A MYRIAD OF SIERPINSKI CURVE JULIA SETS

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This paper is a survey of the numerous Sierpinski curve Julia sets that arise in the family of rational maps given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d} \text{ where } n \geq 2, d \geq 1.$$

While these Julia sets are all the same from a topological point of view (they are all homeomorphic), the dynamics on these sets are almost always very different in the sense that no two maps are topologically conjugate.

1. Introduction

The Sierpinski carpet fractal shown in Figure 1 is one of the most important planar, compact, connected sets for several reasons. First of all, it is a *universal plane continuum* in the sense that it contains a homeomorphic copy of any planar, one-dimensional, compact, connected set. (Here we mean one topological dimension, not Hausdorff or fractal or any other dimension.) For example, the complicated curve shown in Figure 2 may be homeomorphically deformed so that it sits inside the Sierpinski carpet fractal. Secondly, there is a topological characterization of this set: any planar set that is compact, connected, nowhere dense, locally connected, and has the property that any pair of complementary domains are bounded by disjoint simple closed curves is homeomorphic to the Sierpinski carpet[18]. Any set that is homeomorphic to the Sierpinski carpet is called a *Sierpinski curve*.

In recent years, we have shown that Sierpinski curves arise as the Julia sets of certain complex rational functions in a variety of different ways. In

*Partially supported by NSF grant 0205779

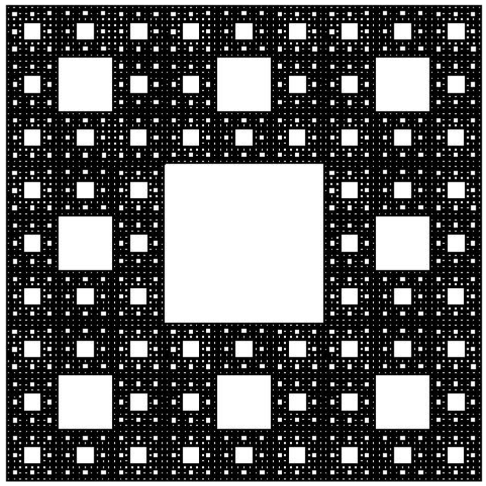

Figure 1. The Sierpinski carpet.

this paper we shall describe several of these possibilities. What is interesting here is that, while these Julia sets are always the same from a topological point of view, the dynamics on these sets are often quite different.

For simplicity, we shall restrict attention to functions of the form

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d} \text{ where } n \geq 2, d \geq 1$$

although many other families of rational maps exhibit similar types of Julia sets. We assume here that $n \geq 2$ so that the point at ∞ is a superattracting fixed point. Hence there is an immediate basin of attraction of ∞ which we denote by B_λ.

As we show below, even though the maps in this family have high degree, there really is (up to symmetry) only one "free" critical orbit. As is well known in complex dynamics, the behavior of this critical orbit plays a principal role in determining both the topology of and the dynamics on the Julia sets of these maps. In particular, if the critical orbit eventually enters B_λ, we have the following result[7].

Theorem (The Escape Trichotomy). *If the free critical orbit remains bounded, then the Julia set of F_λ is a connected set. However,*

(1) If the critical value lies in B_λ, then the Julia set of F_λ is a Cantor set;

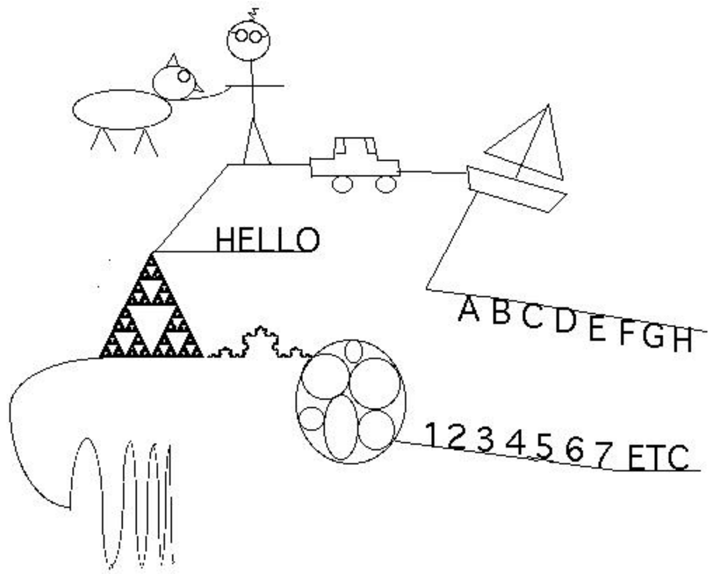

Figure 2. This curve (containing the Sierpinski gasket, the Koch curve, the topologists' sine curve, and many other things) may be homeomorphically realigned to fit into the Sierpinski carpet.

(2) *If the free critical orbit enters B_λ at the second iteration, then the Julia set is a Cantor set of simple closed curves;*
(3) *If the free critical orbit enters B_λ at the third or higher iteration, then the Julia set is a Sierpinski curve.*

Since there is only one free critical orbit, the λ-plane is therefore the natural parameter plane for these families. In Figure 3, we have plotted these planes in the cases $n = d = 2$ and $n = d = 3$. The black points in this picture correspond to parameter values for which the free critical orbit does not escape to ∞. The white regions in this picture represent λ-values for which the critical orbit tends to ∞. The exterior region corresponds to parameter values for which the Julia set is a Cantor set; we call this set the *Cantor set locus.* The small white region in the center of the picture corresponds to parameter values for which the Julia set is a Cantor set of simple closed curves. We call this region the *McMullen domain.* We remark that it is known[11] that McMullen domains exist if and only if $1/n+1/d < 1$. All other white regions in this picture correspond to parameters for which the free critical orbit escapes after three or more iterations so the Julia set is a Sierpinski curve. These regions are called *Sierpinski holes*. It is known[4] and[16] that there are infinitely many such regions in each parameter plane. Hence the Julia set of F_λ is a connected set for all λ-values except those in the Cantor set locus and the McMullen domain. So we call this set of parameters the *connectedness locus*.

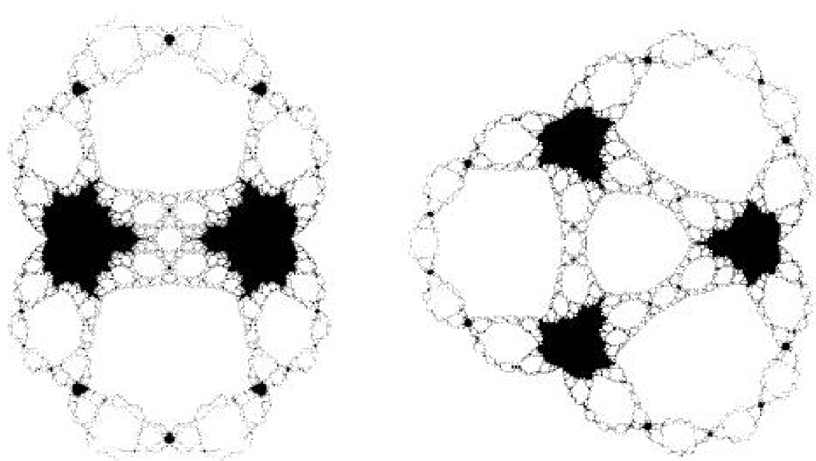

Figure 3. The parameter planes for the cases $n = 3$ and $n = 4$.

It turns out that there are many other ways that the Julia set of F_λ can be a Sierpinski curve besides case 3 in the Escape Trichotomy. The goal of this paper is to give a sketch of the proof of the following results:

Theorem. *Let v_λ denote the image of one of the free critical points of F_λ.*

(1) If $F_\lambda^k(v_\lambda)$ lands on a repelling periodic point of F_λ that does not lie in the boundary of B_λ, then the Julia set is a Sierpinski curve;
(2) If λ lies in the main cardioid of certain "buried" Mandelbrot sets with base period k in the parameter plane, then again the Julia set of F_λ is a Sierpinski curve;
(3) If, as in the Escape Trichotomy, $F_\lambda^k(v_\lambda)$ with $k > 1$ is the first point in the orbit of v_λ that lies in B_λ, then the Julia set of F_λ is a Sierpinski curve.

The dynamical distinctions between these cases are given by:

Corollary. *If λ_1 and λ_2 are any parameter values arising from different cases of this result, then the dynamics on the corresponding Julia sets are not topologically conjugate to one another. Even if these parameter values arise from the same case, but the corresponding k-values are different, then again the dynamics of the two maps are not topologically conjugate.*

This result follows immediately from the fact that, in each case, the behavior of the critical orbits is very different, but a conjugacy between two such maps must preserve this behavior.

2. Preliminaries

In the dynamical plane, the object of principal interest is the *Julia set* of F_λ which we denote by $J(F_\lambda)$. The Julia set is the set of points at which the family of iterates of F_λ, $\{F_\lambda^n\}$, fails to be a normal family in the sense of Montel. It is known that $J(F_\lambda)$ is also the closure of the set of repelling periodic points for F_λ as well as the boundary of the set of points whose orbits escape to ∞ under iteration of F_λ. As a consequence, $J(F_\lambda)$ is the regime where F_λ behaves chaotically. The complement of the Julia set is known as the *Fatou set*. Here the dynamical behavior is quite tame.

Note that B_λ and all of its preimages must lie in the Fatou set. Since the point at ∞ is a superattracting fixed point for F_λ, it is well known[13] that F_λ is conjugate to $z \mapsto z^n$ in a neighborhood of ∞ in B_λ. There is also a pole of order d for F_λ at the origin, so there is a neighborhood of 0 that is mapped into B_λ by F_λ. If this neighborhood is disjoint from B_λ, then we denote the preimage of B_λ that contains 0 by T_λ. So the only preimages of B_λ are B_λ and T_λ. We call T_λ the *trap door* since any orbit that eventually enters the immediate basin of ∞ must "fall through" T_λ enroute to B_λ.

One computes easily that there are $n+d$ critical points for F_λ and that all of them assume the form $\omega^k c_\lambda$ where c_λ is one of the critical points and $\omega = \exp(2\pi i/(n+d))$. We call these points the free critical points. Similarly, the critical values v_λ are arranged symmetrically with respect to $z \mapsto \omega z$, though there need not be $n+d$ of them. There are $n+d$ prepoles at the points $(-\lambda)^{1/(n+d)}$.

The proof of the following Proposition is straightforward.

Proposition (Dynamical Symmetry). *Suppose $\omega = \exp(2\pi i/(n+d))$. Then $F_\lambda(\omega z) = \omega^n F_\lambda(z)$.*

As a consequence of this result, the orbits of points of the form $\omega^j z$ all behave "symmetrically" under iteration of F_λ. For example, if $F_\lambda^i(z) \to \infty$, then $F_\lambda^i(\omega^k z)$ also tends to ∞ for each k. If $F_\lambda^i(z)$ tends to an attracting cycle, then so does $F_\lambda^i(\omega^k z)$. We remark, however, that the cycles involved may be different depending on k and, indeed, they may even have different periods. Nonetheless, all points lying on these attracting cycles are of the form $\omega^j z_0$ for some $z_0 \in \mathbb{C}$. For example, when $n = 2, d = 1$, there are parameters for which some of the critical points tend to an attracting fixed point z_0 on the real line, whereas ωz_0 and $\omega^2 z_0$ lie on an attracting 2-cycle which attracts other critical points.

3. Escape Sierpinski Curve Julia Sets

In this section we assume that the third or higher iterate of the critical point is the first that enters B_λ. We then give a complete proof that, in this case, the Julia set is a Sierpinski curve. In later sections we merely sketch the somewhat similar proofs that the Julia sets in other cases are Sierpinski curves.

In Figure 4, we display four Julia sets drawn from the family where $n = d = 2$. All of these sets are Sierpinski curves, and all have different dynamics since the number of iterates it takes for the critical orbit to reach B_λ is different in each case.

We first observe that, unlike most connected Julia sets of polynomials, for λ-values in the connectedness locus, the complement of the closure of B_λ consists of a single open component. We do not have Julia sets in this case like the well-known Douady rabbit or basilica Julia sets from quadratic dynamics.

Proposition. *Suppose that the free critical orbit tends to ∞, but the critical values do not lie in $B_\lambda \cup T_\lambda$. Then the set $\mathbb{C} - \overline{B_\lambda}$ has a single open, connected component.*

Proof: Suppose first that $\mathbb{C} - \overline{B_\lambda}$ has more than one connected component. Let W_0 be the component of $\mathbb{C} - \overline{B_\lambda}$ that contains the origin. Note that all of T_λ must lie in W_0. We claim that at least one of the prepoles also lies in W_0. Suppose this is not the case. By symmetry, all of the prepoles either lie in the same component of the Fatou set or else they all lie in distinct components. In the latter case, this means that each Fatou component containing a prepole is mapped one-to-one onto W_0. Therefore there must be $n + d$ of these components. Now there are no critical points in the Julia set by assumption, so every point in the boundary of W_0 has $n + d$ preimages, one in each of the boundaries of these components. But there are also d preimages of any such point in the boundary of the trap door

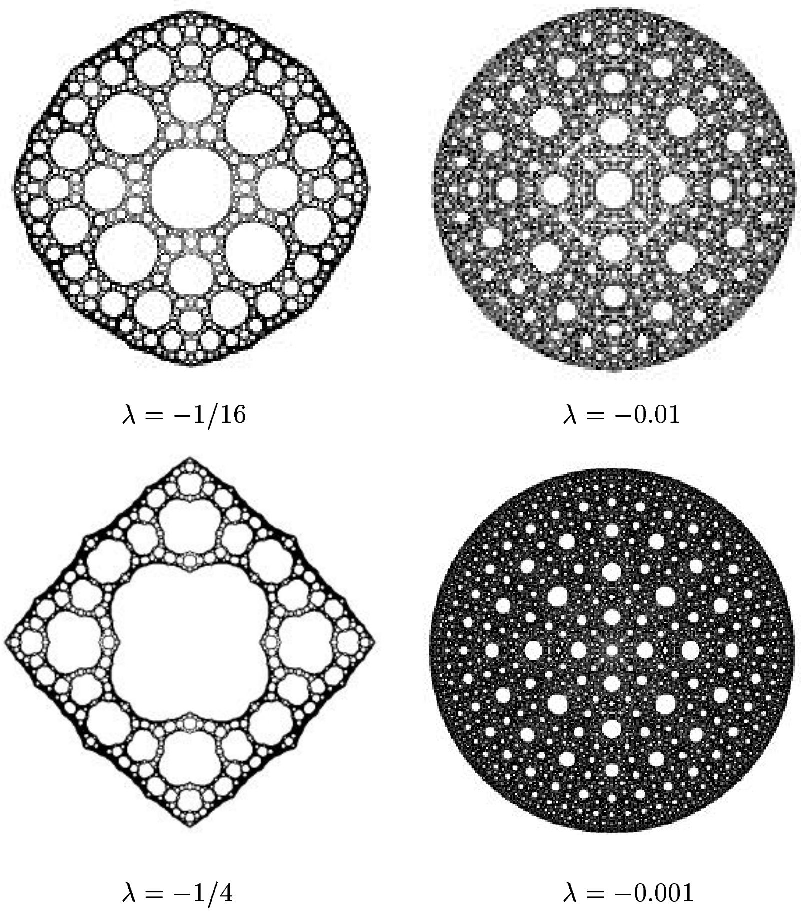

Figure 4. The Julia sets for various values of λ when $n = d = 2$.

which is contained inside $\overline{W_0}$ and mapped to the boundary of B_λ. Since the boundary of T_λ cannot equal the boundary of W_0, this yields too many preimages for any point in the boundary of W_0. Therefore all of the prepoles lie in the same component of the Fatou set, and this component must surround the origin and separate B_λ from $\overline{W_0}$. This, however, is impossible, since the boundary of W_0 is contained in the boundary of B_λ. Hence one and therefore all $n + d$ of the prepoles lie in W_0 and so F_λ is $n + d$ to 1 on W_0. Therefore all of the preimages of points in W_0 must also lie in W_0.

Now suppose that there is a second component W_1 in $\mathbb{C} - \overline{B_\lambda}$. There are no points in W_1 that map into W_0. Consider a point on the boundary of W_1

that does not also lie on the boundary of W_0 and choose a neighborhood of this point that does not meet W_0. By Montel's Theorem, the forward images of this neighborhood map over points in W_0. But this cannot happen, since all preimages of points in W_0 lie in W_0. This proves that W_1 does not exist.

□

Proposition. *The Julia set of F_λ is compact, connected, locally connected, and nowhere dense.*

Proof: Since we are assuming that all of the critical orbits eventually enter the basin of ∞, the Fatou set consists of the union of B_λ and all of its preimages. Hence we have that the Julia set is given by $\mathbb{C} - \cup F_\lambda^{-j}(B_\lambda)$. That is, $J(F_\lambda)$ is $\mathbb{C}$ with countably many disjoint, open, simply connected sets removed. Hence $J(F_\lambda)$ is compact and connected. Since $J(F_\lambda) \neq \mathbb{C}$, $J(F_\lambda)$ cannot contain any open sets, so $J(F_\lambda)$ is also nowhere dense[13]. Finally, since the critical orbits all tend to ∞ and hence do not lie in or accumulate on $J(F_\lambda)$, standard arguments show that $J(F_\lambda)$ is locally connected[13]. □

Thus we have shown that $J(F_\lambda)$ possesses four of the five defining properties of a Sierpinski curve. It suffices to show that the boundaries of the complementary domains are bounded by simple closed curves that are disjoint. This is a little more difficult.

Proposition. *The boundary of B_λ, ∂B_λ, as well as all of the preimages of B_λ are simple closed curves. These boundary curves are pairwise disjoint.*

Proof: By the previous Proposition, $J(F_\lambda)$ is locally connected, so it follows that ∂B_λ is also locally connected. Now recall that, near ∞, F_λ is analytically conjugate to $z \mapsto z^n$. That is, there exists an analytic homeomorphism $\phi_\lambda : B_\lambda \to \overline{\mathbb{C}} - \overline{\mathbb{D}}$ where $\mathbb{D}$ is the open unit disk in the plane. The map ϕ_λ satisfies

$$\phi_\lambda \circ F_\lambda(z) = (\phi_\lambda(z))^n.$$

The preimage under ϕ_λ of the straight ray with argument θ in $\overline{\mathbb{C}} - \overline{\mathbb{D}}$ is called the external ray of angle θ and denoted by $\gamma(\theta)$. Since the boundary of B_λ is locally connected, it is known[15] that all of the external rays land at a point in the boundary of B_λ. Thus, to show that this boundary is a simple closed curve, it suffices to prove that no two external rays land at the same point.

To see this, first recall that W_0 denotes the component of $\mathbb{C} - \overline{B_\lambda}$ that contains the origin, and that W_0 is both connected and simply connected. Suppose that there exists $p \in \partial B_\lambda$ such that $\gamma(t_1)$ and $\gamma(t_2)$ both land on

p. Since these rays together with the point p form a Jordan curve, we have that W_0 lies entirely within one of the two open components created by this Jordan curve. Let $\gamma(t_1, t_2)$ denote the union of all of the external rays whose angles lie between t_1 and t_2 (where we assume that the angle between these two rays is smaller than π). Without loss of generality, assume that W_0 is such that $W_0 \cap \gamma(t_1, t_2) = \emptyset$ (so W_0 is "outside" the sector $\gamma(t_1, t_2)$ between $\gamma(t_1)$ and $\gamma(t_2)$).

We claim that there exist positive integers q and m such that the region

$$\gamma\left(\frac{q}{m}, \frac{q+1}{m}\right) \subset \gamma(t_1, t_2)$$

and neither of the external rays q/m nor $(q+1)/m$ land on ∂W_0. If this were not possible, then all rays in $\gamma(t_1, t_2)$ would land at p. This gives a contradiction because the set of angles $\theta \in \mathbb{R}/\mathbb{Z}$ such that the landing point of the ray with angle θ is p has measure 0[13].

So suppose we have such q and m. As above, let $\gamma(q/m, (q+1)/m)$ denote the union of the external rays contained between q/m and $(q+1)/m$. After m iterations $\gamma(q/m, (q+1)/m)$ is mapped over all of B_λ. In particular, if the external ray of angle θ lands on $\partial(\overline{\mathbb{C}} - \overline{B_\lambda})$, then there is a ray of angle $\phi \in \gamma(q/m, (q+1)/m)$ whose image under F_λ^m is $\gamma(\theta)$. Since $\phi \in \gamma(q/m, (q+1)/m)$ we have that $\gamma(\phi)$ does not land on ∂W_0. Hence there exists a neighborhood N_ϕ of $\gamma(\phi)$ such that $N_\phi \cap W_0$ is empty. However, since $F_\lambda^m(\gamma(\phi))$ lands on the boundary of W_0 we know that $F_\lambda^m(N_\phi) \cap W_0$ is not empty. This is a contradiction since points not in W_0 never enter W_0. Hence, we can never have two rays landing at the same point on ∂B_λ, implying that ∂B_λ is a simple closed curve.

It follows that all of the preimages of B_λ are also bounded by simple closed curves. We claim that no two of these curves can intersect. To see this, suppose first that there exists a point $z_0 \in \partial B_\lambda \cap \partial T_\lambda$. Then there exists an external ray γ in B_λ landing at z_0. In T_λ, there also exists a preimage, η, of an external ray that connects 0 to z_0. But the images of η and γ are the same external ray, and so it follows that z_0 is a critical point. But this contradicts our assumption that all critical orbits tend to ∞. So ∂B_λ and ∂T_λ are disjoint simple closed curves.

Now suppose that two earlier preimages of ∂B_λ intersect, say one preimage in $F_\lambda^{-n}(\partial B_\lambda)$ and one in $F_\lambda^{-m}(\partial B_\lambda)$. If $n \neq m$, then by mapping these preimages forward, we see that ∂B_λ and ∂T_λ also meet, which cannot happen. If $n = m$, then this intersection point must again be a critical point, so this cannot occur either.

□

This shows that the Julia set is a Sierpinski curve when it takes three or more iterations for the critical orbit to enter B_λ. To see that there are infinitely many such sets that are dynamically distinct, note that the number of iterations that it takes for the critical orbit to enter B_λ is a dynamical invariant: a conjugacy between any two such sets must map the invariant boundaries of the basin of ∞ to each other. Hence the j^{th} preimages of the basins must be mapped to each other by the conjugacy. But the only preimages of the basin on which the maps are not one-to-one are those that contain the critical points. Hence, in order for these maps to be conjugate, the critical orbits must all take the same number of iterations to enter the basin.

4. Buried Sierpinski Curves

In this section, we discuss an infinite collection of dynamically distinct Sierpinski curve Julia sets for the family F_λ where the Fatou components are quite different than those described in previous sections. Instead of being just a single superattracting basin at ∞ and its preimages, the Fatou set in these examples consists of a collection of finite attracting basins together with the basin at ∞ as well as all of their preimages. As before, the dynamics on these Julia sets are all distinct from one another as well as from those mentioned above, but again, all of these Julia sets are homeomorphic.

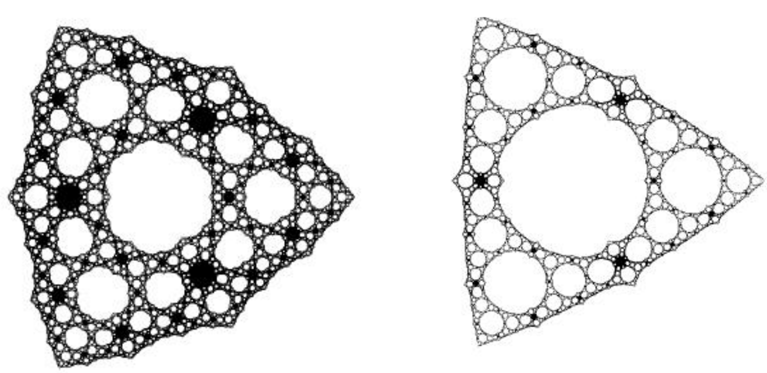

Figure 5. The Julia sets for $F_\lambda(z) = z^2 + \lambda/z$ where $\lambda = -0.327$ and $\lambda = -0.5066$.

For simplicity, we shall restrict attention in this section to the special family $F_\lambda(z) = z^2 + \lambda/z$ with $\lambda \in \mathbb{R}^-$. The examples we give arise in all of the other families, though their construction is a little more complicated in the general case.

In Figure 5, we display the Julia set of F_λ when $\lambda = -0.327$. For this map, there are superattracting basins of period 3 and period 6 together with the basin at ∞. We also display the case where $\lambda = -0.5066$ for which there are three different superattracting basins of period 4 together with the basin at ∞. The basins of the finite cycles in these pictures are displayed in black.

For λ real and negative, the graph of $F_\lambda(x)$ shows that there is a positive real fixed point for F_λ which we denote by $p(\lambda)$. Also, $c(\lambda) = (\lambda/2)^{1/3}$ is a critical point on the real line and

$$v(\lambda) = \frac{3}{2^{2/3}}\lambda^{2/3}$$

is the corresponding critical value.

Let $\lambda^* = -16/27$. Straightforward calculations show that $p(\lambda^*) = 4/3$ and $p(\lambda^*)$ is repelling. Furthermore, the real critical point $c(\lambda^*) = -2/3$ is pre-fixed, i.e., $F_{\lambda^*}(c(\lambda^*)) = 4/3 = p(\lambda^*)$. For λ-values slightly less negative than λ^*, the real critical value lies to the left of $p(\lambda)$ and hence subsequent points on the orbit of the critical value begin to decrease. Graphical iteration shows that there is a sequence of λ-values tending to λ^* for which the critical orbit decreases along the positive axis and then, at the next iteration, lands back at $c(\lambda)$. See Figure 6. Thus, for these λ-values, we have a superattracting cycle. Straightforward analysis[6] of the real dynamics of these functions shows:

Theorem. *There is a decreasing sequence $\lambda_n \in \mathbb{R}^-$ for $n \geq 3$ with $\lambda_n \to \lambda^* = -16/27$ and having the property that F_{λ_n} has a superattracting cycle of period n given by $x_j(\lambda_n) = F_{\lambda_n}(x_{j-1}(\lambda_n))$, where*

(1) $x_0(\lambda_n) = x_n(\lambda_n) = c(\lambda_n)$, and
(2) $x_0 < 0 < x_{n-1} < x_{n-2} < \cdots < x_1 = v(\lambda_n) < p(\lambda_n)$.

Now fix a particular parameter value $\lambda = \lambda_n$ for which F_λ has a superattracting periodic point x_0 lying in $\mathbb{R}^-$ as described in the previous theorem. We say that a basin of attraction of F_λ is *buried* if the boundary of this basin is disjoint from the boundaries of all other basins of attraction (including B_λ). Note that, if the basin of one point on an attracting cycle is buried, then so too are all forward and backward images of this basin, so

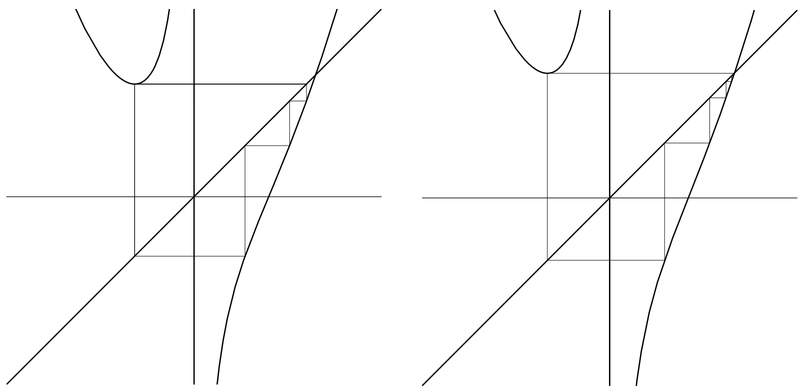

Figure 6. The graphs of $F_\lambda(x) = x^2 + \lambda/x$ where $\lambda = \lambda_4$ and $\lambda = \lambda_7$.

the entire basin of the cycle is buried. As in the previous section, standard arguments show that these Julia sets are compact, connected, locally connected, and nowhere dense. We need only show that the all of the basins of attraction are bounded by disjoint simple closed curves. The case of the basin at ∞ follows as in the previous section. For the finite basins, a different argument is necessary. We refer to[6] for the details. We have:

Theorem. *For the sequence of parameter values $\lambda_n \in \mathbb{R}^-$, all of the basins of F_{λ_n} are buried and so $J(F_\lambda)$ is a Sierpinski curve.*

As discussed earlier, any two Sierpinski curves are homeomorphic. Hence $J(F_{\lambda_n})$ is topologically equivalent to $J(F_{\lambda_m})$ for any n and m. However, each of these Julia sets is dynamically distinct from the others since the periods of the superattracting cycles are different.

5. Structurally Unstable Sierpinski Curves

In this section we turn our attention to the case where the critical orbit eventually lands on a repelling periodic point of F_λ that does not lie in ∂B_λ. Here again the Julia set is a Sierpinski curve and the dynamical behavior on this set is very different from the previous cases.

For simplicity, we shall restrict attention in this section to the case $n = d = 2$, i.e., the family

$$F_\lambda(z) = z^2 + \frac{\lambda}{z}$$

and specifically to the case when λ is negative. In Figure 7 we display the parameter plane for this family. By the results of McMullen[11], there is

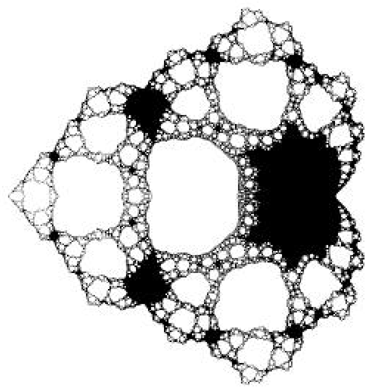

Figure 7. The parameter plane for the family $z^2 + \lambda/z^2$.

no McMullen domain for this family. Indeed, the large white region near the center of this picture is a Sierpinski hole. If we examine the parameter plane along the negative real axis, it appears that there are infinitely many Sierpinski holes. In Figure 8, we display several magnifications of the parameter plane along the negative real axis. In each case we see a large Sierpinski hole flanked by a pair of smaller Serpinski holes which, in turn, are each flanked by a pair of even smaller Sierpinski holes. It appears that the parameter plane along the negative real axis consists of a Cantor set where the removed intervals are the intersections of $\mathbb{R}^-$ with Sierpinski holes. Indeed, this is the case. In fact, along the negative real axis we actually have a *Cantor necklace*.

To define a Cantor necklace, we let Γ denote the Cantor middle thirds set in the unit interval $[0, 1]$. We regard this interval as a subset of the real axis in the plane. For each open interval of length $1/3^n$ removed from the unit interval in the construction of Γ, we replace this interval by an open disk of diameter $1/3^n$ centered at the midpoint of the removed interval. Thus the boundary of this open disk meets the Cantor set at the two endpoints of the removed interval. We call the resulting set the *Cantor middle-thirds necklace*. See Figure 9. Any set homeomorphic to the Cantor middle-thirds

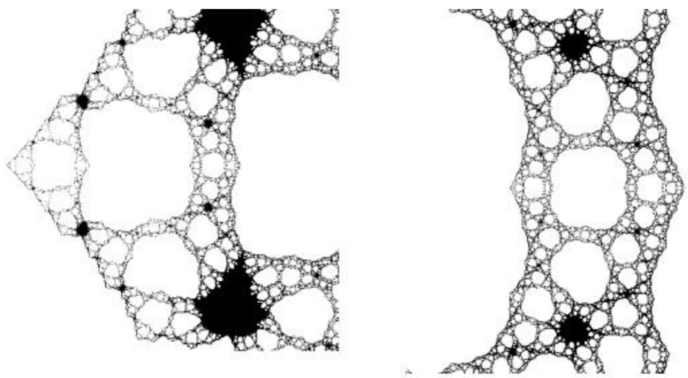

Figure 8. Two magnifications of the parameter plane for the family $z^2 + \lambda/z^2$ along the negative real axis. In the first image, $-0.4 \leq \text{Re}\,\lambda \leq -0.06$ and, in the second, $-0.2 \leq \text{Re}\,\lambda \leq -0.15$

necklace is then called a *Cantor necklace.* We do not include the boundary of the open disks in the Cantor necklace for the following technical reason: it is sometimes difficult in practice to verify that these bounding curves in the parameter plane are simple closed curves.

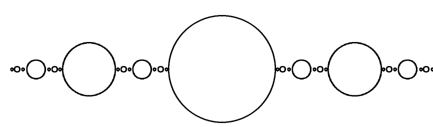

Figure 9. The Cantor middle-thirds necklace.

Cantor necklaces also appear in the dynamical plane. To construct such sets when $\lambda \in \mathbb{R}^-$, we first observe that the intersection of $J(F_\lambda)$ and the real line is an invariant Cantor set that we call Γ_λ. To see this, note that the graph of the real function F_λ shows that F_λ maps the interval $[-p_\lambda, p_\lambda]$ in two-to-one fashion over itself, where p_λ is the fixed point for F_λ on the positive real axis and on the boundary of B_λ. See Figure 10. The fact that such a Cantor set exists follows easily in the case where $|F_\lambda'(x)| > 1$ for all $x \in [-p_\lambda, p_\lambda]$. Unfortunately, this is not always the case, since, as $\lambda \to 0$, the graph of F_λ approaches the graph of x^2. Nonetheless, techniques from complex dynamics involving the Poincaré metric and similar to those used in the case of $z^2 + c$ give this result[1]. Well known facts from dynamics also show that this Cantor set varies continuously as λ varies and that the set of repelling periodic points in Γ_λ is dense in Γ_λ. In standard fashion we may also associate an itinerary $s(\lambda) = (s_0 s_1 s_2 \ldots)$ consisting of 0's and 1's to each point in Γ_λ. Finally, the open intervals in the complement of Γ_λ in $[-p_\lambda, p_\lambda]$ contain points that eventually map into B_λ. These intervals lie in simply connected open sets that are preimages of B_λ, and so the union of these disks together with Γ_λ produces the Cantor necklace.

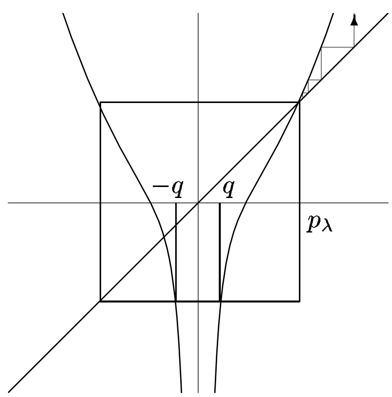

Figure 10. The graph of F_λ on the real line for $\lambda < 0$. The points $\pm q$ bound the trap door on the real axis.

Now consider the second iterate of the critical points. All four of these critical points lie off the real line, but a straightforward calculation shows that the second images of the critical points are all the same and are given by $4\lambda + 1/4$. Let $G(\lambda) = 4\lambda + 1/4$. So $G(\lambda) \in \mathbb{R}$ when λ is negative and $G(\lambda)$ decreases as λ decreases along this axis. A straightforward computation shows that we have $G(0) > 0$ but $G(\lambda) < -p_\lambda$ for λ sufficiently negative. It follows that there is at least one λ-value for which $G(\lambda) = s(\lambda)$ for each possible itinerary. In fact, it is known[6] that there is a unique such λ on the real line for each itinerary and that, in fact, this set of λ-values is a Cantor set in parameter plane. This is the Cantor set portion of the Cantor necklace in the parameter plane.

Now choose a λ-value for which $G(\lambda)$ lands on a periodic point in Γ_λ that is not equal to p_λ (or an eventually periodic point that does not land on

p_λ). These are the "buried" cycles in the Cantor set since they do not lie on the boundary of any complementary interval. For such a λ-value, the Julia set is also a Sierpinski curve. Indeed, the only components of the Fatou set are the basin at ∞ and all of its preimages since all of the critical orbits are eventually periodic. As above, these preimages are again bounded by disjoint simple closed curves. Moreover the Julia set is compact, connected, nowhere dense, and locally connected (this last property follows since we are in the analogue of the Misiurewicz point case for rational maps[17]). Notice that in any neighborhood of such a λ-value, we have infinitely many other parameters for which the critical orbit also lands on a periodic or eventually periodic point in the Cantor set that does not lie in ∂B_λ and whose period is different from the original cycle. Hence the Julia sets for these parameters also are Sierpinski curves, and the dynamics on them are always different provided that the periods of the cycles are different. In addition, there are infinitely many intervals that lie in the complement of Γ_λ in any such neighborhood, so these yield infinitely many dynamically distinct escape Sierpinski curve Julia sets in this neighborhood as well.

We have shown:

Theorem. *Suppose that the critical orbit of F_λ lands on a point in the Cantor set Γ_λ that is periodic or eventually periodic and that this orbit does not lie in ∂B_λ. Then the Julia set of F_λ is a Sierpinski curve. Moreover, in any neighborhood of $\lambda \in \mathbb{R}^-$, there are infinitely many other parameter values whose Julia sets are Sierpinski curves of this type as well as infinitely many other escape Sierpinski curves. All of these parameters have dynamical behavior that is different from that of F_λ.*

Since we have such vastly different dynamical behavior in any neighborhood of such a λ-value, such a map is structurally unstable at that point.

As a remark, if the critical orbit eventually lands on the fixed point p_λ, then the Julia set is what we call a "hybrid" Sierpinski curve. The only difference between this type of set and a Sierpinski curve is that infinitely many of the complementary domains have boundaries that now touch at exactly two points, while infinitely many others have boundaries that are disjoint from all the other bounding curves.

6. Final Comments and Conjectures

While we have shown that there exist infinitely many dynamically different types of Sierpinski curve Julia sets in these families, much more remains to be done. Here are some open problems and conjectures.

We have shown that if two maps have critical orbits that escape to ∞, then their Julia sets are Sierpinski curves. If the escape times of these critical orbits are different for these two parameters, then these maps are not conjugate on thier Julia sets. So the question is: what happens if we have two such maps whose escpae times are the same?

Conjecture. *Excluding any pair of symmetrically located Sierpinski holes in the parameter plane, the maps corresponding to parameter values drawn from different Sierpinski holes are always dynamically different.*

To prove this result, one needs to find a dynamic invariant for these maps that is different from the escape time. We anticipate that the invariant Cantor necklaces described in the previous section will play a role in this.

In the previous section, we discussed the case where the critical orbit eventually lands on a repelling cycle in the Cantor set Γ_λ. But it is known that there is a unique parameter value for which the critical orbit lands on any point in this Cantor set. For example, there are uncountably many parameters for which the critical orbit lands on an orbit which never cycles. For these maps, the Julia set is again a Sierpinski curve. But here we no longer have the period of the cycle on which the critical orbit lands as a dynamical invariant. Nonetheless, we expect that any two such parameters will have distinct dynamics.

Conjecture. *Suppose the critical orbit lands on a point in Γ_λ for which the itinerary is neither periodic nor eventually periodic. Then the dynamics of F_λ is distinct from any other parameter value for which the critical orbit lands on a point in Γ_λ with a different itinerary.*

Besides invariant sets that are Cantor sets, we have shown recently that there are parameter values for which the Julia sets contain a Cantor set of invariant circles. This is not the McMullen domain case, as the critical values of these maps do not lie in T_λ. Rather, for these parameters, the Julia set is connected. We suspect that, just as in the previous section, one can find unique parameter values for which the critical orbit lands on a particular point in this invariant set. This would produce a similar set in the parameter plane.

Conjecture. *There is a Cantor set of simple closed curves in the parameter plane for which the critical orbits eventually land on the invariant set of circles in dynamical plane. For these parameters, the Julia sets are again Sierpinski curves, and we conjecture that all of these maps (excluding symmetric cases) have distinct dynamics.*

We expect that the curves alluded to in this conjecture would actually contain many of the parameters in the Cantor set produced in the previous section. This would produce a much huger array of dynamically distinct Sierpinski curve Julia sets.

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