Geometry of the Antennas in the Mandelbrot Set

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May 4, 2000

Abstract

In the Mandelbrot set, the bulbs attached directly to the main cardioid are called the p/q-bulbs. The reason for this is that the largest component of the interior of these bulbs consists of c-values for which the quadratic function $Q_c(z) = z^2 + c$ admits an attracting cycle with rotation number p/q. In this paper we give a geometric method to read off p/q from the geometry of the antenna attached to the bulb. [1]

1 Introduction

Our goal in this paper is to describe a geometric method of "reading off" dynamical information about the orbits of $Q_c(z) = z^2 + c$ from geometric information about the parameter plane for this family, the well known Mandelbrot set. Recall that the Mandelbrot set $\mathcal{M}$ is given by the set of complex c-values for which the orbit of 0 under the quadratic function $Q_c(z) = z^2 + c$ does not tend to infinity. It is known that many of the components of the interior of $\mathcal{M}$ consist of c-values for which the orbit of 0 is attracted to an

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[1] 2000 *Mathematics Subject Classification.* Primary 37F45; Secondary 37F10.

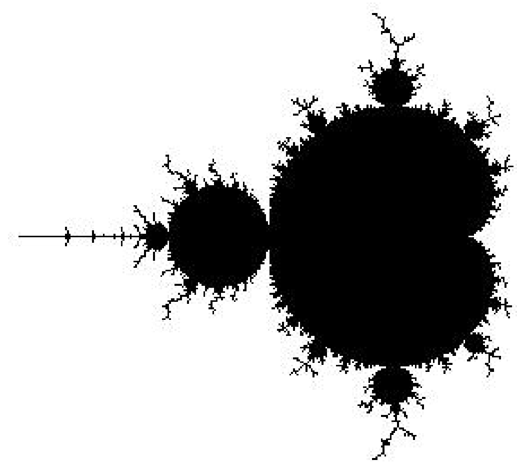

Figure 1: The Mandelbrot set. Note the many bulbs attached to the main cardioid. These are the p/q bulbs.

attracting cycle. In particular, this is true of the bulbs directly attached to the main cardioid in $\mathcal{M}$. In fact, for c inside one of these bulbs, Q_c features an attracting cycle with rotation number p/q. See Figure 1.

One of the fascinating and folkloric features of the Mandelbrot set is that one can often read off p/q directly from the geometry of the corresponding bulb. For the p/q bulb admits an antenna that consists of a junction point from which exactly q spokes emanate. One of these spokes is attached directly to the bulb. This spoke is called the *principal spoke.* For many of these bulbs, the "shortest" spoke attached to the junction point is located exactly p/q turns around the junction point from the principal spoke in the counterclockwise direction. Also, the "longest" spoke is located exactly p/q turns from the principal spoke in the clockwise direction. While this fact is not always true, nevertheless an "expert" in complex dynamics can usually judge where the shortest and longest spokes should lie and thereby read off p/q.

For example, in Figure 2 we display the 2/5 bulb. Note the 5 spokes attached to the junction point and that the shortest spoke is located roughly 2/5 of a turn from the principal spoke in the counterclockwise direction while

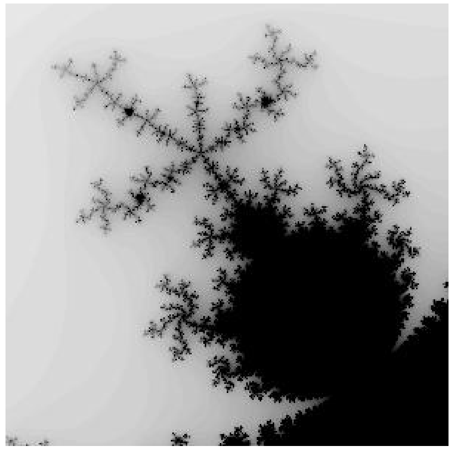

Figure 2: The 2/5 bulb.

the longest is located 2/5 of a turn in the opposite direction. Of course, the natural question is what is meant by the "shortest" and "longest" spoke. As in [4], we will use harmonic measure on the boundary of $\mathcal{M}$ to determine the size of these spokes. This measure is determined by the length of the interval of external rays that land on each spoke. Our goal then is to present an algorithm for computing exactly the length of this interval. With this algorithm, we will see that the p/q spoke is indeed the shortest. One can in fact use dynamical information about the Julia sets drawn from the p/q bulb to accomplish this. However, we will use a different, more combinatorial approach, which is more elementary.

As another example, in Figure 3 we display the 3/7 bulb. Here we see that the shortest spoke is located 3/7 of a turn from the principal spoke. The longest spoke is more difficult to determine. This illustrates why this "visual" method is not completely accurate.

2 The Mandelbrot set

In this section we recall some of the remarkable results of Douady and Hubbard (see [5]). The Mandelbrot set is given by

$$\mathcal{M} = \{c \in C \mid Q_c^n(0) \not\to \infty\}.$$

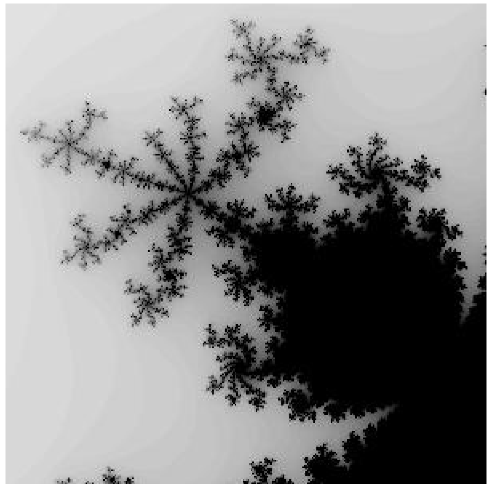

Figure 3: The 3/7 bulb.

See Figure 1. The following facts are well known. The largest cardioid-shaped region in $\mathcal{M}$ consists of c-values for which Q_c admits an attracting fixed point. Along the boundary of this cardioid, Q_c admits a neutral fixed point whose multiplier is $\exp(2\pi i\theta(c))$. As c moves around the boundary of the cardioid in the counterclockwise direction, starting at the cusp, the value of $\theta(c)$ increases monotonically from 0 to 1. In particular, there is a unique point $c_{p/q}$ on the boundary of the cardioid at which $\theta(c) = p/q$ for each rational in $(0, 1)$. The point $c_{p/q}$ is called the *root point* of the p/q bulb. At the root point, there is a simply connected region in the interior of $\mathcal{M}$ which is attached at this point to the cardioid. This component of the interior of $\mathcal{M}$ is called the *p/q bulb* in $\mathcal{M}$ and is denoted by $\mathcal{B}(p/q)$. If we remove the root point, then $\mathcal{M}$ breaks into two pieces; the component of $\mathcal{M} - c_{p/q}$ containing the p/q bulb is called the *p/q limb.*

The main theorem of Douady and Hubbard [5] asserts that there is a unique uniformizing map Φ that takes the exterior of the unit circle in the extended complex plane isomorphically onto the exterior of $\mathcal{M}$, taking ∞ to ∞ and mapping the positive real axis $x > 1$ onto the line $c > 1/4$. The image under Φ of the straight ray $r\exp(2\pi i\theta)$ for fixed θ and $r > 1$ is called the *external ray* with *external angle* θ. Note that we measure these external angles mod 1.

The Theorem of Douady and Hubbard above states further that each external ray with rational external angle actually lands at a unique point on the boundary of $\mathcal{M}$. By this we mean that

$$\lim_{r \to 1} \Phi(r \exp(2\pi i \theta^*))$$

exists when θ^* is rational. In particular, the 0-ray lies along the real axis and reaches $\mathcal{M}$ at the cusp point of the main cardioid at $c = 1/4$. Moreover, there are exactly two rays that land at the root point $c_{p/q}$ of the p/q bulb. We denote the angles of these two rays by $s_- = s_-(p/q)$ and $s_+ = s_+(p/q)$ where we assume that $s_- < s_+$. We call s_- (resp. s_+) the *lower* (resp. *upper*) external angle at $c_{p/q}$.

Finally, the Theorem also asserts that any external ray that lands at the root point of a bulb of period q is a rational that has prime period q under the (angle) doubling map given by $D(x) = 2x \bmod 1$. For example, the rays that land at $\mathcal{B}(1/3)$ have angles $1/7$ and $2/7$, and each of these has period 3 under angle doubling:

$$\frac{1}{7} \to \frac{2}{7} \to \frac{4}{7} \to \frac{1}{7} \to \cdots$$

Similarly, the rays landing at the root point of $\mathcal{B}(2/5)$ are $9/31$ and $10/31$, each of which have period 5 under D.

One can compute the rays s_- and s_+ using information about the dynamics of Q_c as c passes from the main cardioid into the p/q bulb through $c_{p/q}$. For c in the main cardioid, Q_c has an attracting fixed point. As c enters the p/q bulb, this fixed point becomes neutral as a repelling q-cycle merges with it. Once inside the p/q bulb, the fixed point becomes repelling while the cycle becomes attracting. At the fixed point, exactly q dynamical external rays land, two of which are s_- and s_+. In fact, these rays land at this fixed point for each c inside the p/q limb. From this information, the values of s_- and s_+ can then be computed using the dynamics of Q_c. See [8] or [1] for details. Rather than describe this algorithm in detail, we will discuss a different algorithm for computing these angles in the next section.

3 Computing External Angles

In this section we recall some results from [4] which provide an algorithm for computing upper and lower external angles at $c_{p/q}$. We will give a formula for the binary representation of these angles. Since these angles have period q under angle doubling, their binary representations will be a repeating sequence of 0's and 1's with length q. That is, s_- will be a sequence of the form

$$s_- = \overline{s_1 s_2 \ldots s_q}$$

where $s_j = 0$ or 1. In the sequel we will often denote an infinite repeating sequence simply by $s_1 \ldots s_q$ when the meaning is clear.

To determine the s_j, we let $R_{p/q}$ denote the counterclockwise rotation map of the circle by p/q turns. That is, $R_{p/q}(\theta) = \theta + p/q \bmod 1$. Note that

- $R^q_{p/q}(p/q) = p/q$
- $R^{q-1}_{p/q}(p/q) = 0$
- $R^{q-2}_{p/q}(p/q) = 1 - p/q = -p/q$.

Given p/q we define two partitions of the circle as follows. The *lower partition* is given by

$$I_0^- = (0, 1 - p/q] \text{ and } I_1^- = (1 - p/q, 1]$$

and the *upper partition* by

$$I_0^+ = [0, 1 - p/q) \text{ and } I_1^+ = [1 - p/q, 1).$$

Note that these two partitions differ only at their mutual endpoints 0 and $1 - p/q$. Now given θ in the circle, we define the *lower itinerary* of θ, $\ell(\theta)$ to be the sequence $s_1 s_2 \ldots$ where

$$s_j = 0 \text{ if } R^{j-1}_{p/q}(\theta) \in I_0^-.$$

Otherwise we set $s_j = 1$. So $\ell(\theta)$ records the position of the orbit of θ under $R_{p/q}$ relative to the lower partition of the circle. We define the *upper itinerary* $u(\theta)$ similarly by using the upper partition. Note that the upper and lower itinerary of θ are both repeating sequences of 0's and 1's of length q. The upper and lower itineraries of p/q play an important role below. From the above observations, we have

$$\ell(p/q) = \overline{s_1 \ldots s_{q-2} 01}$$

and

$$u(p/q) = \overline{s_1 \dots s_{q-2} 10}.$$

In particular, $\ell(p/q)$ and $u(p/q)$ differ in their fundamental blocks only in the last two digits.

The main result in [4] is that the upper and lower itineraries of p/q give the binary expansions of the upper and lower external angles of the p/q bulb.

Theorem 3.1 *Let $p/q \in (0,1)$. Then $u(p/q) = s^+(p/q)$ and $\ell(p/q) = s_-(p/q)$.*

For example, in the case $p/q = 1/4$, then $\ell(1/4) = \overline{0001}$ and $u(1/4) = \overline{0010}$, so that $s^-(1/4) = 1/15$ ($= \overline{0001}$ in binary) and $s^+(1/4) = 2/15 =$ ($\overline{0010}$ in binary).

We now turn our attention to the rays landing on the antenna of the p/q bulb. The junction point in the antenna is a *Misiurewicz point*, meaning that the orbit of the critical point at this c-value is eventually periodic.

Proposition 3.2 *There is a parameter value $w = w_{p/q}$ in the antenna attached to the p/q bulb that has the following properties:*

- *The orbit of w under z^2+w lands on a repelling fixed point after exactly q iterations.*
- *Two of the rays landing at w are $s_-\overline{s_+}$ and $s_+\overline{s_-}$.*

Proof. We apply the tuning procedure of Douady (see [6]). There is an orientation preserving homeomorphism that takes the entire Mandelbrot set into (but not onto) the p/q limb. The homeomorphism takes c-values in $\mathcal{M}$ that are the landing points of the external ray $s_1 s_2 s_3 \dots$ to c-values in the p/q limb that are landing points of the ray $t_1 t_2 t_3 \dots$ where each t_j is a block of 0's and 1's of length q and $t_j = s_-$ if $s_j = 0$ and $t_j = s_+$ if $s_j = 1$.

Consider $c = -2$. This is a Misiurewicz point in $\mathcal{M}$ since the orbit of 0 is fixed under Q_{-2} after two iterations. The external ray landing at $c = -2$ is the $1/2$ ray, since $c = -2$ lies on the negative real axis. This ray has two binary representations: $(0\overline{1})$ and $(1\overline{0})$. Thus $c = -2$ is mapped to a point in the p/q limb which is the landing point of the rays $s_-\overline{s_+}$ and $s_+\overline{s_-}$. The image of $c = -2$ is the point w. Since w is the landing point of an external ray that becomes periodic after q binary digits, it follows from [6] that w becomes periodic after q iterations of Q_w. Since the repeating part of these

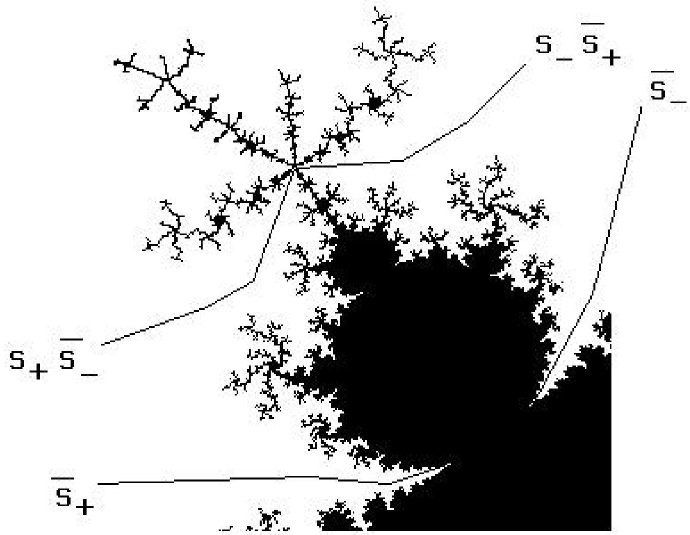

Figure 4: External rays landing on the 2/5 bulb.

angles involves s_- and s_+, the rays in the dynamical plane that land on the repelling fixed point, it follows that w must in fact land on the repelling fixed point alluded to above. ∎

By the above result, we now have four rays landing on the p/q bulb. In order, they are

$$\overline{s_-} < s_-\overline{s_+} < s_+\overline{s_-} < \overline{s_+}.$$

Note that the length of the arc of rays landing between $\overline{s_-}$ and $s_-\overline{s_+}$ is at most $1/2^{2q-2}$ since these two rays agree in the first $2q-2$ binary digits. Similarly, the length of the arc of rays landing between $\overline{s_+}$ and $s_+\overline{s_-}$ is also at most $1/2^{2q-2}$. On the other hand, the length of the arc of rays landing between $\overline{s_-}$ and $\overline{s_+}$ is exactly $1/(2^q-1)$ (see [4].) Thus many more rays land between the two rays landing at w than at other points on the p/q limb. This fact is illustrated for the 2/5 bulb in Figure 4.

Since there are q external rays landing at the repelling fixed point on which w lands, it follows that there an additional $q-2$ external rays landing at w. We will show in the next section that each of these rays lies between the rays $s_-\overline{s_+}$ and $s_+\overline{s_-}$. This indicates that the point w is indeed the junction point so evident in the above figures.

4 Farey numbers

Given any p/q in $(0,1)$ in lowest terms, recall that there are a pair of rationals α/β and γ/δ that serve as the *Farey parents* of p/q. This means that p/q is the rational between the Farey parents that has smallest denominator. It is known (see [7]) that

$$\frac{p}{q} = \frac{\alpha+\gamma}{\beta+\delta}$$

and that $p\beta - q\alpha = 1$ and $p\delta - q\gamma = -1$. One can inductively obtain the sequence of Farey parents. Begin with $0/1$ and $1/1$. These are the Farey parents of $1/2$ via the above "addition". Then $0/1$ and $1/2$ yield $1/3$, while $1/2$ and $1/1$ yield $2/3$. Continuing in this fashion produces the *Farey tree*, a list of all Farey parents and their progeny.

Proposition 4.1 *Suppose p/q has Farey parents $\alpha/\beta, \gamma/\delta$. Then*

$$R^{\beta}_{p/q}(p/q) = \frac{p+1}{q}.$$

Proof: Using the fact that $p\beta - q\alpha = 1$, we have

$$p/q - \alpha/\beta = \frac{1}{q\beta}$$

and so

$$R^{n}_{p/q}(p/q) - R^{n}_{\alpha/\beta}(\alpha/\beta) = (n+1)(\frac{p}{q} - \frac{\alpha}{\beta}) = \frac{n+1}{q\beta}.$$

Since $R^{\beta}_{\alpha/\beta}(\alpha/\beta) = \alpha/\beta$ we obtain

$$R^{\beta}_{p/q}(p/q) = \frac{\alpha}{\beta} + \frac{\beta+1}{q\beta} = \frac{\alpha}{\beta} + \frac{1}{q} + \frac{1}{q\beta} = \frac{p+1}{q}.$$

■

Proposition 4.2 *Given p/q in lowest terms, we have*

$$\ell(\frac{p+1}{q}) = u(\frac{p}{q}).$$

Proof: Note first that the distance between fractions $(p+1)/q$ and p/q remains constant under iteration of $R_{p/q}$ since for any integer $n > 0$

$$R^n_{p/q}(\frac{p+1}{q}) - R^n_{p/q}(\frac{p}{q}) = (n+1)\frac{p+1}{q} - (n+1)\frac{p}{q} = \frac{1}{q}.$$

In particular it follows that the $R_{p/q}$ orbit of $(p+1)/q$ is always just ahead of the orbit of p/q in the counterclockwise direction. Suppose $\ell((p+1)/q) = t_1 t_2 \ldots t_q$ and $u(p/q) = s_1 s_2 \ldots s_q$. We wish to show that $s_k = t_k$ for all $k = 1, \ldots, q$. Thus we have two cases:

- If $s_k = 0$ then
$$0 \leq R^{k-1}_{p/q}(p/q) = \frac{kp}{q} < 1 - \frac{p}{q}.$$
Consequently, $0 < (kp+1)/q \leq 1 - p/q$ so that $R^{k-1}_{p/q}((p+1)/q) \in I_0^-$ and $t_k = 0$.

- If $s_k = 1$ then
$$1 - \frac{p}{q} \leq R^{k-1}_{p/q}(\frac{p}{q}) < 1$$
$$1 - \frac{p}{q} < \frac{kp}{q} + \frac{1}{q} \leq 1$$
so that $R^{k-1}_{p/q}((p+1)/q) \in I_1^-$ and $t_k = 1$. ■

Let σ denote the shift map of the set of sequences of 0's and 1's. That is,

$$\sigma(s_1 s_2 s_3 \ldots) = (s_2 s_3 \ldots).$$

By Proposition 4.1 we have $\sigma^\beta(\ell(p/q)) = \ell((p+1)/q)$ and by Proposition 4.2 $\ell((p+1)/q) = u(p/q)$. Thus we have $\sigma^\beta(\ell(p/q)) = u(p/q)$. Furthermore,

$$\ell(p/q) = \sigma^q(\ell(p/q)) = \sigma^\delta(\sigma^\beta(\ell(p/q))) = \sigma^\delta(u(p/q)).$$

Therefore we have shown

Proposition 4.3 *Suppose p/q has Farey parents $\alpha/\beta < \gamma/\delta$. Then*

$$\sigma^\beta(s_-(p/q)) = s_+(p/q) \quad \text{and} \quad \sigma^\delta(s_+(p/q)) = s_-(p/q).$$

5 Order of Rays

In this section we determine the external angles of the q rays that land at the junction point $w_{p/q}$ of the antenna in the p/q limb. Two of these angles are given by $s_-\overline{s_+}$ and $s_+\overline{s_-}$. Since the fundamental blocks of s_- and s_+ are adjacent in the ordering of binary blocks of length q, it follows that the angles of the rays that land at $w_{p/q}$ must begin with either s_- or s_+. This string is called the *first part* of the angle. The *second part* is the repeating string corresponding to shifting the digits in $s_\pm$, since these angles correspond to the periodic rays landing at the repelling fixed point.

Moreover, since s_+ and s_- only differ in the last two digits, it follows immediately that $\sigma^n(s_-) < \sigma^n(s_+)$ for all n with the exception of $n \bmod q \equiv -1$. This exception comes from the fact that

$$\begin{aligned} \sigma^{-1}(s_1 \cdots s_{q-2}10) &= 0s_1 \cdots s_{q-2}1 \\ \sigma^{-1}(s_1 \cdots s_{q-2}01) &= 1s_1 \cdots s_{q-2}0 \end{aligned}$$

Lemma 5.1 *Given p/q, its Farey parents $\alpha/\beta < \gamma/\delta$, and the ray $s_+ = s_+(p/q)$, we have*

$$\begin{aligned} \sigma^{p\beta}(s_+) &= \sigma(s_+) \\ \sigma^{(p-1)\beta}(s_+) &= \sigma(s_-) \end{aligned}$$

Proof: Since we are shifting the q digits in s_+ in blocks of β digits, then we should consider the composition of iterations mod q. Using the Farey fraction property $p\beta - q\alpha = 1$, we have $p\beta \bmod q = (1 + q\alpha) \bmod q$ which implies the first equation. Moreover,

$$\begin{aligned} (p-1)\beta \bmod q &= 1 - \beta \bmod q \\ &= 1 + (q - \beta) \bmod q \\ &= 1 + \delta. \end{aligned}$$

By Proposition 4.3, we have $\sigma^\delta(s_+) = s_-$. Hence $\sigma^{(p-1)\beta}(s_+) = \sigma(s_-)$. ∎

The next proposition provides us with an increasing ordering for the second part of the angles. Notice that $(q-p)\beta \bmod q \equiv -1$. We claim that $\sigma^{(q-p)\beta}(s_+) = \sigma^{-1}(s_+)$ represents an exterior angle that is smaller than s_+. This can be seen comparing the two sequences digit by digit. Let $s_+ = s_1 s_2 \ldots s_{q-2}10$. Then $\sigma^{-1}(s_+) = 0s_1 s_2 \ldots s_{q-2}1$ and

- If $s_1 = 1$ clearly $\sigma^{-1}(s_+) < s_+$. If not, compare the next two digits
- If $s_2 = 1$ we're done (since $s_1 = 0$). If not, keep comparing subsequent digits

This process ends, for if we reach the last two digits of both itineraries, this would imply that $s_{q-2} = 0$ has to be compared to 1. From this it follows that

$$s_-\overline{\sigma^{(q-p)\beta}(s_+)} < s_-\overline{s_+}$$

This means that the external ray $s_-\overline{\sigma^{(q-p)\beta}(s_+)}$ does not land at the junction point w.

Proposition 5.2 *Given p/q, β and s_+, we have the following ordering under the shift map:*

$$s_+ < \sigma^{\beta}(s_+) < \sigma^{2\beta}(s_+) < \cdots < \sigma^{(q-p-1)\beta}(s_+),$$
$$\sigma^{(q-p)\beta}(s_+) < \sigma^{(q-p+1)\beta}(s_+) < \cdots < \sigma^{(q-2)\beta}(s_+) < s_-$$

Proof: We wish to compare $\sigma^{k\beta}(s_+)$ with $\sigma^{(k-1)\beta}(s_+)$. To do this, rewrite these two itineraries in terms of $\sigma(s_+)$ and $\sigma(s_-)$ as

$$\sigma^{k\beta}(s_+) = \sigma^{-(p-k)\beta}(\sigma^{p\beta}(s_+)) = \sigma^{-(p-k)\beta}(\sigma(s_+))$$

and similarly

$$\sigma^{(k-1)\beta}(s_+) = \sigma^{-(p-k)\beta}(\sigma^{(p-1)\beta}(s_+)) = \sigma^{-(p-k)\beta}(\sigma(s_-)).$$

Consider the right hand side of the equations above. As we are applying the shift map $-(p-k)\beta + 1$ times to s_+ and s_-, the inequality

$$\sigma^{(k-1)\beta}(s_+) < \sigma^{k\beta}(s_+)$$

holds unless $k\beta \bmod q \equiv -1$. We claim that this happens precisely when $k = q - p$ for $\beta > 1$. First, we will show that $k\beta \bmod q$ generates all the equivalence classes mod q. This is true if and only if $\beta \nmid q$. But $q = \delta + \beta$ so this condition reduces to show $\beta \nmid \delta$. This follows easily since the Farey property $\beta\gamma - \alpha\delta = 1$ indicates that β and δ are relatively prime. Thus, the first ordering is achieved for $k = 1, \ldots, q-p-1$ while, for the second ordering, $k = q-p+1, \ldots, q-1$, avoiding only the case $k = q - p$.

When $\beta = 1$ then we simply have to avoid the case when $k \bmod q \equiv -1$, i.e., when $k = -1$ or $k = q + 1$, but neither of these cases occur above. ∎

Using this result we are now able to state

Theorem 5.3 *Let $w_{p/q}$ be the junction point of the bulb $\mathcal{B}(p/q)$. Then there exist q preperiodic rays contained in the interval $[s_-\overline{s_+}, s_+\overline{s_-}]$ that land on $w_{p/q}$. These rays are give in increasing order by:*

$$s_-\overline{s_+}, s_-\overline{\sigma^{\beta}(s_+)}, \ldots, s_-\overline{\sigma^{(q-p-1)\beta}(s_+)}, s_+\overline{\sigma^{(q-p)\beta}(s_+)}, \ldots, s_+\overline{s_-}.$$

6 Length of Spokes

In this section we identify the shortest and longest spokes of the antenna at $w_{p/q}$. Recall that this means that the gap between the two external rays landing at w and cutting off this spoke is smallest and largest respectively.

Proposition 6.1 *Among the q rays landing at w, the rays which have q-periodic second part given by $\sigma^{(p-1)\beta}(s_+)$ and $\sigma^{p\beta}(s_+)$ are closest together.*

Proof: We first show that both rays have the same first part. Recall that the change in the first part of the angle from s_- to s_+ occurs when we shift s_+ exactly $(q-p)\beta$ times. Thus, if $p < q-p$, then the first part for both rays is s_-. When $p > q-p$, the worst possible case occurs when $p-1 = q-p$, but then again the first part is s_+ for both rays.

Now, when $p = q-p$ then $p/q = 1/2$ and the Farey parents are $0/1$ and $1/1$. This is a very special case where the list of rays is given simply by $s_-\overline{s_+}, s_+\overline{s_-}$.

Finally, we only need to prove that the difference $\sigma^{p\beta}(s_+) - \sigma^{(p-1)\beta}(s_+)$ is the smallest among consecutive rays. By Lemma 5.1, we know $\sigma^{p\beta}(s_+) = s_2 \cdots s_{q-2}10s_1$ and $\sigma^{(p-1)\beta}(s_+) = s_2 \cdots s_{q-2}01s_1$. Thus, the difference between the blocks of length q is

$$\sigma^{p\beta}(s_+) - \sigma^{(p-1)\beta}(s_+) = \frac{1}{2^{q-2}} - \frac{1}{2^{q-1}} = \frac{1}{2^{q-1}}.$$

The difference of the repeating part of the sequences is given by

$$\begin{aligned}\overline{\sigma^{p\beta}(s_+)} - \overline{\sigma^{(p-1)\beta}(s_+)} &= \left(\frac{1}{2^{q-2}} + \frac{1}{2^{2q-2}} + \cdots\right) - \left(\frac{1}{2^{q-1}} + \frac{1}{2^{2q-1}} + \cdots\right) \\ &= \sum_{k=1}^{\infty} \frac{4}{2^{kq}} - \sum_{k=1}^{\infty} \frac{2}{2^{kq}}\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{k=1}^{\infty}(\frac{1}{2^q})^k \\
&= \frac{1}{2^{q-1}}\frac{2^q}{2^q-1}.
\end{aligned}
$$

Thus, the entire difference is

$$
\begin{aligned}
s_\pm\overline{\sigma^{p\beta}(s_+)} - s_\pm\overline{\sigma^{(p-1)\beta}(s_+)} &= (\frac{1}{2^{2q-2}} + \frac{1}{2^{3q-2}} + \cdots) - (\frac{1}{2^{2q-1}} + \frac{1}{2^{3q-1}} + \cdots) \\
&= (\sum_{k=1}^{\infty}\frac{4}{2^{kq}} - \frac{1}{2^{q-2}}) - (\sum_{k=1}^{\infty}\frac{2}{2^{kq}} - \frac{1}{2^{q-1}}) \\
&= \frac{1}{2^{q-1}}\frac{2^q}{2^q-1} - \frac{1}{2^{q-1}} \\
&= \frac{1}{2^{q-1}}\frac{1}{2^q-1}
\end{aligned}
$$

since both rays have the same first parts (thus no contribution is made) and both series start with indices shifted q places.

Now, assume there exists an integer m such that

$$\sigma^{m\beta}(s_+) - \sigma^{(m-1)\beta}(s_+) < \sigma^{p\beta}(s_+) - \sigma^{(p-1)\beta}(s_+).$$

In this case, $\sigma^{m\beta}(s_+)$ and $\sigma^{(m-1)\beta}(s_+)$ must differ in the last two digits, i.e., they are s_+ and s_- respectively. However, the rays with these two second parts do not share the same first parts and consequently they are further apart. This completes the proof. ∎

Remark. Note that this result verifies the observation in the introduction that the smallest spoke is located p/q turns in the counterclockwise direction from the principal spoke in the antenna.

We now identify the longest spoke. From the ordering given by Theorem 5.3, the largest gap between rays must be located when the first part changes, that is, between the rays $s_-\sigma^{(q-p-1)\beta}(s_+)$ and $s_+\sigma^{(q-p)\beta}(s_+)$. To compute the length between these rays, first recall that

$$\sigma^{(q-p)\beta}(s_+) = \sigma^{-1}(s_+).$$

On the other hand,

$$\sigma^{(q-p-1)\beta}(s_+) = \sigma^{(q-p)\beta}(\sigma^{-\beta}(s_+))$$

$$\begin{aligned} &= \sigma^{-1-\beta}(s_+) \\ &= \sigma^{-1-\beta}(\sigma^{\beta}(s_-)) \end{aligned}$$

by Proposition 4.3. Thus $\sigma^{(q-p-1)\beta}(s_+) = \sigma^{-1}(s_-)$. As above we have

$$\sigma^{(q-p)\beta}(s_+) - \sigma^{(q-p-1)\beta}(s_+) = \frac{1-2^{q-1}}{2^q}.$$

Thus, the largest distance between consecutive rays is given by

$$\begin{aligned} s_+\overline{\sigma^{(q-p)\beta}(s_+)} - s_-\overline{\sigma^{(q-p-1)\beta}(s_+)} &= s_+\overline{\sigma^{-1}(s_+)} - s_-\overline{\sigma^{-1}(s_-)} \\ &= \frac{1}{2^q} + \sum_{k=2}^{\infty}\frac{1}{2^{kq}} - \sum_{k=1}^{\infty}\frac{1}{2^{kq+1}} \\ &= \frac{1}{2^q} + \frac{1-2^{q-1}}{2^q-1} - \frac{1-2^{q-1}}{2^q} \\ &= \frac{1}{2}\frac{1}{2^q-1}. \end{aligned}$$

The last computation exhibits an interesting feature of the largest spoke: since the measure of the bulb is $1/(2^q-1)$ (see [4]), the subset of rays that land on the largest spoke has exactly half the measure of the total set. Similarly, the subset of rays landing at the shortest spoke is $1/2^{q-1}$th of the total measure.

Similar calculations allow us to compute the length of any spoke in the antenna. The proof of the following is therefore left to the reader.

Proposition 6.2 *Given any two consecutive rays with the same first parts and second parts $\sigma^{k\beta}(s_+)$ and $\sigma^{(k-1)\beta}(s_+)$, their difference is given by*

$$s_\pm\overline{\sigma^{k\beta}(s_+)} - s_\pm\overline{\sigma^{(k-1)\beta}(s_+)} = \frac{1}{2^{\Gamma+1}}\frac{1}{2^q-1}$$

where $\Gamma = k\delta - 1 \bmod q$.

Remark: When $k = p$ in Proposition 6.2, the length of the shortest spoke or, equivalently, the distance between the rays $s_\pm\sigma^{(p-1)\beta}$ and $s_\pm\sigma^{p\beta}$ agrees with our previous computation as $\Gamma(p) = (p\delta - 1) \bmod q = (q\gamma - 1) - 1 \bmod q = -2 \bmod q = q - 2$. In the case of the largest spoke, we have $k = q - p$. Even though our assumption above regarding equal first parts does not hold in this case, the formula does, since

$$\begin{aligned}\Gamma(q-p) &= (q-p)\delta - 1 \bmod q \\ &= -p\delta - 1 \bmod q \\ &= 1 - q\alpha - 1 \bmod q \\ &= 1 - 1 = 0.\end{aligned}$$

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