Limiting Behavior of Julia Sets of Singularly Perturbed Rational Maps

Robert L. Devaney*

Boston University
Boston, MA 02215 USA
bob@bu.edu

February 10, 2012

*This work was partially supported by grant #208780 from the Simons Foundation.

1 Introduction

In recent years there have been a number of papers dealing with singular perturbations of complex dynamical systems. Most of these papers deal with maps of the form $z^n + c + \lambda/z^d$ where $n \geq 2$ and $d \geq 1$ and c is the center of a hyperbolic component of the Multibrot set, i.e., the connectedness locus for the family $z^n + c$. These maps are called singular perturbations because, when $\lambda = 0$, the map is just the polynomial $z^n + c$ and the dynamical behavior for this map is completely understood. When $\lambda \neq 0$, the degree of the map changes and the dynamical behavior suddenly explodes.

Our goal in this paper is to give a survey of the behavior of these maps as the parameter λ tends to 0. By far the most interesting (and complicated) subfamily of these maps is the family $z^2 + c + \lambda/z^2$. The interesting fact here is that the Julia sets of these rational maps converge in the Hausdorff metric to the filled Julia set of the quadratic polynomial $z^2 + c$ as $\lambda \to 0$. This is somewhat surprising since it is known that, if the Julia set of a rational map ever contains an open set, then that Julia set must in fact be the entire complex plane. Here the limiting set always contains an open set when c is the center of a hyperbolic component, but this set is never the entire complex plane. So, as $\lambda \to 0$, the Julia sets of these rational maps come arbitrairly close to subsets of $\mathbb{C}$ that contain open sets. The actual Julia set for $\lambda = 0$ is, of course, much different.

For example, in Figure 1, we display several Julia sets in the family $z^2 + \lambda/z^2$ where λ is small. The white regions lie in the complement of the Julia set. Note how these disks become smaller as λ moves closer to 0. The limiting set is the unit disk which is the filled Julia set of z^2, but the actual Julia set when $\lambda = 0$ is just the unit circle.

In the more general case of the family $z^n + c + \lambda/z^d$ where $n, d \geq 2$ (but not both equal to 2), the situation is very different. For example, when $c = 0$, it is known that the Julia set is a Cantor set of simple closed curves, at least if $\lambda \neq 0$ is small enough. It is also known that there is always a round annulus of some definite width in the complement of the Julia set, so the Julia sets do not converge to the unit disk in this case (i.e., to the filled Julia set of z^n). When c is the center of some other hyperbolic component of the Multibrot set, the Julia set again contains a Cantor set of simple closed curves, but now infinitely many of these curves are "decorated," so this situation is quite different.

In Figure 2, we display Julia sets from the family $z^n + \lambda/z^n$ where λ is

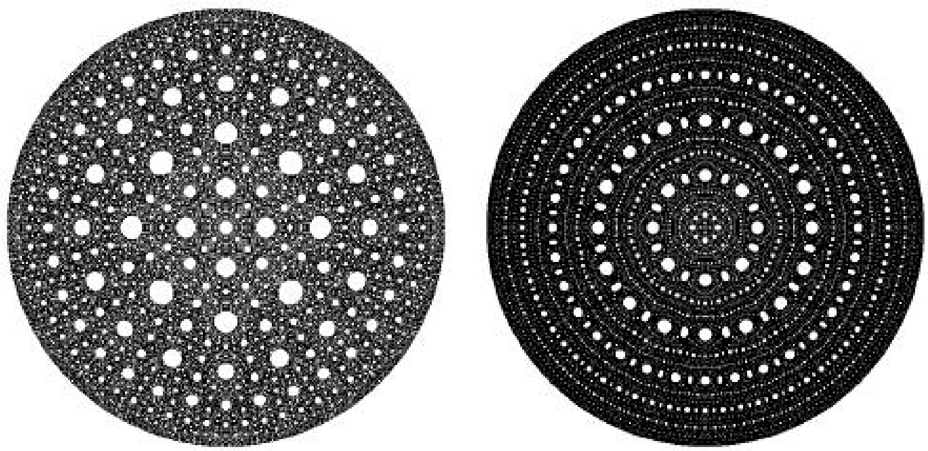


Figure 1: The Julia sets for $n = 2$ and $\lambda = -0.001$ and $\lambda = -0.00001$.

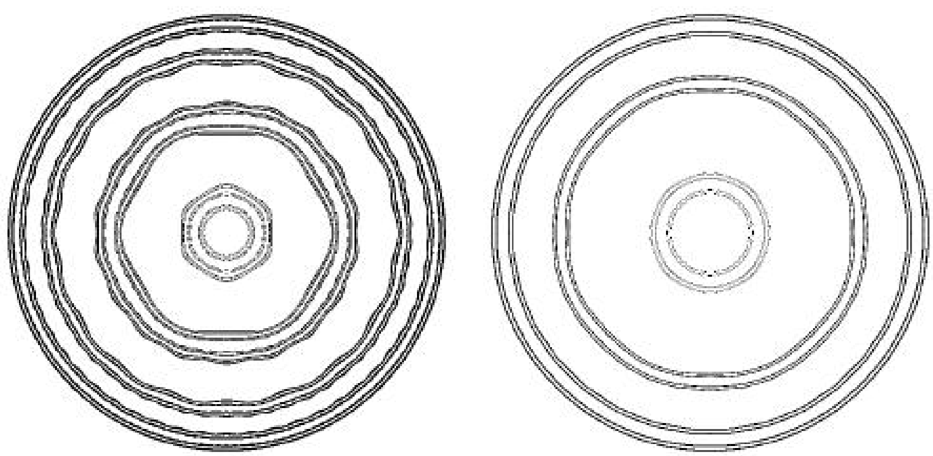


Figure 2: The Julia sets for $z^3 - 0.001/z^3$ and $z^4 - 0.001/z^4$ are both Cantor sets of circles.

small and $n = 3, 4$. Note that the complement of the Julia set in this case is a collection of annuli, and one of these annuli seems to have relatively large width. This is always the case as λ approaches the origin.

This paper is dedicated to Jack Milnor whose books, papers, and lectures have been an inspiration to me from the very beginning of my mathematical career.

2 Elementary Mapping Properties

For simplicity, for most of this paper, we will deal with the special case

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

where $n \geq 2$. At the end of the paper we discuss the differences that occur when we add the parameter c.

In the dynamical plane, the object of principal interest is the *Julia set* of F_λ, which we denote by $J(F_\lambda)$. The Julia set is the set of points at which the family of iterates $\{F_\lambda^n\}$ fails to be a normal family in the sense of Montel. It is known that $J(F_\lambda)$ is also the closure of the set of repelling periodic points for F_λ as well as the boundary of the set of points whose orbits escape to ∞ under iteration of F_λ. See [13].

The point at ∞ is a superattracting fixed point for F_λ and we denote the immediate basin of ∞ by B_λ. It is well known that F_λ is conjugate to $z \mapsto z^n$ in a neighborhood of ∞ in B_λ [16], [13]. There is also a pole of order n for F_λ at the origin, so there is a neighborhood of 0 that is mapped into B_λ by F_λ. If this preimage of B_λ is disjoint from B_λ (which it is when $|\lambda|$ is sufficiently small [6]), then we denote this preimage of B_λ by T_λ. So F_λ maps both B_λ and T_λ in n-to-one fashion onto B_λ. We call T_λ the *trap door* since any orbit that eventually enters the immediate basin of ∞ must pass through T_λ en route to B_λ.

The map F_λ has $2n$ free critical points given by $c_\lambda = \lambda^{1/2n}$. (We say "free" here since ∞ is also a critical point, but it is fixed, and 0 is also a critical point, but 0 is immediately mapped to ∞.) There are, however, only two critical values, and these are given by $v_\lambda = \pm 2\sqrt{\lambda}$. The map also has $2n$ prepoles given by $(-\lambda)^{1/2n}$. Note that all of the critical points and prepoles lie on the circle of radius $|\lambda|^{1/2n}$ centered at the origin. We call this circle the *critical circle.*

The map F_λ has some very special properties when restricted to circles centered at the origin. The following are straightforward computations:

1. F_λ takes the critical circle $2n$-to-one onto the straight line segment connecting the two critical values $\pm 2\sqrt{\lambda}$ and passing through 0;

2. F_λ takes any other circle centered at the origin to an ellipse whose foci are the two critical values.

We call the image of the critical circle the *critical segment*. Also, the straight ray connecting the origin to ∞ and passing through one of the critical points is called a *critical point ray*. Similar straightforward computations show that each of the critical point rays is mapped in two-to-one fashion onto one of the two straight line segments of the form tv_λ, where $t \geq 1$ and v_λ is the image of the critical point on this ray. So the image of a critical point ray is one of two straight rays connecting $\pm v_\lambda$ to ∞. Therefore the critical segment together with these two rays forms a straight line through the origin.

We now turn to the symmetry properties of F_λ in both the dynamical and parameter planes. Let ν be the primitive $2n^{\text{th}}$ root of unity given by $\exp(\pi i/n)$. Then, for each j, we have $F_\lambda(\nu^j z) = (-1)^j F_\lambda(z)$. Hence, if n is even, we have $F_\lambda^2(\nu^j z) = F_\lambda(z)$. Therefore the points z and $\nu^j z$ land on the same orbit after two iterations and so have the same eventual behavior for each j. If n is odd, the orbits of $F_\lambda(z)$ and $F_\lambda(\nu^j z)$ are either the same or else they are the negatives of each other. In either case it follows that the orbits of $\nu^j z$ behave symmetrically under $z \mapsto -z$ for each j. Hence the Julia set of F_λ is symmetric under $z \mapsto \nu z$. In particular, each of the free critical points eventually maps onto the same orbit (in case n is even) or onto one of two symmetric orbits (in case n is odd). Thus these orbits all have the same behavior (up to the symmetry) and so the λ-plane is a natural parameter plane for each of these families. That is, like the well-studied quadratic family $z^2 + c$, there is only one free critical orbit for this family up to symmetry.

Let $H_\lambda(z)$ be one of the n involutions given by $H_\lambda(z) = \lambda^{1/n}/z$. Then we have $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, so the Julia set is also preserved by each of these involutions. Note that each H_λ maps the critical circle to itself and also fixes a pair of critical points $\pm\sqrt{\lambda^{1/2n}}$. H_λ also maps circles centered at the origin outside the critical circle to similar circles inside the critical circle and vice versa. It follows that two such circles, one inside and one outside the critical circle, are mapped onto the same ellipse by F_λ.

Since there is only one free critical orbit, we may use the orbit of any critical point to plot the picture of the parameter plane. In Figure 3 we have plotted the parameter planes in the cases $n = 3$ and $n = 4$. The parameter planes for F_λ also possess several symmetries. First of all, we have

$$\overline{F_\lambda(z)} = F_{\overline{\lambda}}(\overline{z})$$

so that F_λ and $F_{\overline{\lambda}}$ are conjugate via the map $z \mapsto \overline{z}$. Therefore the parameter plane is symmetric under complex conjugation.

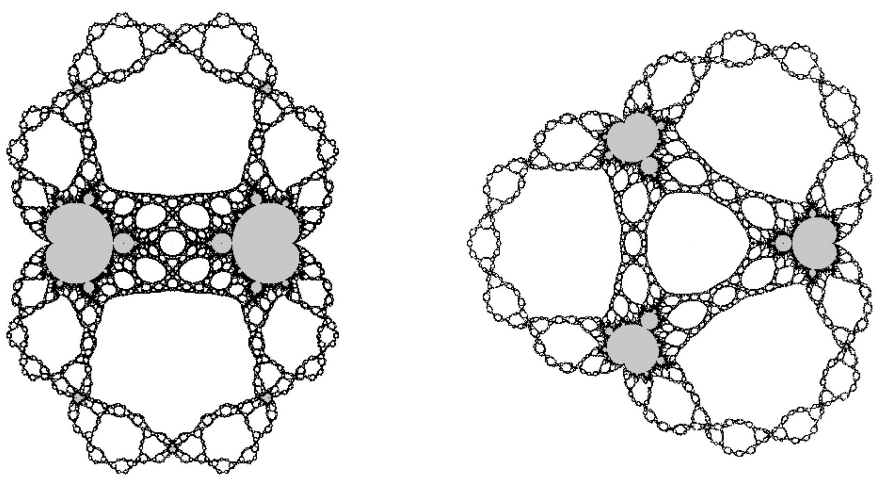


Figure 3: The parameter planes for the cases $n = 3$ and $n = 4$.

We also have $(n-1)$-fold symmetry in the parameter plane for F_λ. To see this, let ω be the primitive $(n-1)^{\text{st}}$ root of unity given by $\exp(2\pi i/(n-1))$. Then, if n is even, a straightforward computation shows that

$$F_{\lambda\omega}(\omega^{n/2} z) = \omega^{n/2}(F_\lambda(z)).$$

As a consequence, for each $\lambda \in \mathbb{C}$, the maps F_λ and $F_{\lambda\omega}$ are conjugate under the linear map $z \mapsto \omega^{n/2} z$. When n is odd, the situation is a little different. We now have

$$F_{\lambda\omega}(\omega^{n/2} z) = -\omega^{n/2}(F_\lambda(z)).$$

Since $F_\lambda(-z) = -F_\lambda(z)$, we therefore have that $F^2_{\lambda\omega}$ is conjugate to F^2_λ via the map $z \mapsto \omega^{n/2}z$. This means that the dynamics of F_λ and $F_{\lambda\omega}$ are "essentially" the same, though subtly different. For example, if F_λ has a fixed point, then, under this conjugacy, this fixed point and its negative are mapped to a 2-cycle for $F_{\lambda\omega}$. To summarize the symmetry properties of F_λ, we have:

Proposition (Symmetries in the dynamical and parameter plane). *The dynamical plane for F_λ is symmetric under the map $z \mapsto \nu z$ where ν is a primitive $(2n)^{\text{th}}$ root of unity as well as the involution $z \mapsto \lambda^{1/n}/z$. The parameter plane is symmetric under both $z \mapsto \overline{z}$ and $z \mapsto \omega z$ where ω is a primitive $(n-1)^{\text{st}}$ root of unity.*

Recall that, for the quadratic family, if the critical orbit escapes to ∞, the Julia set is always a Cantor set. For F_λ, it turns out that there are three different possibilities for the Julia sets when the free critical orbit escapes. The following result is proved in [6].

Theorem (The Escape Trichotomy). *For the family of functions*

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

with $n \geq 2$ and $\lambda \in \mathbb{C}$:

1. *If the critical values lie in B_λ, then the Julia set is a Cantor set.*
2. *If the critical values lie in T_λ (and, by assumption, T_λ is disjoint from B_λ), then the Julia set is a Cantor set of simple closed curves.*
3. *If the critical values lie in any other preimage of T_λ, then the Julia set is a Sierpinski curve.*

A *Sierpinski curve* is a planar set that is characterized by the following five properties: it is a compact, connected, locally connected and nowhere dense set whose complementary domains are bounded by simple closed curves that are pairwise disjoint. It is known from work of Whyburn [18] that any two Sierpinski curves are homeomorphic. In fact, they are homeomorphic to the well-known Sierpinski carpet fractal. From the point of view of topology, a Sierpinski curve is a universal set in the sense that it contains a homeomorphic copy of any planar, compact, connected, one-dimensional set. The first example of a Sierpinski curve Julia set was given by Milnor and Tan Lei [14].

Case 2 of the Escape Trichotomy was first observed by McMullen [12], who showed that this phenomenon occurs in each family provided that $n \neq 2$ and $|\lambda|$ is sufficiently small.

In the parameter plane pictures, the white regions consist of parameters for which the critical orbit escapes to ∞. The external white region is the set of parameters for which the Julia set is a Cantor set. The small central disk is the region containing parameters for which the Julia set is a Cantor set of simple closed curves. This is the McMullen domain, $\mathcal{M}$. And all of the other white regions contain parameters whose Julia sets are Sierpinski curves. These are the Sierpinski holes.

In Figure 4 we display three Julia sets drawn from the family $F_\lambda(z) = z^4 + \lambda/z^4$, one corresponding to each of the three cases in the Escape Trichotomy.

3 Julia Sets Converging to the Unit Disk

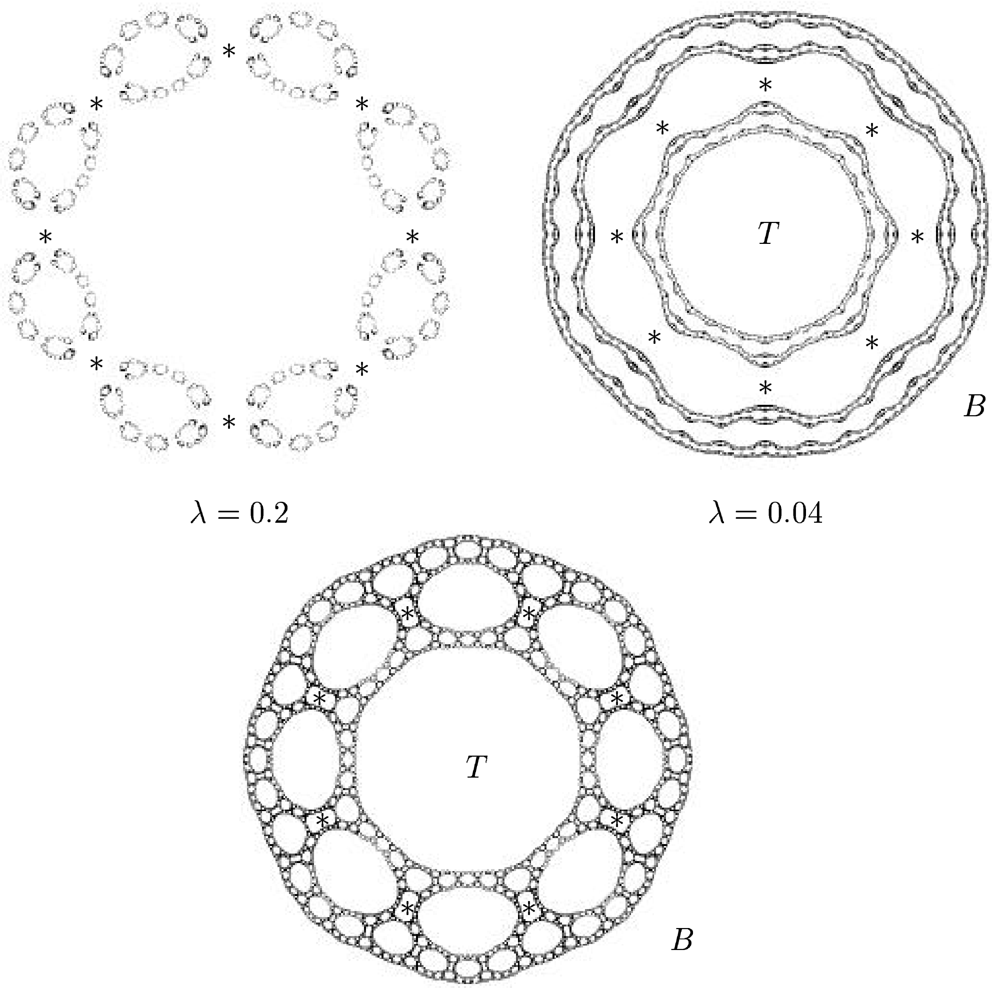
In this section we describe the interesting limiting behavior of the family

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2}$$

as $\lambda \to 0$. In [5], the following result was proved:

Theorem: *If λ_j is a sequence of parameters converging to 0, then the Julia sets of F_{λ_j} converge in the Hausdorff metric to the closed unit disk.*

Here is a sketch of the proof that the Julia sets of F_λ converge to the unit disk as $\lambda \to 0$. It is known that if c_λ does not lie in B_λ or T_λ, then $J(F_\lambda)$ is a connected set [4]. It has also been proved in that paper that, if $|\lambda| < 1/16$, then the Julia set always contains an invariant *Cantor necklace.* A Cantor necklace is a set that is a continuous and one-to-one image of the following subset of the plane. Place the Cantor middle thirds set on the real axis. Then adjoin a circle of radius $1/3^j$ in place of each of the 2^j removed intervals at the j^{th} level of the construction of the Cantor middle thirds set. The union of the Cantor set and the adjoined circles is the model for the Cantor necklace. See Figure 5. We remark that the existence of a Cantor necklace holds for any λ for which $J(F_\lambda)$ is connected, not just those with $|\lambda| < 1/16$ [4]. The only difference is that the boundaries of the open regions



$\lambda = -0.1$

Figure 4: Some Julia sets for $z^4 + \lambda/z^4$: if $\lambda = 0.2$, $J(F_\lambda)$ is a Cantor set; if $\lambda = 0.04$, $J(F_\lambda)$ is a Cantor set of circles; and if $\lambda = -0.1$, $J(F_\lambda)$ is a Sierpinski curve. Asterisks indicate the location of critical points.

now may not be simple closed curves — they may just be the boundary of a bounded, simply connected, open set (which need not be a simple closed curve).

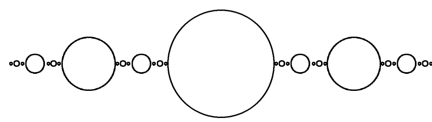


Figure 5: The Cantor middle-thirds necklace.

In the Julia set of F_λ, the invariant Cantor necklace has the following properties: the simple closed curve corresponding to the largest circle in the model is the boundary of the trap door. All of the closed curves corresponding to the circles at level j correspond to the boundaries of preimages of ∂B_λ that map to this set after j iterations. The Cantor set portion of the necklace is an invariant set on which F_λ is conjugate to the one-sided shift map on two symbols. The two extreme points in this set correspond to a fixed point and its negative, both of which lie in ∂B_λ. Hence the Cantor necklace stretches completely "across" $J(F_\lambda)$. Moreover, it is known that the Cantor necklace is located in a particular subset of the Julia set. Specifically, let $c_0(\lambda)$ be the critical point of F_λ that lies in the sector $0 \leq \operatorname{Arg} z < \pi/2$ when $0 \leq \operatorname{Arg} \lambda < 2\pi$. Let c_j be the other critical points arranged in the clockwise direction around the origin as j increases. Let I_0 denote the sector bounded by the two critical point rays connecting the origin to ∞ and passing through c_0 and c_3. Let I_1 be the negative of this sector. Then, as shown in [4], the Cantor set portion of the necklace is the set of points in $J(F_\lambda)$ whose orbits remain in $I_0 \cup I_1$ for all λ with $0 \leq \operatorname{Arg} \lambda < 2\pi$. The appropriate preimages of T_λ all lie in $I_0 \cup I_1$ as well.

It is easy to check that, when λ is small, the boundary of B_λ is close to the unit circle, so $J(F_\lambda)$ is contained in a region close to the unit disk. We now show why the Julia sets of F_λ converge to the closed unit disk $\mathbb{D}$ as $\lambda \to 0$. Here convergence to the closed unit disk means convergence in the Hausdorff metric.

Proposition. *Let $\epsilon > 0$ and denote the disk of radius ϵ centered at z by*

$B_\epsilon(z)$. There exists $\mu > 0$ such that, for any λ with $0 < |\lambda| \leq \mu$, $J(F_\lambda) \cap B_\epsilon(z) \neq \emptyset$ for all $z \in \mathbb{D}$.

Proof: Suppose that this is not the case. Then, given $\epsilon > 0$, we may find a sequence of parameters $\lambda_j \to 0$ and another sequence of points z_j in the unit disk $\mathbb{D}$ such that $J(F_{\lambda_j}) \cap B_{2\epsilon}(z_j) = \emptyset$ for each j. Since $\mathbb{D}$ is compact, there is a subsequence of the z_j that converges to some point $z^* \in \mathbb{D}$. This point z^* does not lie in T_λ since one checks easily that T_λ shrinks to the origin as $\lambda \to 0$. For each parameter in the corresponding subsequence, we then have $J(F_{\lambda_j}) \cap B_\epsilon(z^*) = \emptyset$ if j is sufficiently large. Hence we may assume at the outset that we are dealing with a sequence $\lambda_j \to 0$ such that $J(F_{\lambda_j}) \cap B_\epsilon(z^*) = \emptyset$.

Now consider the circle of radius $|z^*|$ centered at the origin. This circle meets $B_\epsilon(z^*)$ in an arc γ of length ℓ. Choose k so that $2^k\ell > 2\pi$. Since $\lambda_j \to 0$, we may choose j large enough so that $|F^i_{\lambda_j}(z) - z^{2^i}|$ is very small for $1 \leq i \leq k$, provided z lies outside the circle of radius $|z^*|/2$ centered at the origin. In particular, it follows that $F^k_{\lambda_j}(\gamma)$ is a curve whose argument increases by approximately 2π, i.e., the curve $F^k_{\lambda_j}(\gamma)$ wraps at least once around the origin. As a consequence, the curve $F^k_{\lambda_j}(\gamma)$ must meet the Cantor necklace in the dynamical plane. But this necklace lies in $J(F_{\lambda_j})$. Hence $J(F_{\lambda_j})$ must intersect this curve. Since the Julia set is backward invariant, it follows that $J(F_{\lambda_j})$ must intersect $B_\epsilon(z^*)$. This then yields a contradiction, and so the result follows.

$\square$

Remark: A similar result concerning the convergence to the unit disk occurs in the family of maps $G_\lambda(z) = z^n + \lambda/z$. See [15]. The difference here is that the Julia sets only converge to the unit disk if λ approaches the origin along the straight rays given by

$$\operatorname{Arg} \lambda = \frac{(2k+1)\pi}{n-1}$$

where $k \in \mathbb{Z}$. In Figure 6 we display the parameter plane for the family $z^5 + \lambda/z$. Note that there are four accesses to the origin where the parameter plane is "interesting." It is along these rays that the Julia sets converge to the unit disk. On any other ray, G_λ always has attracting cycles whose basins extend from the boundary of T_λ to the boundary of B_λ.

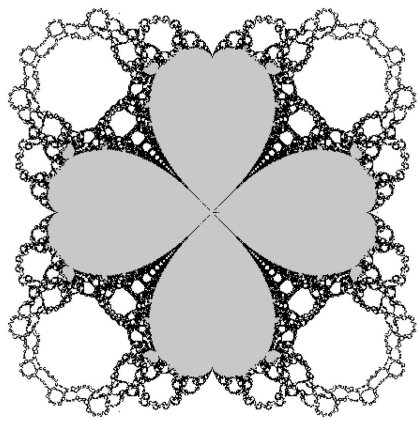


Figure 6: The parameter plane for $z^5 + \lambda/z$.

4 The Case $n > 2$

In this section we show that the case $n > 2$ is quite different from the case $n = 2$. In particular, there is a McMullen domain whenever $n > 2$ and, moreover, the Julia sets no longer converge to the unit disk as $\lambda \to 0$.

Recall from the Escape Trichotomy that, if the critical values lie in T_λ, then the Julia set of F_λ is a Cantor set of simple closed curves. This situation does not occur when $n = 2$. To see this, we need to specify the location of these critical values of $z^n + \lambda/z^n$. Let $\lambda^* = 4^{-n/(n-1)}$. Then one checks easily that, if $|\lambda| = \lambda^*$, then $|v_\lambda| = |c_\lambda|$ so both the critical points and critical values lie on the critical circle. Then, if $|\lambda| < \lambda^*$, we have $|v_\lambda| < |c_\lambda|$, and so F_λ maps the critical circle strictly inside itself. So a slightly larger circle is mapped to an ellipse that lies strictly inside this circle. Then, using quasiconformal surgery, one can glue the map $z \mapsto z^n$ into the disk bounded on the outside by this circle. See [3] for details. This new map is then conjugate to $z \mapsto z^n$ and the boundary of this map's basin of ∞ is then our original ∂B_λ. It then follows that B_λ is bounded by a simple closed curve lying strictly outside this disk. In particular, there is a preimage of B_λ surrounding the origin inside this circle. This is the trap door T_λ which is therefore disjoint from B_λ.

Next we compute that

$$F_\lambda(v_\lambda) = 2^n \lambda^{n/2} + \frac{1}{2^n \lambda^{n/2-1}}.$$

When $n > 2$, as $\lambda \to 0$, we have $v_\lambda \to 0$ and so $F_\lambda(v_\lambda) \to \infty$. Thus, when $|\lambda|$ is small, v_λ does indeed lie in the trap door when $n > 2$. But when $n = 2$, $F_\lambda(v_\lambda) \to 1/4$ as $\lambda \to 0$. The point $1/4$ is not in B_λ for $|\lambda|$ small since the boundary of B_λ is close to the unit circle. Hence v_λ does not lie in T_λ in this case.

There is another way to see why this is true. Suppose both critical values lie in T_λ. It is easy to see that T_λ is an open disk, so the question is: what is the preimage of T_λ? A natural first thought would be that the preimage of T_λ is a collection of open disks, one surrounding each preimage of $\pm v_\lambda$. But there are $2n$ such preimages, namely the critical points, and so each of these disks would then necessarily be mapped two-to-one onto T_λ. But this would then mean that the map would have degree $4n$. But the degree of F_λ is $2n$, so the preimages of T_λ cannot be a collection of disjoint disks. Therefore some of the preimages of T_λ must overlap. But then, by the symmetries discussed earlier, all of these preimages must overlap, and so the preimage of T_λ is a connected set. By the Riemann-Hurwitz formula, we know that

$$\mathrm{conn}\,(\mathrm{F}_\lambda^{-1}(\mathrm{T}_\lambda)) - 2 = (\deg \mathrm{F}_\lambda)(\mathrm{conn}\,(\mathrm{T}_\lambda) - 2) + (\text{number of critical points})$$

where $\mathrm{conn}(X)$ denotes the number of boundary components of the set X. But both the degree of F_λ and the number of critical points in this formula is $2n$, and $\mathrm{conn}(T_\lambda) = 1$. So it follows that the preimage of T_λ has two boundary components. That is, $F_\lambda^{-1}(T_\lambda)$ is an annulus.

This then is the beginning of McMullen's proof [12] that the Julia set in this case is a Cantor set of simple closed curves. We know that the complement of the Julia set contains the disks B_λ and T_λ as well as the annulus $F_\lambda^{-1}(T_\lambda)$. The entire preimage of B_λ is the union of B_λ and T_λ, while the entire preimage of T_λ is the annulus $F_\lambda^{-1}(T_\lambda)$. So what is the preimage of $F_\lambda^{-1}(T_\lambda)$? This preimage must lie in the two annular regions between $F_\lambda^{-1}(T_\lambda)$ and B_λ or T_λ. Call these annuli $A_{\rm in}$ and $A_{\rm out}$. See Figure 7. Since the preimages of $F_\lambda^{-1}(T_\lambda)$ cannot contain a critical point, it follows that the preimages must be mapped as a covering onto $F_\lambda^{-1}(T_\lambda)$, in fact, as an n-to-one covering since F_λ is n-to-one on both B_λ and T_λ. So the preimage of $F_\lambda^{-1}(T_\lambda)$ consists of a pair of disjoint annuli, one in $A_{\rm in}$ and the other in $A_{\rm out}$.

Then the preimages of these annuli consist of four annuli, and so forth. What McMullen shows is that, when you remove all of these preimage annuli, what is left is a Cantor set of simple closed curves, each surrounding the origin.

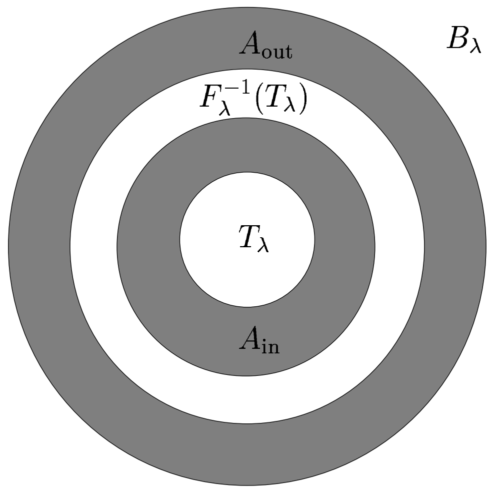


Figure 7: The annuli A_{in} and A_{out}.

Here then is the other reason why there is no McMullen domain when $n = 2$. From the above, we have that each of the annuli A_{in} and A_{out} is mapped as an n-to-one covering onto the annulus A which is the union of $F_\lambda^{-1}(T_\lambda)$, A_{in}, and A_{out}. Then the modulus of A_{in} is equal to $\text{mod}\,(A)/n$ and similarly for the modulus of A_{out}. But then, when $n = 2$, we have

$$\text{mod}\,A_{\text{in}} + \text{mod}\,A_{\text{out}} = \text{mod}\,A.$$

So this leaves no room for the intermediate annulus, $F_\lambda^{-1}(T_\lambda)$, so this picture cannot occur when $n = 2$.

So the question is: can these simple closed curves in the Julia set converge to the closed unit disk as $\lambda \to 0$. This, in fact, does not happen. The proof makes use of an important fact proved by Ble, Douady, and Henriksen concerning round annuli. We call an annulus of the form $0 < r_1 < |z| < r_2$ a round annulus. Then in [1] it is shown that any annulus in the plane that surrounds the origin and has modulus $\alpha > 1/2$ must contain a round annulus of modulus at least $\alpha - 1/2$.

As $\lambda \to 0$, we have that the annulus A stretches from ∂B_λ to ∂T_λ. Since ∂B_λ approaches the unit circle and ∂T_λ approaches 0 as $\lambda \to 0$, it follows that the modulus of A tends to ∞. So the moduli of A_{in} and A_{out} also tend to ∞. Then there is a subannulus, A_1 in A_{out} that is mapped n-to-one onto A_{out}. Then $\text{mod}\, A_1 = \text{mod}\, A_{\text{out}}/n$. Then A_1 contains a subannulus A_2 that is mapped n-to-one onto A_1, so $\text{mod}\, A_2 = \text{mod}\, A_1/n = \text{mod}\, A_{\text{out}}/n^2$. Continuing in this fashion, we find a sequence of annuli A_j whose moduli are given by $\text{mod}\, A_{\text{out}}/n^j$, and each of these annuli has one boundary in ∂B_λ. Adjacent to each A_j is another annulus E_j that eventually maps to T_λ and hence lie in the complement of the Julia set. One can estimate in similar fashion the moduli of these annuli, and note that they also lie "close" to ∂B_λ. Eventually we can find an annulus E_j whose modulus is larger than one and that lies outside the circle of some given radius centered at the origin. Then, as shown in [5], this annulus must contain a round annulus of modulus at least 1/2 and so the Julia sets do not converge to the unit disk as $\lambda \to 0$.

5 Other c-values

In this section we describe some other recent results involving the more general family

$$F_\lambda(z) = z^n + c + \frac{\lambda}{z^n}$$

where c is now the center of some other hyperbolic component of the Multibrot set. As in the previous sections, the situation when $n = 2$ is quite different from that when $n > 2$. When $n = 2$ it has been shown in [10] that the Julia sets of F_λ converge to the filled Julia sets of $z^2 + c$ as $\lambda \to 0$. The proof here is a little more complicated since we no longer can show that Cantor necklaces lie in the Julia set. However, it can be shown that, for λ sufficiently small, $J(F_\lambda)$ is connected and ∂B_λ is homeomorphic to the Julia set of $z^2 + c$. The latter involves a holomorphic motions argument. In Figures 8 and 9 we display the quadratic Julia sets known as the basilica and the Douady rabbit together with small singular perturbations of these maps.

When $n > 2$ for these families, the situation is a little different from the case when $c = 0$. The reason is that the interior of the filled Julia set of $z^n + c$ now consists of infinitely many disjoint disks. Only finitely many of these disks, say k, contain the single superattracting cycle. When λ is small a similar holomorphic motions argument shows that ∂B_λ is again homeomorphic to the Julia set of the unperturbed polynomial, so we have k similar disks that surround the former superattracting cycle. If we consider

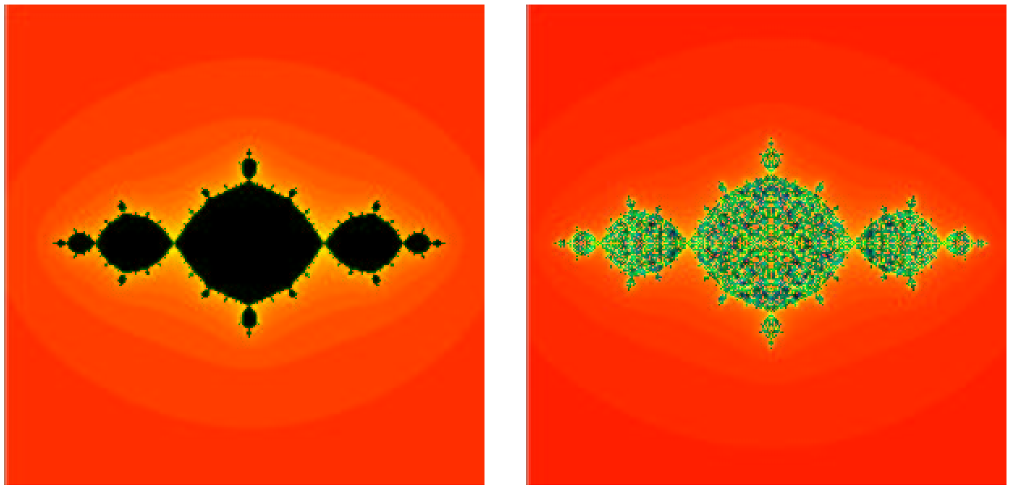


Figure 8: The Julia sets for $z^2 - 1 + \lambda/z^2$ where $\lambda = 0$ and $\lambda = -.00001$.

just the points whose orbits remain in the union of these k closed disks, then similar arguments as in the case $c = 0$ show that this set consists of k different Cantor sets of simple closed curves, each surrounding points on the former superattracting cycle. Then all of the infinitely many other preimages of these disks also contain Cantor sets of simple closed curves. However, none of these additional curves contain periodic points, as they all eventually map onto the original k Cantor sets of simple closed curves. So there must be more to the Julia sets than just these curves.

Indeed, in [2] it was shown that there are additional Cantor sets of point components in the Julia sets. These can be characterized by specifying how the points move around the disks that lie in the complement of ∂B_λ. In addition, countably many of the simple closed curves in the original k disks actually map onto the boundaries of the periodic disks. From the point of view of the entire Julia set, these boundaries are just a part of the entire set that makes up ∂B_λ. Hence these are no longer simple closed curves; rather, each of them has infinitely many "decorations" attached, i.e., preimages of the entire boundary of the basin of ∞. In Figures 10 and 11 we display the Julia set of the map $z^3 - i$ and its singular perturbation. Note that the annuli in the complement of the Julia set now have boundary curves with infinitely many attachments.

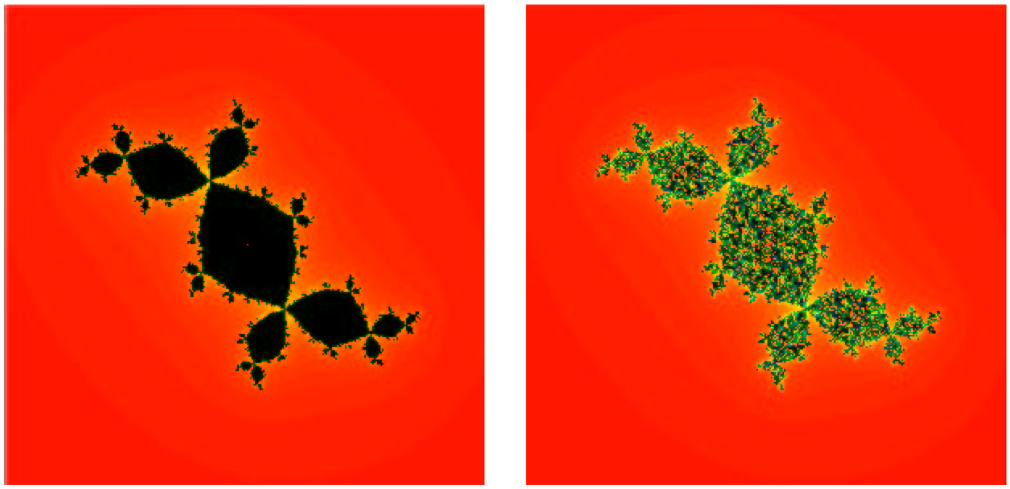


Figure 9: The Julia sets for $z^2 - 0.122 + 0.745i + \lambda/z^2$ where $\lambda = 0$ and $\lambda = -.000001$

Finally, for most of this paper, we considered singular perturbations by which a pole was inserted in place of the critical point of $z^n + c$. There have been a number of papers that address other types of singluar perturbations. For example, in [7], maps of the form $z^n + \lambda/(z-a)^d$ were investigated. When a is nonzero but close to 0, the McMullen domain disappears. The Julia set now contains infinitely many closed curves, but they are no longer concentric. In fact, only one surrounds the origin. In addition, there are uncountably many point components in the Julia set. Similar phenomena occur in the family $z^2 + c + \lambda/z^2$ where c is in a hyperbolic component of the Mandelbrot set but not at its center. See [11].

We also remark that convergence of Julia sets to objects that are different from the Julia set of the limiting map is not restricted to singularly perturbed maps. Indeed, Douady [9] has shown that, when a family of polynomials approaches a map with a parabolic point, there are many possible limiting sets while the limiting polynomial's Julia set is quite different (and much tamer).

Acknowledgement. The author would like to thank the referee for pointing out many infelicities in the original version of this paper.

Figure 10: The Julia set for the unperturbed map $z^3 - i$.

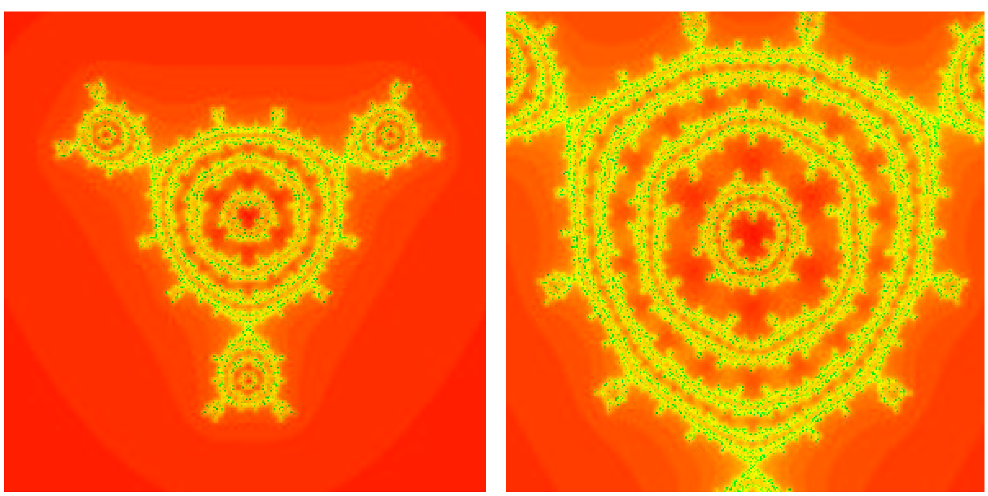


Figure 11: The Julia set for $z^3 - i + 0.0001/z^3$ and a magnification.

References

[1] Ble, G., Douady, A., and Henriksen, C. Round Annuli. *Contemporary Mathematics* **355** (2004), 71-76.

[2] Blanchard, P., Devaney, R. L., Garijo, A., and Russell, E. D. A Generalized Version of the McMullen Domain. *Int'l J. Bifurcation and Chaos* **18** (2008), 2309-2318.

[3] Blanchard, P., Devaney, R. L., Look, D. M., Moreno Rocha, M., Seal, P., Siegmund, S., Uminsky, D. Sierpinski Carpets and Gaskets as Julia Sets of Rational Maps. In *Dynamics on the Riemann Sphere*. European Math Society (2006), 97-119.

[4] Devaney, R. L. Cantor Necklaces and Structurally Unstable Sierpinski Curve Julia Sets for Rational Maps. *Qual. Theory Dynamical Systems* **5** (2006), 337-359.

[5] Devaney, R. L. and Garijo, A. Julia Sets Converging to the Unit Disk. *Proc. Amer. Math. Soc.* **136** (2008), 981-988.

[6] Devaney, R. L., Look, D. M., and Uminsky, D. The Escape Trichotomy for Singularly Perturbed Rational Maps. *Indiana University Mathematics Journal* **54** (2005), 1621-1634.

[7] Devaney, R. L. and Marotta, S. Evolution of the McMullen Domain for Singularly Perturbed Rational Maps. *Topology Proceedings* **32** (2008), 301-320.

[8] Devaney, R. L. and Pilgrim, K. Dynamic Classification of Escape Time Sierpinski Curve Julia Sets. *Fundamenta Mathematicae* **202** (2009), 181-198.

[9] Douady, A. Does the Julia Set Depend Continuously on the Polynomial? *Proc. Symp. Appl. Math.* **49** (1994), 91-138.

[10] Kozma, R. and Devaney, R. L. Julia Sets of Perturbed Quadratic Maps Converging to the Filled Quadratic Julia Sets. To appear.

[11] Marotta, S. Singular Perturbations in the Quadratic Family. *J. Difference Equations and Applications* **4** (2008), 581-595.

[12] McMullen, C. The Classification of Conformal Dynamical Systems. *Current Developments in Mathematics*. International Press, Cambridge, MA, (1995) 323-360.

[13] Milnor, J. *Dynamics in One Complex Variable*. Third Edition. Annals of Mathematics Studies. Princeton University Press, (2006).

[14] Milnor, J. and Tan Lei. A “Sierpinski Carpet” as Julia Set. Appendix F in Geometry and Dynamics of Quadratic Rational Maps. *Experiment. Math.* **2** (1993), 37-83.

[15] Morabito, M. and Devaney, R. L. Limiting Julia Sets for Singularly Perturbed Rational Maps. *International Journal of Bifurcation and Chaos* **18** (2008), 3175-3181.

[16] Petersen, C. and Ryd, G. *Convergence of Rational Rays in Parameter Spaces*, The Mandelbrot set: Theme and Variations, London Mathematical Society, Lecture Note Series 274, Cambridge University Press, 161-172, 2000.

[17] Roesch, P. On Capture Zones for the Family $f_\lambda(z) = z^2 + \lambda/z^2$. In *Dynamics on the Riemann Sphere*. European Mathematical Society, (2006), 121-130.

[18] Whyburn, G. T. Topological Characterization of the Sierpinski Curve. *Fundamenta Mathematicae* **45** (1958), 320-324.