Limiting Julia Sets for Singularly Perturbed Rational Maps

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1 Introduction

In this paper we consider the family of complex rational maps of the form

$$F_\lambda(z) = z^n + \frac{\lambda}{z}$$

where the parameter $\lambda \in \mathbb{C}$ and $n \geq 2$. We will primarily consider the case where $|\lambda|$ is small, so these maps may be regarded as singular perturbations of the well understood map $z \mapsto z^n$. For this map, it is known that the Julia set (the chaotic set) is the unit circle. But when $\lambda \neq 0$, the degree of this map increases and consequently the Julia set changes dramatically.

Our goal is to show that, as λ approaches the origin along $n-1$ special rays in the parameter plane, the Julia set of F_λ converges as a set to the unit disk, not to the unit circle. This is an interesting phenomenon since it is well known that, if a Julia set contains an open set, it must necessarily be the entire Riemann sphere. So here we have the situation where Julia sets can converge to a set that contains an open set but is not the entire sphere.

For example, in Figure 1, we display several Julia sets drawn from the family $z^2 + \lambda/z$. Colors in these pictures indicate how quickly the orbit of the point enters the immediate basin of ∞, with red points escaping most quickly, followed by orange, yellow, green and blue. So the colored points are not in the Julia set in these examples. Black points lie in the Julia set, though the black points are difficult to see since an open and dense set of points are known to escape in the cases depicted; basically, the Julia sets lies at the points where there is an abrupt change of color. Note how the red regions shrink in size in these pictures as $|\lambda|$ decreases.

This paper continues a study begun in [3] concerning the family

$$G_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

where $n, d \geq 2$. In that paper it was shown that, if $n = d = 2$, then, as $\lambda \to 0$,

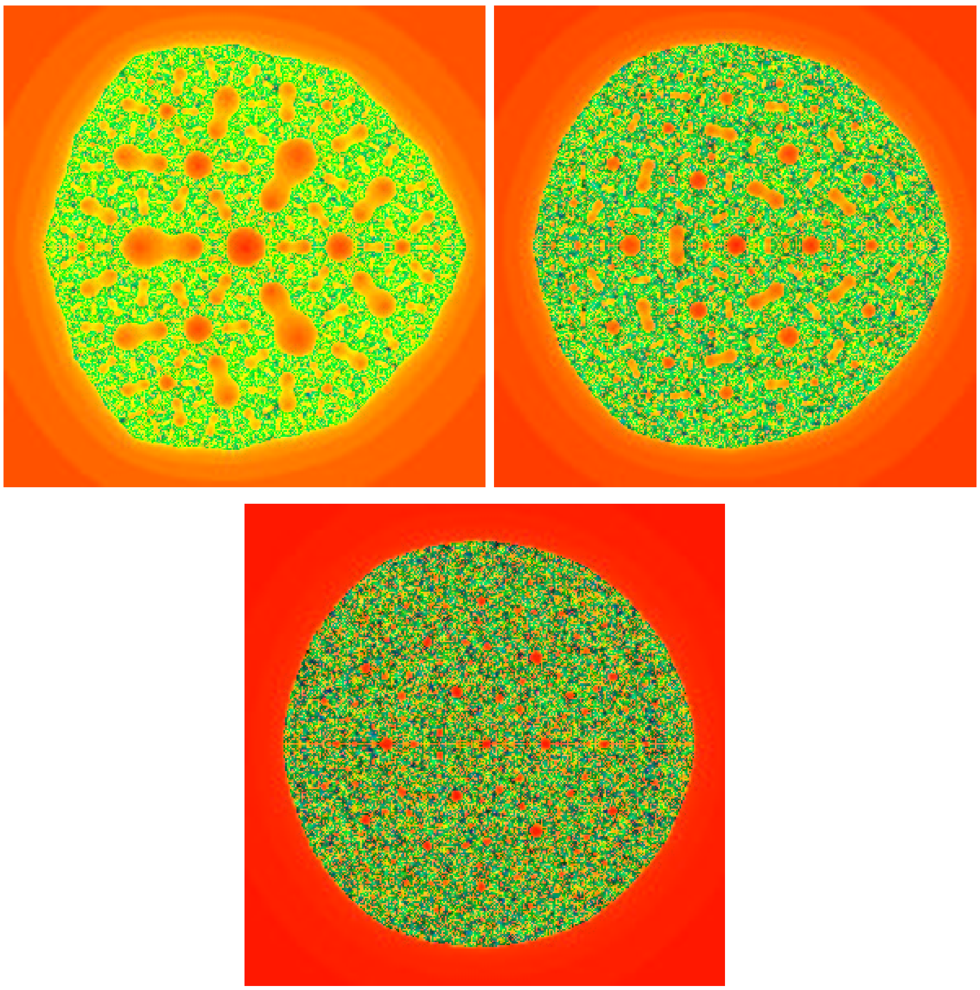

Figure 1: Julia sets for $z^2 + \lambda/z$ where $\lambda = -0.1, -0.05$, and -0.025. Reddish shaded regions contain points that escape to ∞, so these points form the complement of the Julia set.

the Julia set of G_λ tends to the unit disk no matter what the direction of approach to the origin. On the other hand, when $n, d \geq 2$ (but not both equal to 2), the Julia set does not converge to the unit disk. Indeed, in this case, it is known that the Julia set is a Cantor set of simple closed curves (when $|\lambda|$ is sufficiently small). Moreover, as $\lambda \to 0$, there is always a "round" annulus of some specific width that lies between some of these curves but does not lie in the Julia set.

2 Preliminaries

Let

$$F_\lambda(z) = z^n + \frac{\lambda}{z}$$

where $\lambda \in \mathbb{C}$ and $n \geq 2$. The map F_λ has critical points at

$$c_\lambda = \big(\lambda/n\big)^{1/(n+1)}.$$

We call these points the "free" critical points. The point at ∞ is also a critical point of order $n-1$. Hence we have an immediate basin of ∞ which we denote by B_λ. There is a neighborhood of 0 that is then mapped to B_λ. We call this set the trap door and denote it by T_λ. It is known [1] that B_λ and T_λ are disjoint if $|\lambda|$ is sufficiently small.

In this paper we shall be primarily interested in the structure of the *Julia set* of F_λ. The Julia set of F_λ, denoted by $J(F_\lambda)$, has several equivalent definitions [6]:

1. $J(F_\lambda)$ is the closure of the set of repelling periodic points;

2. $J(F_\lambda)$ is the boundary of the full basin of attraction of ∞;

3. $J(F_\lambda)$ is the chaotic regime in the sense that any neighborhood of a point in $J(F_\lambda)$ is eventually mapped over the entire Riemann sphere, excluding at most two points.

The complement of the Julia set is called the *Fatou set*. The dynamical behavior on the Fatou set is usually quite simple: most often, all orbits in the Fatou set simply tend to one of finitely many attracting cycles.

The behavior of the free critical orbit often determines the structure of of the Julia set. For example, the following Theorem was proved in [4].

Theorem (The Escape Trichotomy).

1. *If one and hence all of the critical values of F_λ lie in B_λ, then the Julia set of F_λ is a Cantor set;*

2. *If one and hence all of the critical values lie in T_λ, then the Julia set is a Cantor set of simple closed curves;*

3. *If the critical values all lie in preimages of T_λ under F_λ^j for some $j > 0$, then the Julia set is a Sierpinski curve.*

A *Sierpinski curve* is a planar set that is characterized by the following five properties: it is a compact, connected, locally connected and nowhere dense set with two or more complementary domains that are all bounded by simple closed curves that are pairwise disjoint. It is known from work of Whyburn [9] that any two Sierpinski curves are homeomorphic. In fact, they are homeomorphic to the well-known Sierpinski carpet fractal. From the point of view of topology, a Sierpinski curve is a universal set in the sense that it contains a homeomorphic copy of any planar, compact, connected, one-dimensional set. The first example of a Sierpinski curve Julia set was given by Milnor and Tan Lei [7]. In Figure 1, all of the Julia sets depicted are Sierpinski curves.

We remark that case 2 in the above theorem does not occur in the family $z^n + \lambda/z$; it does occur in the more general family of maps $z^n + \lambda/z^d$ where $n, d \geq 2$ (but not both equal to 2).

There is an $(n+1)$-fold symmetry in the dynamical plane. Let ω be an $(n+1)^{\text{st}}$ root of unity. Then we have

$$F_\lambda(\omega z) = \omega^n F_\lambda(z) = \omega^{-1} F_\lambda(z).$$

It follows that points that are symmetric under $z \mapsto \omega z$ have orbits that also behave symmetrically. Hence if one such orbit escapes to ∞, all of the $n+1$ symmetrically arranged points have this property. Similarly, if one such orbit tends to an attracting cycle, then all of the symmetrically arranged orbits also tend to such a cycle, though, as we shall see, the periods of these symmetric cycles may be different. Therefore $J(F_\lambda)$ is symmetric under rotation by ω. As a consequence, we really have only one free critical orbit since the orbits of c_λ all behave symmetrically. Thus we can paint the picture of the parameter plane for this family by following the orbit of any of the $n+1$ free critical points.

There are several symmetries in the parameter planes for these maps. First of all, the parameter plane is symmetric under complex conjugation since we have

$$F_\lambda(\overline{z}) = \overline{F_{\overline{\lambda}}(z)}.$$

Secondly, let α satisfy $\alpha^{n-1} = 1$. Then we have

$$F^2_{\alpha\lambda}(\alpha^{1/n+1} z) = \alpha^{1/n+1} F^2_\lambda(z).$$

So the second iterate of F_λ is conjugate to the second iterate of $F_{\alpha\lambda}$ and hence the parameter plane is symmetric under $\lambda \mapsto \alpha\lambda$. Incidentally, we use the second iterate in the above conjugacy because it is not true that the first iterates of these maps are conjugate. For example, consider the case where $n = 3$. One checks easily that, for example, when $\lambda = 3/16$, this map has a pair of superattracting fixed points at $\pm 1/2$ and a superattracting 2-cycle at $\pm i/2$. The dynamical behavior of the map $F_{\alpha\lambda}$ is a little different. This map has a pair of superattracting 2-cycles, one given by $i^{1/2}/2$ and $i^{3/2}/2$ and the other given by the negatives of these two points. Note that the second iterates of both of these maps have four superattracting fixed points, so the conjugacy works for the second iterates of the maps but not for the first.

For the first iterates of F_λ, we have

$$F_{\alpha^2\lambda}(\alpha z) = \alpha F_\lambda(z).$$

So $F_{\alpha^2\lambda} \approx F_\lambda$. Consequently, when n is even, $F_\lambda \approx F_{\alpha\lambda}$, but when n is odd, there are two distinct conjugacy classes of maps of the form $F_{\alpha^j\lambda}$.

We will be primarily concerned with parameters that lie along the $n-1$ *dividing rays* in the parameter plane. The dividing rays are the straight rays given by

$$\operatorname{Arg}\lambda = \frac{(2k+1)\pi}{n-1}$$

for $k = 0, 1, \ldots, n-1$. So, when $n = 2$, the only dividing ray is the negative real axis, and when $n = 3$ the dividing rays are the positive and negative imaginary axes. In Figures 2 and 3, we display the parameter planes in the cases $n = 2, 3, 4$ and 5. Note the large black regions surrounding the origin. These regions contain parameters for which F_λ has attracting fixed points and/or attracting cycles, and these regions are separated from one another by the dividing rays. More precisely, we have

Proposition. *The set of parameters for which F_λ has a neutral fixed point with derivative $e^{i\theta}$ is given by the curve*

$$\lambda(\theta) = \left(\frac{e^{i\theta}+1}{n+1}\right)^{2/n-1} \left(\frac{n-e^{i\theta}}{n+1}\right).$$

These curves bound the regions in parameter plane for which the corresponding maps have (at least one) attracting fixed point and so all of the critical orbits tend to an attracting cycle. Moreover, each of these regions approaches

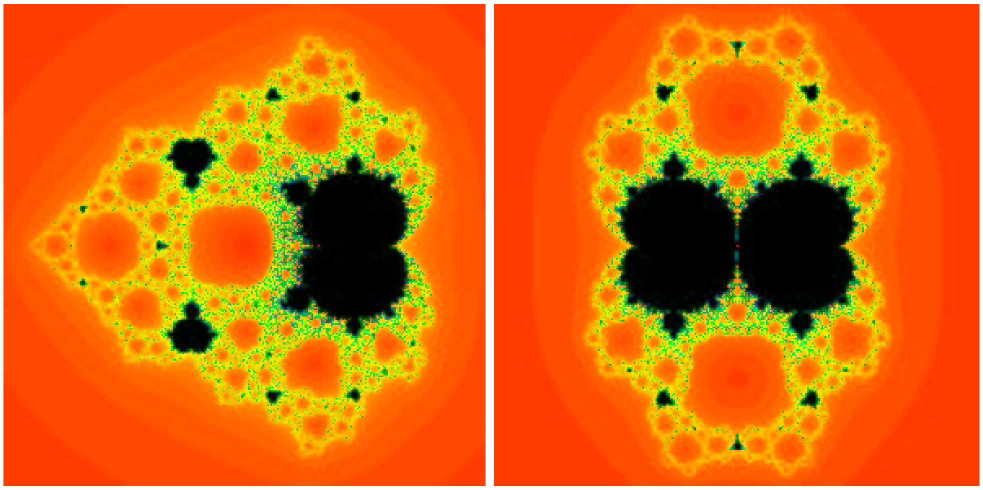

Figure 2: The parameter planes in the cases $n = 2$ and $n = 3$.

the origin tangentially to a pair of adjacent dividing rays in the parameter plane.

Proof: The neutral fixed points of F_λ are determined by the equations

$$\begin{aligned} z &= z^n + \frac{\lambda}{z} \\ e^{i\theta} &= nz^{n-1} - \frac{\lambda}{z^2} \end{aligned}$$

so that

$$\lambda = \lambda(\theta) = z^2 \left(\frac{n - e^{i\theta}}{n+1} \right).$$

Therefore

$$z = z(\theta) = \left(\frac{e^{i\theta} + 1}{n+1} \right)^{1/n-1}.$$

Inserting the formula for z into the equation for λ yields the formula for $\lambda(\theta)$.

As $\theta \to \pm\pi$, we have

$$\mathrm{Arg} \left(\frac{\mathrm{e}^{\mathrm{i}\theta} + 1}{\mathrm{n} + 1} \right) \to \pm\frac{\pi}{2}.$$

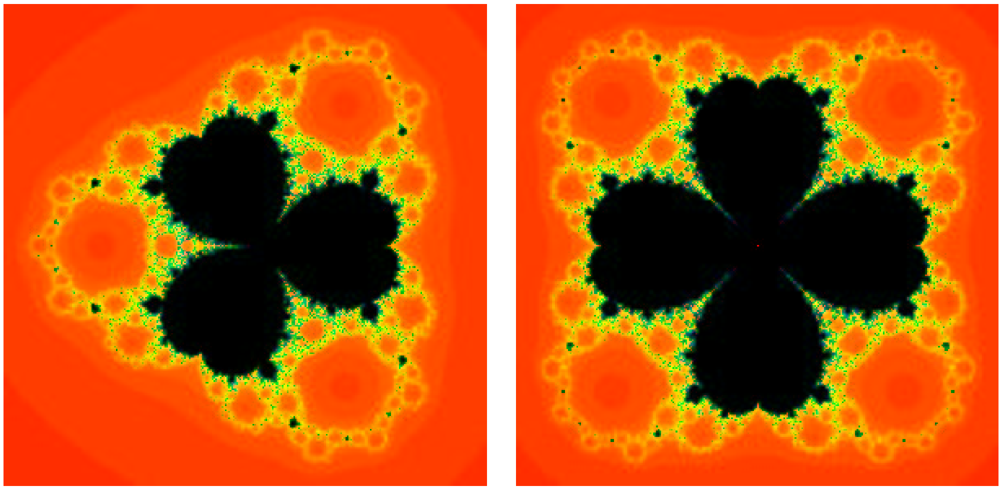

Figure 3: The parameter planes in the cases $n = 4$ and $n = 5$.

From the formula for $z(\theta)$, we then have that, as $\theta \to \pm\pi$,

$$\text{Arg}\, z(\theta) \to \pm\frac{\pi}{2(n-1)} + \frac{2k\pi}{n-1}$$

for $k = 0, 1, \ldots, n-1$. But, as $\theta \to \pm\pi$, $\text{Arg}\,\lambda(\theta) \to 2\text{Arg}\, z(\theta)$ so that

$$\text{Arg}\,\lambda \to \pm\frac{\pi}{n-1} + \frac{4k\pi}{n-1}.$$

But this set of rays is the same as the set of rays given by

$$\text{Arg}\,\lambda = \frac{\pi}{n-1} + \frac{2k\pi}{n-1},$$

i.e., the set of dividing rays.

□

3 The Main Theorem

Our goal in this section is to prove the following result:

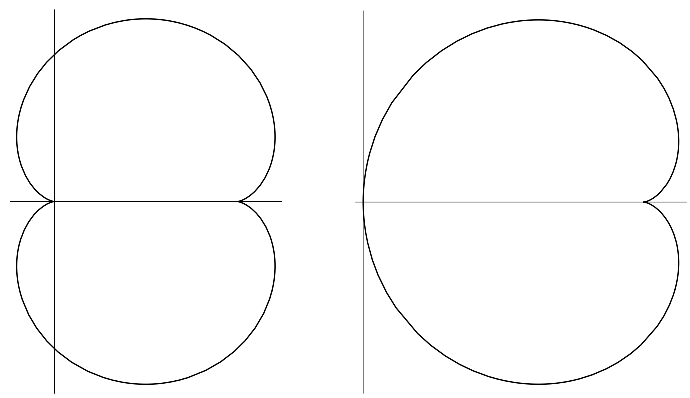

Figure 4: On the left is the double-cusped "cardioid" in the parameter plane for $n = 2$. The maps corresponding to parameters inside this region each have an attracting fixed point and an attracting 2-cycle. On the right is the cardioid in the parameter plane for $n = 3$ containing parameters with two attracting fixed points and an attracting 2-cycle.

Theorem. *$J(F_\lambda)$ converges as a set to the closed unit disk as λ tends to 0 along each of the dividing rays in the parameter plane.*

When we say that $J(F_\lambda)$ converges as a set to the unit disk, we mean convergence in the Hausdorff topology. To be precise, let $B_\epsilon(z)$ denote the ball of radius ϵ about z and let $\mathbb{D}$ denote the closed unit disk in $\mathbb{C}$. Then $J(F_\lambda)$ converges as a set to $\mathbb{D}$ if, given any $\epsilon > 0$, there exists λ^* such that, for all λ with $0 < |\lambda| < \lambda^*$, we have:

1. each z with $|z| > 1 + \epsilon$ lies in B_λ and so not in $J(F_\lambda)$;
2. for each $z \in \mathbb{D}$, $J(F_\lambda) \cap B_\epsilon(z) \neq \emptyset$.

Thus, for each such λ, any point in the Julia set of F_λ is within ϵ of $\mathbb{D}$, and any point in $\mathbb{D}$ is within ϵ of $J(F_\lambda)$.

Proof: Given $\epsilon > 0$, we first prove that, if $|\lambda| \leq \epsilon$ and z satisfies $|z| > 1 + \epsilon$, then $z \in B_\lambda$. (Note that we do not need λ to lie on the dividing rays for this part of the proof.)

We have

$$|z|^{n+1} - |z|^2 = |z|^2(|z|^{n-1} - 1) \geq |z|^2(n-1)\epsilon \geq |z|^2\epsilon > \epsilon \geq |\lambda|.$$

Therefore we have

$$|z|^n - \frac{|\lambda|}{|z|} > |z|.$$

Hence

$$|F_\lambda(z)| = \left| z^n + \frac{\lambda}{z} \right| \geq |z|^n - \frac{|\lambda|}{|z|} > |z|.$$

Therefore z lies in B_λ so $z \notin J(F_\lambda)$.

Now we turn to the second requirement for convergence as a set to $\mathbb{D}$. For simplicity, we shall first consider the case where $n = 2$, so

$$F_\lambda(z) = z^2 + \frac{\lambda}{z}.$$

At the end of the proof, we discuss the straightforward modifications needed for the case where $n > 2$.

When $n = 2$, the dividing ray in the parameter plane is the negative real axis, so we assume that $\lambda \in \mathbb{R}^-$. We now have two symmetries in the dynamical plane. First, let ω be the cube root of unity lying in the upper half plane, and recall that $F_\lambda(\omega z) = \omega^2 F_\lambda(z)$. So the Julia set is symmetric under $z \mapsto \omega z$. Second, since $\lambda \in \mathbb{R}^-$, we now have $F_\lambda(\overline{z}) = \overline{F_\lambda(z)}$. So the Julia set is also symmetric under complex conjugation.

Since λ is real, the real line is invariant under F_λ. By the $z \mapsto \omega z$ symmetry, it follows that F_λ interchanges the two straight lines $\omega\mathbb{R}$ and $\omega^2\mathbb{R}$. We call the three lines $\mathbb{R}, \omega\mathbb{R}$, and $\omega^2\mathbb{R}$ the symmetry lines in the dynamical plane.

Now assume that $J(F_\lambda)$ does not converge to $\mathbb{D}$ as $\lambda \to 0$ along $\mathbb{R}^-$. Therefore, given any $\epsilon > 0$, we may find a sequence of parameters $\lambda_j \in \mathbb{R}^-$

with $\lambda_j \to 0$ and a sequence of points $z_j \in \mathbb{D}$ such that $J(F_{\lambda_j}) \cap B_{2\epsilon}(z_j) = \emptyset$ for each j. Since $\mathbb{D}$ is compact, there is then a subsequence of the points z_j that converges to some point $z^* \in \mathbb{D}$. For each parameter λ_j in the corresponding subsequence with j sufficiently large, we then have $J(F_{\lambda_j}) \cap B_\epsilon(z^*) = \emptyset$. Hence we may assume at the outset that we are dealing with a subsequence $\lambda_j \to 0$ such that $J(F_{\lambda_j}) \cap B_\epsilon(z^*) = \emptyset$, provided λ_j is close enough to 0.

We first claim that $z^* \neq 0$. To see this, suppose $|\lambda| < 1/8$. Then, if z lies on the circle of radius $|\lambda|^{1/3}$ centered at the origin, we have

$$|F_\lambda(z)| \leq 2|\lambda|^{2/3} < |\lambda|^{1/3} = |z|.$$

So this circle is mapped strictly inside itself. It follows that the boundary of T_λ lies within this circle for each such λ and hence ∂T_λ tends to 0 as $\lambda \to 0$. So, for $|\lambda|$ small, there are points in the Julia set arbitrarily close to 0 and hence inside $B_\epsilon(0)$. Consequently $z^* \neq 0$.

Now consider the circle of radius $|z^*|$ centered at the origin. This circle meets $B_\epsilon(z^*)$ in an arc γ of whose argument has total length ℓ. Choose k so that $2^k\ell > 2\pi$. Since $\lambda_j \to 0$, we may choose j large enough so that $|F^i_{\lambda_j}(z) - z^{2^i}|$ is very small for $1 \leq i \leq k$, provided z lies outside the circle of radius $|z^*|/2$ centered at the origin. In particular, it follows that $F^k_{\lambda_j}(\gamma)$ is a curve whose argument increases by approximately 2π, i.e., the curve $F^k_{\lambda_j}(\gamma)$ wraps at least once around the origin.

As a consequence, the curve γ meets all three of the symmetry lines in the dynamical plane. By the $z \mapsto \overline{z}$ symmetry, it follows that $F^k_{\lambda_j}(B_\epsilon(z^*))$ contains an annulus that lies in the Fatou set and surrounds the origin. Let U_j be the component of the Fatou set that contains this curve. By the No Wandering Domains Theorem [8], the set U_j must eventually map onto a component of the Fatou set that is periodic. Call this component V_j. But the points in U_j that lie on the three symmetry lines remain on these lines for all iterations. Hence these points cannot converge to a finite attracting or neutral cycle, so V_j cannot be a basin of attraction of a finite cycle. Similarly, V_j cannot be a Siegel disk or a Herman ring. It follows that V_j must be B_{λ_j}. Therefore the preimage of V_j is the trap door and the second preimage of B_{λ_j} is a Fatou component that contains an annulus that surrounds the origin. Call this component W_j.

Since T_{λ_j} is a disk and W_j is not simply connected, it follows that W_j must contain at least one critical point of F_{λ_j}. By symmetry, W_j must therefore contain all of the free critical points. But then, by the Riemann-Hurwitz formula, W_j must in fact be an annulus that is mapped $n+1$ to one onto T_{λ_j}.

We claim that this cannot happen. Let A_j denote the open annulus that lies between the trap door and the immediate basin of ∞. The annulus W_j separates A_j into two annuli, an outer annulus A_j^{out}, and an inner annulus A_j^{in}. F_{λ_j} maps A_j^{out} n to 1 onto A_j, while F_{λ_j} maps A_j^{in} one-to-one onto A_j. Hence we have

$$\operatorname{mod} A_j = \operatorname{mod} A_j^{\text{in}}.$$

Since the inner boundaries of A_j and A_j^{in} are the same, this leaves no room for A_j^{out} and hence we have a contradiction. Thus we cannot have such a disk $B_\epsilon(z^*)$ in the Fatou set when $n = 2$.

For the case $n > 2$, we need to produce similar symmetry lines. To do this, first assume that n is even. In this case, the negative real axis is one of the dividing lines. By the symmetry in the parameter plane, we therefore only need to deal with this case. So we assume that $\lambda \in \mathbb{R}^-$. We claim that the straight lines through the origin with arguments $j\pi/(n+1)$ and $\pi - (j\pi/(n+1))$ are mapped to themselves by the second iterate of F_λ. These give the symmetry lines in this case.

To see this, suppose that

$$\operatorname{Arg} z = \frac{j\pi}{n+1}.$$

Then

$$\operatorname{Arg} z^n = \frac{nj\pi}{n+1}$$

and

$$\operatorname{Arg} \frac{\lambda}{z} = -\frac{j\pi}{n+1} + \pi = \frac{(n+1-j)\pi}{n+1}.$$

Therefore

$$\operatorname{Arg} z^n - \operatorname{Arg} \frac{\lambda}{z} = \frac{(n+1)j - (n+1)}{n+1}\pi = (j-1)\pi.$$

So, depending on j, either z^n and λ/z lie on the same ray or on opposite rays, but at least these points lie on the same straight line passing through the origin and containing the ray whose argument is

$$\frac{(n+1-j)\pi}{n+1} = \pi - \frac{j}{n+1}\pi.$$

A similar argument then shows that points that lie on the line through the origin containing the ray with argument

$$\pi - \frac{j}{n+1}\pi$$

is mapped to the original line, i.e., the line passing through the origin that contains the ray with argument $j\pi/(n+1)$. So we have a similar situation to the case where $n = 2$ and the above proof then carries over to this case.

When n is odd, the situation is a little different. The negative real axis is no longer a dividing line. However, the ray with argument given by $\pi/(n-1)$ is one of these lines. So similar arguments as the above, using parameters on this ray, show that the symmetry lines are now given by

$$\operatorname{Arg} z = \frac{\pi}{n^2-1} + \frac{j(n-1)\pi}{n^2-1}.$$

In this case there is no fixed symmetry line; rather a pair of these lines are always interchanged by F_λ. We leave these details to the reader. $\square$

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