Fractal Patterns and Chaos Games

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1. Introduction

One of the most wonderful ways to introduce middle or secondary school students to the beauty and excitement of contemporary mathematics is to involve them in the many variations of the "chaos game." While the fact that this game produces such fractal patterns as the Sierpinski triangle and Koch curve has been known for many years by a handful of research mathematicians, it was only in the late 1980s that the widespread availability of computer graphics enabled everyone to see the intriguing patterns that arise from this game.

Since that time, many teachers have incorporated the chaos game and the concept of a fractal into various areas of the algebra and geometry curriculum. Most of these efforts are centered around the computation of fractal dimension or the deterministic construction of these fractals. We believe that the chaos game approach to fractals also provides teachers with an opportunity to help students comprehend the geometry of affine transformations. For, given the fractal output of a chaos game, you can "go backwards." That is, with a knowledge of the geometry of transformations (contractions, rotations, and translations) and a keen eye for geometry, you can determine the rules of the chaos game that produces a particular fractal image.

We have used the following activities in "chaos clubs" that were organized in middle and high schools as well as in summer math camps for underprivileged as wel as talented high schoolers. The students involved invariably become quite excited when they encounter these ideas.

2. The "Classical" Chaos Game

The easiest chaos game to understand is played as follows. Start with three points at the vertices of a triangle. Color one vertex red, one green, and one blue. Take a die and color two sides red, two sides green, and two sides blue. Then pick any point whatsoever in the triangle; this is the *seed.* Now roll the die. Depending upon which color comes up, move this point half the distance to the similarly colored vertex. Then repeat this procedure, each time moving the previous point half the distance to the vertex whose color turns up when the die is rolled. After a dozen rolls, start marking where these points land.

When you repeat this process many thousands of times, the pattern that emerges is a surprise: it is not a "random mess," as most first-time players would expect. Rather, the image that unfolds is one of the most famous fractals of all, the Sierpinski triangle. Note that it does not matter which shape the original triangle assumes: you can produce a Sierpinski triangle of any desired shape via the chaos game, as shown in Figure 1. In these

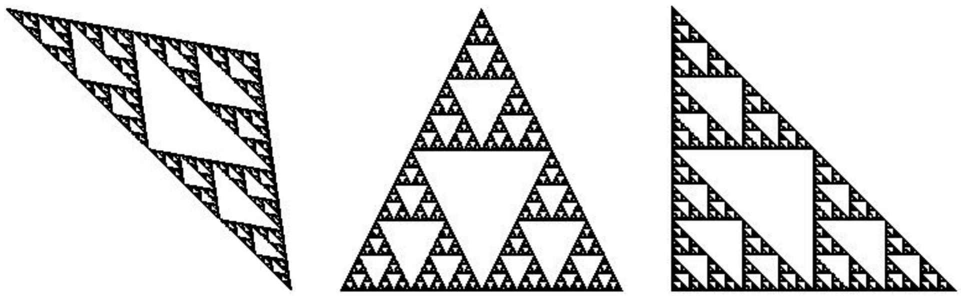

Figure 1: Sierpiski triangles.

cases the three original vertices are located at the vertices of the resulting Sierpinski triangle.

Now here is the observation that leads to the geometry: each of these Sierpinski triangles consists of three self-similar pieces, each of which is exactly one half the size of the original triangle in terms of the lengths of the sides. These are precisely the numbers that we used to play the game: three vertices and move half the distance to the vertex after each roll. So, just by looking at this object, you can read off the rules of the game we played to produce it. This is what we mean when we say you can go backwards.

3. Other Chaos Games

When students first see the output of the chaos game, they immediately start asking: What if you change the number of vertices? What happens if you change the distance you move to the vertex? What happens if this and what happens if that...? This is the beginning of the connection with the geometry of transformations. For example, if we modify the previous chaos game by assuming that we contract toward one of the vertices by a factor of three rather than two, then the image in Figure 2 results. Note that we have three self-similar pieces here. Two are half the size of the original figure, but the top self-similar piece is only one-third the size of the entire figure. As

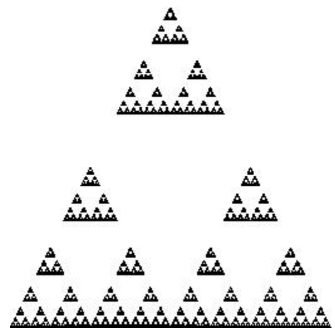

Figure 2: In this chaos game we have contracted toward the top vertex by a factor of three, not two.

before, we can read off the rules of the chaos game that we played directly from the resulting pattern.

Let's try another chaos game. Put six points at the vertices of a regular hexagon. Number them one through six and erase the colors on the die. We change the rules a bit here: instead of moving the point half the distance to the appropriate vertex after each roll, we now, as in the previous example, "compress the distance by a factor of three." By this we mean we move the point so that the resulting distance from the moved point to the chosen vertex is one third the original distance. We say that the *compression ratio* for this game is three.

Again we get a surprise: after rolling the die thousands of times the resulting image is a "Sierpinski hexagon" as shown in Figure 3. And again

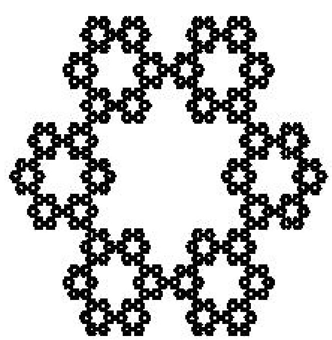

Figure 3: The Sierpinski hexagon.

we can go backwards: this image consists of six self-similar pieces, each of which is exactly one third the size of the full Sierpinski hexagon — the same numbers we used to design the game. By the way, there is much more to this picture than meets the eye at first: notice that the interior white regions of this figure are all bounded by the well known Koch snowflake fractal!

The reason for the choice of compression ratio three here is that if we place six equal-sized smaller hexagons inside a larger hexagon so that their vertices meet as shown in Figure 4, then each of these smaller hexagons has sides whose length is exactly one-third the original length. So if we choose our initial point anywhere inside the large hexagon, then after one roll of the die, the next point lands inside one of the six smaller hexagons. After the next roll, the point moves into one of the 36 even smaller hexagons similarly arranged inside the six previous hexagons. After a dozen or so rolls, we are inside a tiny hexagon too small to see because of the resolution of the image. At that point we begin randomly jumping around through all of these small hexagons, eventually visiting all of them, at least up to the resolution of the image at hand.

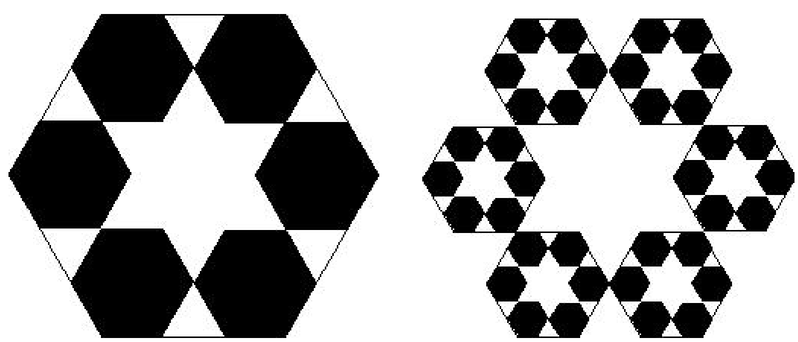

Figure 4: Construction of the Sierpinski hexagon.

Here now is a "reverse surprise." Play the chaos game with four vertices at the corners of a square and a compression ratio of two. After all of the previous games, the result of this game is — surprise! — a square. But this is not really a surprise, since the square consists of four self-similar subsquares, each of which is exactly one half the size of the original (in length and width). While the square is not a fractal, it is indeed a self-similar object.

4. Fractals

Clearly, self-similarity is only one component in the definition of a fractal. A line segment and a square are self-similar sets, but they are definitely not fractals. The missing ingredient here is fractal dimension: a fractal set must also have fractal dimension that exceeds the set's topological dimension. Without going into details, topological dimension is the "usual" dimension of a set; it is always a nonnegative integer. Fractal dimension gives finer information about the roughness or complexity of a set. Sets like the Sierpinski triangle or hexagon which have intricate geometries therefore have fractal dimension larger than one, which is the topological dimension of both. For more details, we refer to [1] or [3]. Incidentally, many people believe that a fractal is a set whose fractal dimension is not an integer. This is incorrect: there are many fractals that have integer fractal dimension. The Sierpinski tetrahedron (a tetrahedral analogue of the triangle) has fractal dimension two (but topological dimension one).

5. Rotations

Now let's add rotations to the mix. This is where the geometry of transformations becomes more important. Start with the vertices of a triangle as in the case of the Sierpinski triangle. For two of the vertices, the rules are as before: just move half the distance to that vertex when that vertex is called. For the remaining vertex, the rule is: first move the point half the distance to that vertex, and then rotate the point 90 degrees about the vertex in the clockwise direction. The result of this chaos game is shown in Figure 5a: note that there are basically three self-similar pieces in the fractal, each of which is half the size of the original, but the top one is rotated by 90 degrees in the clockwise direction. Again, as before, we can go backwards and determine the rules of the chaos game that produced the image.

Changing the rotation at this vertex to 180 degrees yields the image in Figure 5b. This time, the top self-similar piece is rotated 180 degrees. For the fractal in Figure 5c, we rotated twenty degrees in the clockwise direction around the lower left vertex, twenty degress in the counterclockwise direction around the lower right vertex, and there was no rotation around the top vertex.

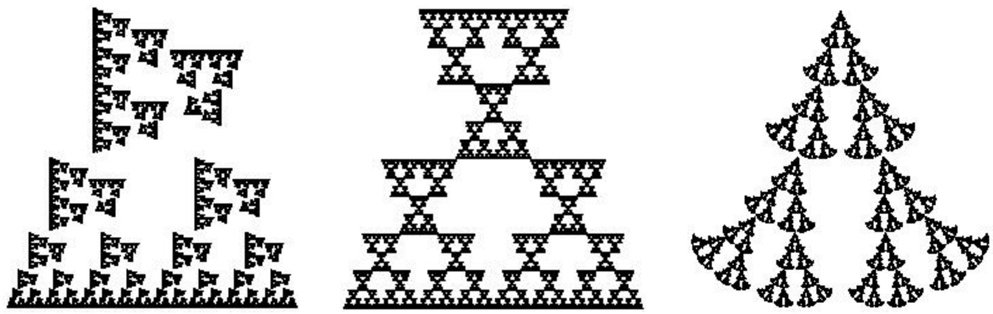

Figure 5: Sierpinski with rotations.

6. Role of Technology

Many teachers introduce the chaos game using pencil and paper or markers and transparencies. This enables students to comprehend the basic ideas behind the chaos game. Piling student-constructed transparencies one atop another enables students to visualize how the Sierpinski begins to emerge. But the basic fact is that this activity really requires technology. You simply cannot see these images in their full glory without iterating thousands and thousands of times. There is a website called the Dynamical Systems and Techology Project (DS & T) that provides free software that allows students to play the chaos game in various configurations. Called Fractalina, this software is a java applet that enables students to choose various configurations for the vertices as well as different compression ratios and rotations. Since the software is written in java, it runs on all computers with java-enabled browsers. The URL is `math.bu.edu/DYSYS/applets`.

7. Challenging the Students

We often challenge our students to figure out how we made various fractal images. The students must determine the number of vertices (read number of self-similar pieces), the compression ratio, and the rotations, if any. Figure 6 displays a couple of challenging images made by chaos games. To determine the rules of the chaos game, students must be able to decipher the geometry of the contractions and rotations that break the image into self-similar pieces. This is not always easy, as the fractals in Figure 6 illustrate. Each of these images consists of three self-similar pieces, each with a compression ratio of two, the first with a rotation of 90 degrees in the clockwise direction about each vertex, and the second with the same rotation, but in the counterclockwise direction. In particular, determing where to place the vertices involves both geometry and trigonometry when rotations are involved. The vertices associated to the fractal in Figure 6a are shown in Figure 7.

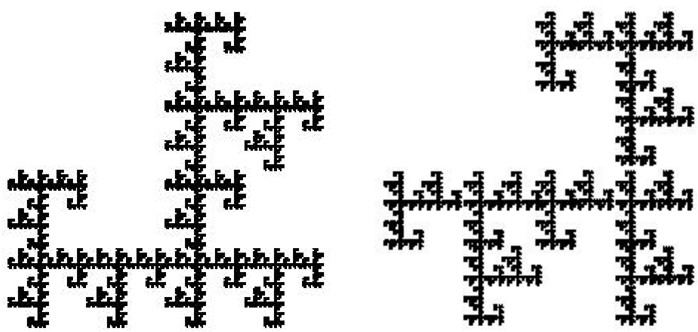

Figure 6: Challenging chaos game images.

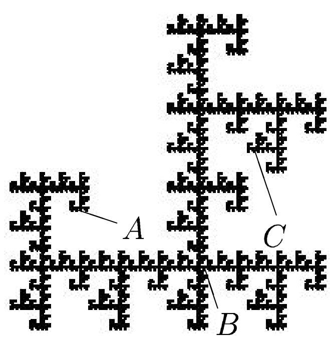

Figure 7: The location of the three vertices.

Another activity that greatly motivates students is fractal movie-making. Once you know how to create a single fractal pattern via the chaos game, you can slowly vary some of the rotations, compression ratios, or locations of the vertices to create a fractal movie. We challenge our students to make a movie that is "beautiful" and that we cannot figure out how they made it. Students often work for hours to make these animations. Of course, beautiful here means "with a lot of symmetry," so there really is a lot of geometry in this activity. While a journal is not the best place to display a movie, several frames from the movie "Dancing Sierpinski" are displayed in Figure 8. An applet called Fractanimate is available to make these movies at the DS & T website. A number of fractal movies created by students are also posted at this site.

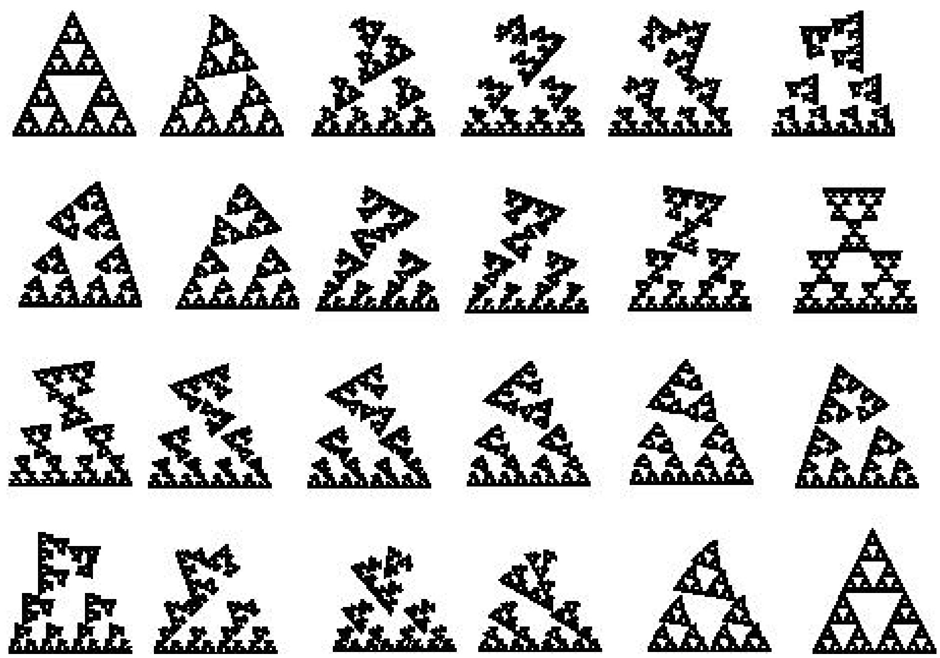

Figure 8: How did we produce this fractal "movie"?

8. The Chaos Game "Game"

Another challenge arises as an actual game. To play this game, we begin with the outline of the Sierpinski triangle down to some level. That is, we begin with the original triangle and successively remove groups of sub-triangles at each level. The first level is defined to be the case where only one triangle has been removed from the original triangle; the second level occurs when the three smaller triangles are removed, and so forth. Highlight one of the remaining small triangles at the given level. This triangle is the *target*. Now place a point at the lower right vertex of the original triangle.

This is the *starting point*. The goal of the game is to move the starting point into the *interior* of the target. The moves are just the moves of our original chaos game: At each stage the point is moved half the distance to one of the original vertices. The chaos game setup for a level three game is displayed in Figure 9.

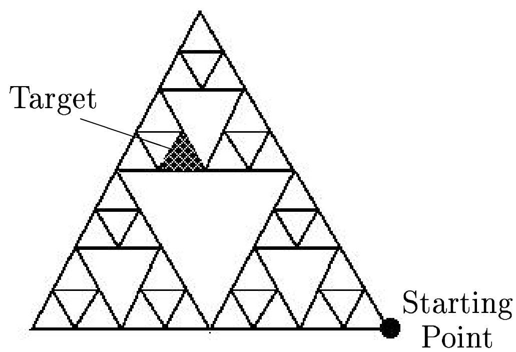

Figure 9: Level three of the chaos game.

At a given level, it is always possible to move the starting point into the interior of the given target in the same number of moves, no matter where the target is placed. For example, for the three targets available at level one, three moves are necessary to hit any target. At level two, four moves suffice, and at level n, exactly $n + 2$ moves can be found to hit any target. The challenge to students is to figure out the algorithm for hitting any possible target. Students can usually come up fairly quickly with a way to hit a specific target, but the algorithm necessary to hit any target is much more difficult both to formulate and to explain.

For example, in Figure 9, the moves to hit the prescribed target are, in order: top, left, right, left, and top. There is only one other way to hit this target in five moves: left, top, right, left, top. This in general is the case: there are exactly two sequences of moves that allow you to hit the target in the minimum number of moves. As a hint as to how to proceed, the first two moves are necessary to move the starting point from the boundary of the original triangle into the interior of this triangle. Thereafter, hitting the target involves determining the "address" of that target. An interactive version of this game is available at the DS & T website.

9. Some Applications

Our students never seem to worry about applications of these ideas when they see the fascinating shpes that arise from the chaos game. Nonetheless there are many ways that these are currently being used. One involves data compression. Think about how much data we need to feed into the computer to generate the Sierpinski triangle: just three vertices, a compression ratio of two, and the total number of iterations. That tiny amount of data allows us to store the incredibly complicated set of points making up the fractal. Furthermore, many objects from nature (trees, clouds, ferns, etc.) are fractals, and a slightly more sophisticated form of the chaos game allows us to capture these images as very small data sets. These ideas have been used to great advantage in such diverse arenas as digital encyclopedias and Hollywood movies to construct and store lifelike, fractal images.

References

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