boundary of V_+ is mapped onto the entire straight line passing through $\pm v_\lambda$ and the origin. Therefore F_λ maps V_+ univalently onto one of the half planes bounded by this line. Similarly, F_λ maps V_- univalently onto the opposite half plane.

Let ℓ_λ be the straight line given by $2t\sqrt{\lambda}$ where $t \in (-\infty, 1]$. So ℓ_λ is the straight line that starts at $2\sqrt{\lambda}$ at $t = 1$ and passes through the origin and $-2\sqrt{\lambda}$ enroute to ∞ as $t \to \infty$. Note that the boundary of V_λ is mapped two-to-one onto ℓ_λ by F_λ. Hence F_λ maps the interior of V_λ univalently onto $\mathbb{C} - \ell_\lambda$. Now, for each $\lambda \in \mathcal{O}'$, the critical segment lies outside V_λ since neither V_+ nor V_- meets the interior of the critical circle. Also, the portion of ℓ_λ extending from $-2\sqrt{\lambda}$ to ∞ lies in the left half plane, so the entire line ℓ_λ does not intersect V_λ. So we have:

Proposition. *For each* $\lambda \in \mathcal{O}'$, F_λ *maps the interior of* V_λ *univalently onto* $\mathbb{C} - \ell_\lambda$ *and so the image of* V_λ *contains* V_λ.

Recall that the k^{th} ring in the dynamical plane is parametrized by $\gamma_\lambda^k(\theta)$ and is periodic with period $2n^k\pi$.

Proposition. *For each* $k \geq 1$, *the portion of the ring* $\gamma_\lambda^k(\theta)$ *with* $|\theta| \leq n^{k-1}\pi$ *lies in the region*

$$-\frac{3\pi}{2n} < \operatorname{Arg} z < \frac{3\pi}{2n}.$$

Proof: We deal first with the case $0 \leq \theta \leq n^{k-1}\pi$; the other case is handled by applying the $z \mapsto \nu^{-1}z$ symmetry, as we describe below.

We claim that the portion of the ring $\gamma_\lambda^k(\theta)$ with $0 \leq \theta \leq n^{k-1}\pi$ actually lies in the smaller region

$$-\frac{\pi}{2n} < \operatorname{Arg} z < \frac{3\pi}{2n}.$$

To see this, we first consider the simplest case where $\lambda \in \mathbb{R}^+$. In this case, V_+ is bounded by $\mathbb{R}^+$ and $\nu \cdot \mathbb{R}^+$ and F_λ maps V_+ univalently onto $\operatorname{Im} z \geq 0$. Recall that $\gamma_\lambda^0(\theta)$ lies in the region $\operatorname{Im} z \geq 0$ if $\theta \in [0, \pi]$. Hence there is a continuous preimage of $\gamma_\lambda^0(\theta)$ lying in V_+. This preimage is, by definition, $\gamma_\lambda^1(\theta)$ for $\theta \in [0, \pi]$. So $\gamma_\lambda^1(\theta)$ lies in the region $0 \leq \operatorname{Arg} z \leq \pi/n$ and thus the result is true when $k = 1$.

Next note that $\gamma_\lambda^1(\pi)$ lies on the line $\nu \cdot \mathbb{R}^+$ and is given by $\nu\gamma_\lambda^1(0)$. So we can use the symmetry in the dynamical plane to extend the definition of $\gamma_\lambda^1(\theta)$ to a continuous curve defined for $\theta \in [0, n\pi]$ as follows: if $\theta \in [j\pi, (j+1)\pi]$, let $\gamma_\lambda^1(\theta) = \nu^j \gamma_\lambda^1(\theta - j\pi)$ for $j = 1, \ldots, n-1$. So $\gamma_\lambda^1(\theta)$ lies in $\operatorname{Im} z \geq 0$ for $\theta \in [0, n\pi]$. Then the sector V_+ is again mapped over $\gamma_\lambda^1(\theta)$ for these θ-values, so we have a continuous preimage $\gamma_\lambda^2(\theta)$ lying in V_+, mapped onto $\gamma_\lambda^1(\theta)$, and defined for $\theta \in [0, n\pi]$.

Then we extend the definition of $\gamma_\lambda^2(\theta)$ to $[0, n^2\pi]$ as above using the symmetry in the dynamical plane. So we have that $\gamma_\lambda^3(\theta)$ lies in V_+ for all $\theta \in [0, n^2\pi]$. Continuing in this fashion proves the stronger result that $\gamma_\lambda^k(\theta)$ in fact lies in V_+ for $\theta \in [0, n^{k-1}\pi]$ for all k as long as $\lambda \in \mathbb{R}^+$.

Now suppose that $0 < \operatorname{Arg} \lambda < \pi$. We no longer have the fact that V_+ is mapped over $\gamma_\lambda^0(\theta)$ for $0 \leq \theta \leq \pi$. Indeed, the point $\gamma_\lambda^1(0)$ now lies in V_-. This follows from the fact that the critical point ray through c_0 is mapped to a line whose argument is strictly larger than that of c_0, so the preimage of c_0 must lie below this critical point line. By the previous Proposition, we have that F_λ maps the interior of the entire region V_λ univalently onto $\mathbb{C} - \ell_\lambda$. Let ℓ'_λ denote the portion of ℓ_λ lying in the lower half plane. Then

$$\pi < \frac{\operatorname{Arg} \lambda}{2} + \pi = \operatorname{Arg} \ell'_\lambda < \frac{3\pi}{2}.$$

Since, for $\theta \in [0, \pi]$, we have

$$0 < \operatorname{Arg} c_0 \leq \operatorname{Arg} \gamma_\lambda^0(\theta) \leq \operatorname{Arg} c_0 + \pi < \frac{\operatorname{Arg} \lambda}{2} + \pi = \operatorname{Arg} \ell'_\lambda,$$

it follows that the entire line ℓ_λ never meets $\gamma_\lambda^0(\theta)$ for these θ-values. Hence there is a continuous preimage of $\gamma_\lambda^0(\theta)$ in $V_+ \cup V_-$ for each $\theta \in [0, \pi]$. This defines $\gamma_\lambda^1(\theta)$ over this interval. Note that $\gamma_\lambda^1(\pi) = \nu\gamma_\lambda^1(0)$ must lie in V_+. In fact, we can say more:

$$-\frac{\pi}{2n} < \frac{\operatorname{Arg} \lambda}{2n} - \frac{\pi}{2n} \leq \operatorname{Arg} \gamma_\lambda^1(\theta)$$

for $0 \leq \theta \leq \pi$. This follows since F_λ maps the prepole line in V_- to a line perpendicular to ℓ_λ in $-\pi/2 < \operatorname{Arg} z < 0$. This line does not intersect the curve $\gamma_\lambda^0(\theta)$ for $\theta \in [0, \pi]$. So $\gamma_\lambda^1(\theta)$ does not meet the prepole line in V_-. We therefore have

$$-\frac{\pi}{2n} < \operatorname{Arg} \gamma_\lambda^1(\theta) < \frac{3\pi}{2n}$$

for $\theta \in [0, \pi]$, so this proves the case $k = 1$ when $0 < \operatorname{Arg} \lambda < \pi$.

Now we extend the definition of $\gamma_\lambda^1(\theta)$ to $\theta \in [0, n\pi]$ as in the previous case using symmetry. Then we have, for $0 \le \theta \le n\pi$,

$$-\frac{\pi}{2n} < \operatorname{Arg} \gamma_\lambda^1(\theta) \le \operatorname{Arg} c_0 + \pi.$$

But $\operatorname{Arg} c_0 + \pi < \operatorname{Arg} \lambda/2 + \pi = \operatorname{Arg} \ell'_\lambda$. So again ℓ_λ does not meet the extension of $\gamma_\lambda^1(\theta)$. So we have that $\gamma_\lambda^2(\theta)$ lies in the interior of $V_+ \cup V_-$ for $0 \le \theta \le n\pi$ and so $\operatorname{Arg} \gamma_\lambda^2(\theta) < 3\pi/2n$. As above we in fact also have $-\pi/2n \le \operatorname{Arg} \gamma_\lambda^2(\theta)$, so this proves the case $k = 2$. Continuing inductively proves the result for all k-values when $0 < \operatorname{Arg} \lambda < \pi$ and $0 \le \theta \le n^{k-1}\pi$.

The case of negative values of θ is handled by symmetry as follows. We again assume that $0 < \operatorname{Arg} \lambda < \pi$. For each k we have, since $\gamma_\lambda^k(\theta)$ is $2n^k\pi$ periodic,

$$\begin{aligned} F_\lambda(\nu^{-1}\gamma_\lambda^k(\theta)) &= -F_\lambda(\gamma_\lambda^k(\theta)) \\ &= -\gamma_\lambda^{k-1}(\theta) \\ &= \gamma_\lambda^{k-1}(\theta - n^{k-1}\pi) \\ &= F_\lambda(\gamma_\lambda^k(\theta - n^{k-1}\pi)). \end{aligned}$$

Therefore

$$\nu^{-1}\gamma_\lambda^k(\theta) = \gamma_\lambda^k(\theta - n^{k-1}\pi)$$

follows since $\gamma_\lambda^k(\theta)$ is continuous in θ. Therefore we have that, when $\theta \in [-n^{k-1}\pi, 0]$, $\gamma_\lambda^k(\theta)$ lies in the region

$$-\frac{3\pi}{2n} < \operatorname{Arg} z < \frac{\pi}{2n}.$$

So altogether the curve $\gamma_\lambda^k(\theta)$ lies in the region $|\operatorname{Arg} z| < 3\pi/2n$ for all $|\theta| \le n^{k-1}\pi$. This concludes the proof when $0 \le \operatorname{Arg} \lambda < \pi$.

If $-\pi < \operatorname{Arg} \lambda < 0$, we invoke the $z \mapsto \overline{z}$ symmetry in the parameter plane. Since F_λ is conjugate to $F_{\overline{\lambda}}$ via $z \mapsto \overline{z}$, it follows that the curves $\gamma_\lambda^k(\theta)$ are mapped to $\gamma_{\overline{\lambda}}^k(-\theta)$ by the conjugacy. Hence these curves lie in the same region when $-\pi < \operatorname{Arg} \lambda < 0$. This concludes the proof.

5 Rings in Parameter Plane

Before turning to the proof of the existence of the Mandelpinski necklaces in the parameter plane, we need to examine more carefully the parametrizations of the rings in the dynamical plane in two of the special cases discussed earlier, namely when $\lambda \in \mathbb{R}^+$ and $\lambda \in \omega \cdot \mathbb{R}^+$.

First suppose that $\lambda \in \mathbb{R}^+$. For the special parameters λ_k among the superstable parameters in $\mathbb{R}^+$, we have seen that $F_{\lambda_k}(c_0)$ always lies in $\mathbb{R}^+$ and satisfies

$$0 < F_{\lambda_k}(c_0) < c_0 = F^k_{\lambda_k}(c_0) < F^{k-1}_{\lambda_k}(c_0) < \ldots < F^2_{\lambda_k}(c_0).$$

Hence $F^2_{\lambda_k}(c_0)$ lies on $\gamma^{k-2}_{\lambda_k} \cap \mathbb{R}^+$ and $F^j_{\lambda_k}(c_0)$ lies on $\gamma^{k-j}_{\lambda_k} \cap \mathbb{R}^+$ for $j = 2, \ldots, k$.

In particular, since the definition of the parametrization requires that $F_\lambda(\gamma^j_\lambda(0)) = \gamma^{j-1}_\lambda(0)$, it follows that, for the special parameter value λ_k, we have

$$\begin{aligned}
\gamma^0_{\lambda_k}(0) &= c_0 \\
\gamma^{k-2}_{\lambda_k}(0) &= F^2_{\lambda_k}(c_0) \\
\gamma^{k-3}_{\lambda_k}(0) &= F^3_{\lambda_k}(c_0) \\
&\vdots \\
\gamma^1_{\lambda_k}(0) &= F^{k-1}_{\lambda_k}(c_0)
\end{aligned}$$

Next we turn attention to the special parameter values $\lambda_k\omega$ lying along the line $\omega \cdot \mathbb{R}^+$ in the parameter plane. Here the situation is somewhat more complicated. For simplicity of notation, we fix a value of k and set $\mu = \lambda_k\omega$.

As we showed earlier, the line $\omega^{n/2} \cdot \mathbb{R}^+$ contains the critical point c_{n+1} and is either invariant under F_μ (if n is even) or interchanged with the symmetric line $-\omega^{n/2} \cdot \mathbb{R}^+$ by F_μ (if n is odd). In either case the symmetric line $-\omega^{n/2} \cdot \mathbb{R}^+$ is mapped to this line by F_μ and contains the critical point $c_1 = -c_{n+1}$. Also, the critical point line through c_0 is mapped to $-\omega^{n/2} \cdot \mathbb{R}^+$ by F_μ and then to $\omega^{n/2} \cdot \mathbb{R}^+$ by F_μ^2.

We have, by definition, $\gamma_\mu^0(0) = c_0$. Since $c_1 = \nu c_0$ where, as usual, $\nu = \exp(\pi i/n)$, we also have

$$\begin{aligned} c_1 &= \gamma_\mu^0\left(\frac{\pi}{n}\right) \\ c_{n+1} &= \gamma_\mu^0\left(\frac{\pi}{n} + \pi\right). \end{aligned}$$

Consider the portion of the critical point sector containing c_0 and lying on or outside C_λ. $\gamma_\mu^1(0)$ is the unique point in this region that is mapped to c_0 by F_μ. Since F_μ takes the critical point line through c_0 to the critical point line through c_1, it follows that $\gamma_\mu^1(0)$ lies below this line and that $\gamma_\mu^1(\pi/n)$, the preimage of c_1, lies on the critical point line through c_0. By symmetry, $\gamma_\mu^1((\pi/n) + \pi)$ then lies on the critical point line through c_1 and, since γ_μ^1 is $2n\pi$-periodic, the point

$$\gamma_\mu^1\left(\frac{\pi}{n} + \pi + n\pi\right)$$

lies on the line $\omega^{n/2} \cdot \mathbb{R}^+$ containing c_{n+1}.

Continuing, we have that $\gamma_\mu^2((\pi/n) + \pi)$ lies on the critical point line through c_0 and is mapped by F_μ to $\gamma_\mu^1((\pi/n) + \pi)$. The point

$$\gamma_\mu^2\left(\frac{\pi}{n} + \pi + n\pi\right)$$

then lies on the critical point line through c_1 and is mapped to

$$\gamma_\mu^1\left(\frac{\pi}{n} + \pi + n\pi\right)$$

on $\omega^{n/2} \cdot \mathbb{R}^+$.

Continuing inductively, we see that the critical point line through c_0 contains the points

$$
\begin{aligned}
&c_0 = \gamma_\mu^0(0)\\
&\gamma_\mu^1\left(\frac{\pi}{n}\right)\\
&\gamma_\mu^2\left(\frac{\pi}{n}+\pi\right)\\
&\quad\vdots\\
&\gamma_\mu^j\left(\frac{\pi}{n}+\pi+n\pi+\ldots+n^{j-2}\pi\right) = \gamma_\mu^j\left(\frac{\pi}{n}\left(1+n+\ldots+n^{j-1}\right)\right).
\end{aligned}
$$

and the critical point line through c_1 contains the points

$$
\begin{aligned}
&c_1 = \gamma_\mu^0\left(\frac{\pi}{n}\right)\\
&\gamma_\mu^1\left(\frac{\pi}{n}+\pi\right)\\
&\gamma_\mu^2\left(\frac{\pi}{n}+\pi+n\pi\right)\\
&\quad\vdots\\
&\gamma_\mu^j\left(\frac{\pi}{n}+\pi+n\pi+\ldots+n^{j-1}\pi\right) = \gamma_\mu^j\left(\frac{\pi}{n}\left(1+n+\ldots+n^{j}\right)\right).
\end{aligned}
$$

Equivalently, $\gamma_\mu^j(\theta)$ lies on the critical point line through c_1 for

$$\theta = \frac{\pi}{n}\left(\frac{n^{j+1}-1}{n-1}\right).$$

Now consider the corresponding points on the critical point line through c_{-1}. Since the parametrization corresponding to points on this line and γ_μ^j is obtained by subtracting $n^{j-1}\pi$ from the corresponding critical point line through c_0, we find the following points on this critical point line:

$$\begin{aligned}
&c_{-1} = \gamma_\mu^0\left(-\frac{\pi}{n}\right)\\
&\gamma_\mu^1\left(\frac{\pi}{n} - \pi\right)\\
&\gamma_\mu^2\left(\frac{\pi}{n} + \pi - n\pi\right)\\
&\quad\vdots\\
&\gamma_\mu^j\left(\frac{\pi}{n} + \pi + n\pi + \ldots + n^{j-2}\pi - n^{j-1}\pi\right).
\end{aligned}$$

Equivalently, $\gamma_\mu^j(\theta)$ lies on the critical point line through c_{-1} for

$$\theta = \frac{\pi}{n}\left(1 + n + n^2 + \ldots + n^{j-1} - n^j\right) = \frac{\pi}{n}\left(\frac{n^j - 1}{n-1}\right) - n^{j-1}\pi.$$

For later use, this value of θ is called $\theta_{n,j}$. See Figure 8.

We now turn to the proof of the existence of the rings $\mathcal{S}^k$ in parameter plane for $k > 1$. For simplicity, we consider only the case when $n \geq 5$ in this section; the special cases $n = 3, 4$ are described in [11].

Recall that, from the results of the previous section, we have that, when $k \geq 1$, the portion of the curve $\gamma_\lambda^k(\theta)$ for $|\theta| \leq n^{k-1}\pi$ lies in the region

$$-\frac{3\pi}{2n} < \operatorname{Arg} z < \frac{3\pi}{2n}.$$

We call this region W_n and note that W_n lies in the right half plane. Let H_λ denote the involution that fixes c_0, i.e.,

$$H_\lambda(z) = \frac{\lambda^{1/n}}{z}.$$

Lemma. *If* $n \geq 5$ *and* $\lambda \in \mathcal{O}'$, *then* $H_\lambda(W_n)$ *lies in the half plane* $\operatorname{Re} z > 0$.

Proof: Since

$$\operatorname{Arg} H_\lambda(z) = \frac{\operatorname{Arg}\lambda}{n} - \operatorname{Arg} z,$$

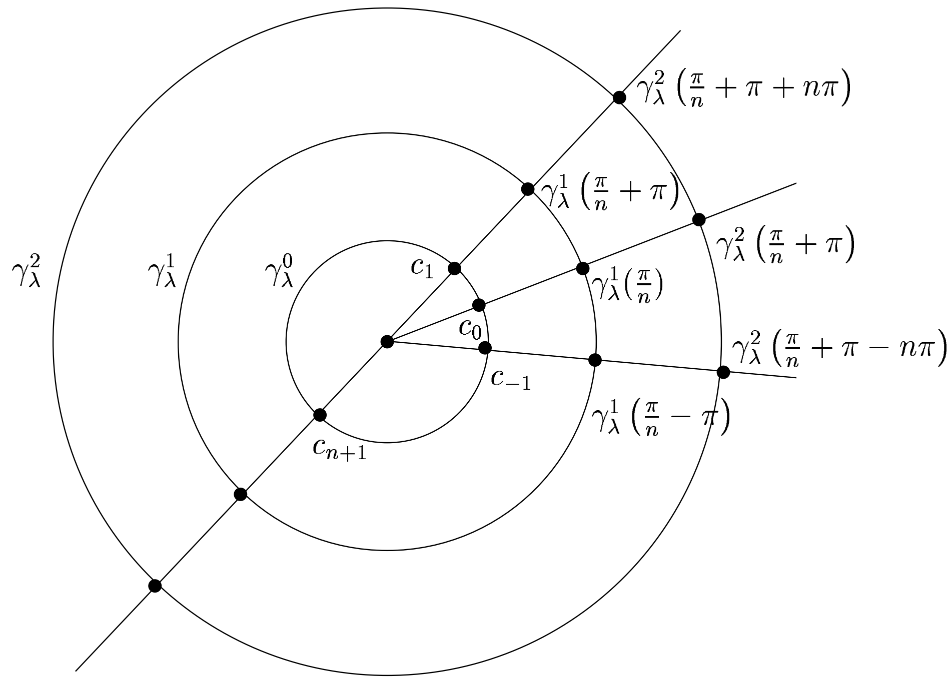


Figure 8: Parametrization of $\gamma_\lambda(\theta)$ when $\lambda = \lambda_k\omega$.

we have, if $z \in W_n$ and $n \geq 5$,

$$-\frac{\pi}{2} \leq -\frac{5\pi}{2n} \leq -\frac{3\pi}{2n} + \frac{\operatorname{Arg}\lambda}{n} < \operatorname{Arg} H_\lambda(z) < \frac{3\pi}{2n} + \frac{\operatorname{Arg}\lambda}{n} \leq \frac{5\pi}{2n} \leq \frac{\pi}{2}.$$

□

We remark that this result is false when $n = 3, 4$; that is the reason why these are special cases.

Now consider the curves

$$\xi_\lambda^k(\theta) = H_\lambda(\gamma_\lambda^k(\theta)).$$

Since the involution H_λ interchanges the inside and outside of C_λ, each of the curves ξ_λ^k is a simple closed curve lying inside the critical circle. We have

$$F_\lambda(\xi_\lambda^k(\theta)) = \gamma_\lambda^{k-1}(\theta)$$

since $F_\lambda(H_\lambda(z)) = F_\lambda(z)$. By the Lemma, we also have that $\xi_\lambda^k(\theta)$ lies in $\operatorname{Re} z > 0$ for $|\theta| \leq n^{k-1}\pi$, at least if $n \geq 5$.

Theorem. *For each $k \geq 1$ and any θ satisfying $|\theta| \leq n^{k-1}\pi$, there exists a unique parameter $\lambda = \lambda_{\theta,k}$ such that*

$$v_\lambda = 2\sqrt{\lambda} = \xi_\lambda^k(\theta).$$

Proof: The function $G(\lambda) = v_\lambda = 2\sqrt{\lambda}$ takes the subset $\mathcal{O}'$ of the parameter plane univalently onto an open subset of $\operatorname{Re} z > 0$. For each $\lambda \in \mathcal{O}'$, $G(\lambda)$ lies inside C_λ, but for λ on the dividing circle (which is the circular boundary of $\mathcal{O}'$), $G(\lambda)$ lies on the critical circle. Hence G maps $\mathcal{O}'$ univalently onto the interior of a half disk in the right half plane that contains the region inside C_λ in $\operatorname{Re} z > 0$ for each $\lambda \in \mathcal{O}'$. Call this half disk D.

Also, for fixed θ, the function $\lambda \mapsto \xi_\lambda^k(\theta)$ is analytic on $\mathcal{O}'$ and takes this set strictly inside the portion of the critical circle bounded by the rays $|\operatorname{Arg} z| = 3\pi/2n$. Hence, for each θ, the set of points $\xi_\lambda^k(\theta)$ lies inside a compact sector in D. That is, this set of points can possibly accumulate on the boundary of D only at the origin. Hence we may consider the composition $Q(\lambda) = G^{-1}(\xi_\lambda^k(\theta))$. As a function of λ, Q is analytic and maps the simply connected region $\mathcal{O}'$ inside itself. By the Schwarz Lemma, Q has a unique fixed point in this set or on its boundary. But the fixed point cannot lie at $\lambda = 0$ since 0 is surrounded by the McMullen domain so that the curves ξ_λ^k are bounded away from the origin. Hence there must be a unique fixed point in the interior of D. This fixed point is $\lambda_{\theta,k}$.

□

Note that the fixed points $\lambda_{\theta,k}$ vary continuously with θ, so $\theta \mapsto \lambda_{\theta,k}$ is a curve in the parameter plane.

The following Proposition identifies the specific values of $\lambda_{\theta,k}$ corresponding to the special cases considered earlier.

Proposition. *When $\theta = 0$ and $k \geq 1$, the parameter values $\lambda_{0,k}$ are given by the parameters $\lambda_{k+1} \in \mathbb{R}^+$. When $\theta = \theta_{n,k}$, $\lambda(\theta, k)$ is given by $\omega\lambda_{k+1}$ on the symmetry line $\omega \cdot \mathbb{R}^+$.*

Proof: When $\lambda \in \mathbb{R}^+$, the points $\gamma_\lambda^j(0)$ also lie in $\mathbb{R}^+$ for each j. Since, as shown earlier, the parameter λ_{k+1} has the property that $v_{\lambda_{k+1}} \in \xi^k_{\lambda_{k+1}}$, $F^2_{\lambda_{k+1}}(c_0) \in \gamma^{k-1}_{\lambda_{k+1}} \cap \mathbb{R}^+$ and the forward orbit of this point decreases along $\mathbb{R}^+$ until meeting c_0, it follows from the uniqueness of the parameter $\lambda_{0,j}$ that we must have $\lambda_{0,k} = \lambda_{k+1}$ for each $k \geq 1$.

When $\lambda = \lambda_{k+1}\omega$ and $\theta = \theta_{n,k}$, we know that the point $\gamma^k_\lambda(\theta_{n,k})$ lies on the critical point line through c_{-1}. Hence $H_\lambda(\gamma^k_\lambda(\theta_{n,k}))$ lies on the critical point line through c_1 and is given by $\xi^k_\lambda(\theta_{n,k})$. This point is then mapped by F_λ to the point on $\omega^{n/2} \cdot \mathbb{R}^+$ whose orbit meets c_{n+1} after $k-1$ iterations of F_λ or F^2_λ, depending upon whether n is even or odd. Hence $\lambda_{\theta_{n,k},k} = \lambda_{k+1}\omega$ as claimed.

□

Now the parameters in the previous Proposition are the unique parameters on the corresponding lines in parameter space for which the orbit of the second iterate of the appropriate critical point monotonically decreases along the corresponding line(s) for $k-1$ iterations before returning to itself and becoming periodic. So the curve $\theta \mapsto \lambda_{\theta,k}$ meets each of these two symmetry lines only once. Hence the portion of this curve defined for $0 \leq \theta \leq \theta_{n,k}$ either lies outside the sector

$$0 \leq \operatorname{Arg} \lambda \leq \frac{2\pi}{n-1}$$

for all values of θ or else this entire curve lies inside the sector. But the former cannot occur since this would imply that some $\lambda_{\theta,k}$ would lie in $\mathbb{R}^-$, contradicting the fact that each $\lambda_{\theta,k}$ lies in $\mathcal{O}'$. Hence the portion of the curve $\lambda_{\theta,k}$ defined for $0 \leq \theta \leq \theta_{n,k}$ is a continuous arc connecting $\theta = 0$ and $\theta = 2\pi/(n-1)$. It then follows by the $(n-1)$-fold symmetry that, for each $k \geq 1$, $\lambda_{\theta,k}$ is a simple closed curve in parameter space which is periodic of period

$$\begin{aligned}(n-1)\theta_{n,k} &= (n-1)\left(\frac{\pi}{n}\left(\frac{n^k-1}{n-1}\right) - n^{k-1}\pi\right) \\ &= \frac{\pi}{n}\left(-n^{k+1} + 2n^k - 1\right).\end{aligned}$$

We therefore define the ring $\mathcal{S}^{k+1}$ to be the simple closed curve $\theta \mapsto \lambda_{\theta,k}$. That is, $\mathcal{S}^{k+1}$ consists of parameter values for which the critical orbit has the following behavior:

1. both critical values lie inside the critical circle;
2. $F_\lambda^2(c_\lambda)$ lies on γ_λ^{k-1};
3. subsequent iterates decrease through the γ_λ^j until, at the k^{th} iterate, the critical orbit lands back on the critical circle.

We have shown:

Theorem. *When $n \geq 5$, the ring $\mathcal{S}^{k+1}$ in parameter space is a simple closed curve that is parameterized by $\theta \mapsto \lambda_{\theta,k}$ and is periodic of period*

$$\frac{\pi}{n}\left(n^{k+1} - 2n^k + 1\right) = \frac{\pi}{n}\left((n-2)n^k + 1\right).$$

In particular, since the critical points (resp., prepoles) of F_λ are located on $\gamma_\lambda^0(\theta)$ at $\theta = \pi j/n$ (resp., $(2j+1)\pi/2n$) for $0 \leq j < 2n$, we have the following count of superstable parameters and centers of Sierpinski holes along $\mathcal{S}^{k+1}$:

Corollary. *There are precisely* $(n-2)n^k + 1$ *parameters along* $\mathcal{S}^{k+1}$ *that are superstable parameters. There are the same number of parameters that are centers of Sierpinski holes. These parameters alternate between these two types as the parameter winds around* $\mathcal{S}^{k+1}$.

This proves the existence of the Mandelpinski necklaces when $n \geq 5$.

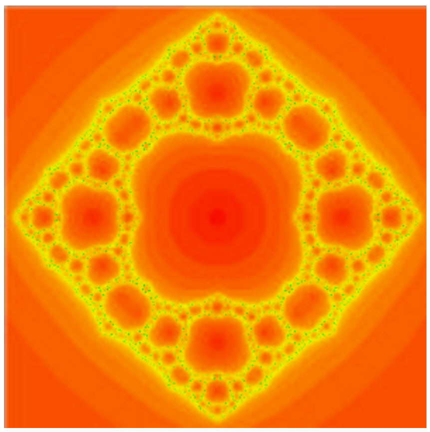
6 The Special Case $n = 2$

In this section we give three examples of how the case $n = 2$ is so much different from the cases where $n > 2$. The first example of this difference is the fact that there is no McMullen domain when $n = 2$. The reason for this is as follows. Recall that the critical values of F_λ are given by $v_\lambda = \pm 2\sqrt{\lambda}$. By McMullen's result [14], the critical values must lie in the trap door if the Julia set is a Cantor set of simple closed curves. But, in the case $n = 2$, we have

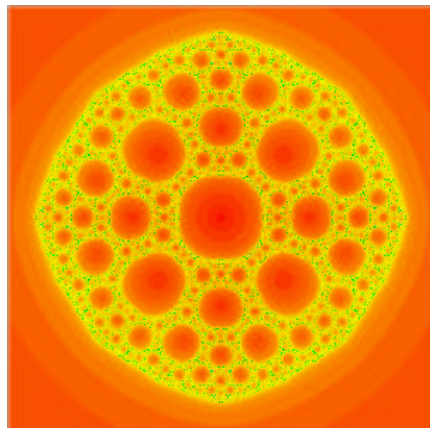
$$F_\lambda(v_\lambda) = 4\lambda + \frac{1}{4}.$$

So, as $\lambda \to 0$, $F_\lambda(v_\lambda) \to 1/4$, which is nowhere near B_λ since, when $|\lambda|$ is small, the boundary of B_λ is close to the unit circle.

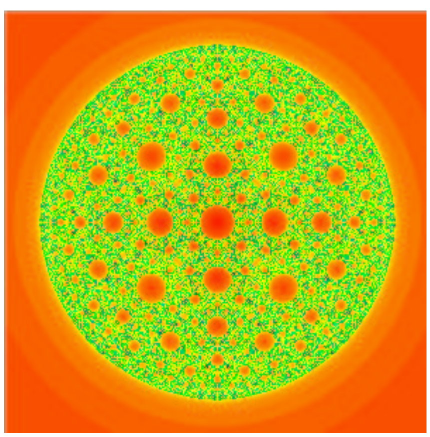
A second reason why the case $n = 2$ is different involves the Julia sets of the maps F_λ when $|\lambda|$ is small. When $n > 2$ these Julia sets are always Cantor sets of simple closed curves surrounding the origin. It is known [6] that there is a round annulus of some given width lying inside the unit circle and separating two of these curves when $|\lambda|$ is small. Hence these Julia sets never converge to the unit disk as $\lambda \to 0$. However, when $n = 2$, it is also shown in [6] that the Julia sets for F_λ do converge to the closed unit disk as $\lambda \to 0$. In Figure 9 we display four Julia sets with λ small and $n = 2$. All of these Julia sets are in fact Sierpinski curves. But notice how the preimages of T_λ get smaller and smaller as $|\lambda|$ decreases.



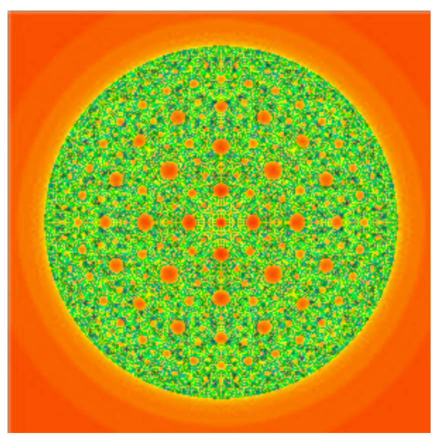
$\lambda = -1/4$



$\lambda = -1/16$



$\lambda = -0.01$



$\lambda = -0.001$

Figure 9: Sierpinski curve Julia sets for various negative values of λ in the case $n = 2$.

The final example of the difference between the cases $n = 2$ and $n > 2$ involves the Mandelpinski necklaces described above. As we showed earlier, when $n > 2$, the ring $\mathcal{S}^k$ passes alternately through exactly $(n-2)n^{k-1} + 1$ centers of baby Mandelbrot sets and centers of Sierpinski holes. Note that, when $n = 2$, this formula yields 1 for each k. And that, in fact, is true. As shown in [5], we do have these special rings $\mathcal{S}^k$ in this case. The single center of the only Mandelbrot set in $\mathcal{S}^k$ now lies along $\mathbb{R}^+$, while the single center of the corresponding Sierpinski hole lies in $\mathbb{R}^-$.

In Figure 10 we display the parameter plane for the case $n = 2$ together with a magnification. The large red central region is not a McMullen domain; rather it is a Sierpinski hole and it does not contain the origin. The ring $\mathcal{S}^1$ is the dividing circle which passes through the center of the main cardioid of the principal Mandelbrot set on the right and the center of that large red region on the left, which is a Sierpinski curve. In the magnification, the ring $\mathcal{S}^2$ then passes through the center of the period 2 bulb of the Mandelbrot set and the center of the large red disk, also a Sierpinski hole, that lies to the left of the origin.

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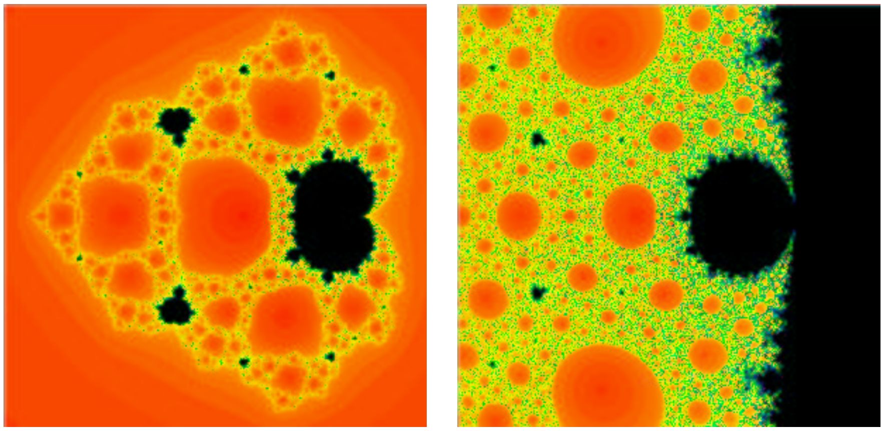

Figure 10: The parameter plane for the family $z^2 + \lambda/z^2$ and a magnification centered at the origin.

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