Rabbits, Basilicas, and Other Julia Sets Wrapped in Sierpinski Carpets

Paul Blanchard, Robert L. Devaney, Antonio Garijo, Sebastian M. Marotta, Elizabeth D. Russell

1 Introduction

In this paper we consider complex analytic rational maps of the form

$$F_\lambda(z) = z^2 + c + \frac{\lambda}{z^2}$$

where $\lambda, c \in \mathbb{C}$ are parameters. For this family of maps, we fix c to be a parameter that lies at the center of a hyperbolic component of the Mandelbrot set, i.e., a parameter such that, for the map

$$F_0(z) = z^2 + c,$$

0 lies on a periodic orbit. We then perturb F_0 by adding a pole at the origin. Our goal is to investigate the structure of the Julia set of F_λ, which we denote by $J(F_\lambda)$, when λ is nonzero.

For these maps, the point at ∞ is always a superattracting fixed point, so we have an immediate basin of attraction of ∞ that we denote by B_λ. As a consequence, we may also define the filled Julia set for these maps to be the set of points whose orbits remain bounded. We denote this set by $K(F_\lambda)$.

In the case where c is chosen so that the map has a superattracting cycle of period 1, the structure of $J(F_\lambda)$ has been well-studied [1], [3], [4], [5]. In this case, $c = 0$ and the map is $z^2 + \lambda/z^2$. This map has four "free" critical points at the points $c_\lambda = \lambda^{1/4}$, but there is essentially only one critical orbit, since one checks easily that $F_\lambda^2(c_\lambda) = 4\lambda + 1/4$. Hence all four of the critical orbits land on the same point after two iterations. Then the following result is proved in [1].

Theorem. *Suppose the free critical orbit of $z^2 + \lambda/z^2$ tends to ∞ but the critical points themselves do not lie in B_λ. Then the Julia set of this map is a Sierpinski curve. In particular, there are infinitely many disjoint* *open sets in any neighborhood of* 0 *in the parameter plane (the* λ*-plane) for which this occurs. Furthermore, two maps drawn from different open sets in this collection are not topologically conjugate on their Julia sets.*

A *Sierpinski curve* is any planar set that is homeomorphic to the well-known Sierpinski carpet fractal. By a result of Whyburn [14], a Sierpinski curve may also be characterized as any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any pair of complementary domains are bounded by simple closed curves that are pairwise disjoint. A Sierpinski curve is an important object from the topological point of view because it is a universal plane continuum among all topologically one-dimensional sets, i.e., it contains a homeomorphic copy of any planar one-dimensionsl set.

When $|\lambda|$ is small, we consider the map $z^2 + \lambda/z^2$ to be a singular perturbation of the simple map $F_0(z) = z^2$. As is well known, the Julia set of F_0 is the unit circle. When $|\lambda|$ is small, it is known that the Julia set of F_λ is also bounded by a simple closed curve that moves continuously as λ varies. Note that, when the hypothesis of the above Theorem is met, the Julia set suddenly changes from a simple closed curve to a much more complicated Sierpinski curve.

We remark that the situation for families of the form

$$z^n + \frac{\lambda}{z^d}$$

where $n, d \geq 2$ but not both are equal to 2 is quite different. For these families, it is known [8] that the Julia sets for $|\lambda|$ small consist of a Cantor set of simple closed curves, each of which surrounds the origin. Sierpinski curves do occur as Julia sets for these maps, but only for larger parameter values.

Our goal in this paper is to describe a related but somewhat different phenomenon that occurs in the family

$$F_\lambda(z) = z^2 + c + \frac{\lambda}{z^2}$$

when the singular perturbation occurs at c-values that are the centers of other hyperbolic components of the Mandelbrot set. A similar explosion in the Julia set takes place when $\lambda \neq 0$. Unlike the case $c = 0$, the boundaries of the components of the basin of ∞ are no longer simple closed curves. Rather, these domains are usually bounded by “doubly” inverted copies of the Julia set of $z^2 + c$. By removing the attachments on these components, we then find infinitely many disjoint Sierpinski curves that now lie in the Julia set whenever the critical orbits eventually escape. In addition, there is a collection of other points in the perturbed Julia set when $c \neq 0$.

To be more precise, suppose now that $n = d = 2$ and c lies at the center of some hyperbolic component with period $k > 1$ in the Mandelbrot set. When $\lambda = 0$, the Julia set and filled Julia set of F_0 are connected sets whose structure is also well understood: the interior of $K(F_0)$ consists of countably many simply connected open sets, each of which is bounded by a simple closed curve that lies in the Julia set. Let C_0 denote the closure of the component of this set that contains 0. Let C_j be the closure of the component that contains $F_0^j(0)$ for $1 \leq j \leq k-1$. Then F_0^k maps each C_j to itself. Moreover, F_0^k on C_j is conjugate to the map $z \mapsto z^2$ on the closed unit disk. For example, the center of the hyperbolic component of period 2 in the Mandelbrot set occurs when $c = -1$; this Julia set is known as the basilica and is displayed in Figure 1.1. Similarly, when $c \approx -0.12256 + 0.74486i$, c lies at the center of a period 3 hyperbolic component and the corresponding Julia set is the Douady rabbit. See Figure 1.1.

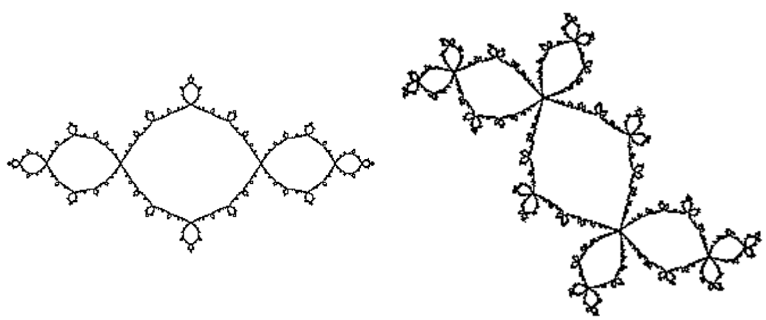

Figure 1.1. The basilica ($c = -1$) and the Douady rabbit ($c = -0.12256+0.74486i$) Julia sets.

Our first goal in this paper is to prove the following result.

Theorem 1. *There exists $\delta > 0$ such that, if $|\lambda| < \delta$, the boundary of B_λ is homeomorphic to $\partial B_0 = J(F_0)$ and F_λ restricted to ∂B_λ is conjugate to F_0 on $J(F_0)$.*

By this result the structure of the Julia set of $z^2 + c$ persists as ∂B_λ when $|\lambda|$ is small. However, the structure of $J(F_\lambda)$ inside ∂B_λ is quite a bit more complex. In Figure 1.2, we display perturbations of the basilica and the Douady rabbit. Note that the boundary of B_λ in these cases is a copy of the original basilica or rabbit, but that there are infinitely many "doubly inverted" basilicas or rabbits inside this set. See Figures 1.3 and 1.4. By a

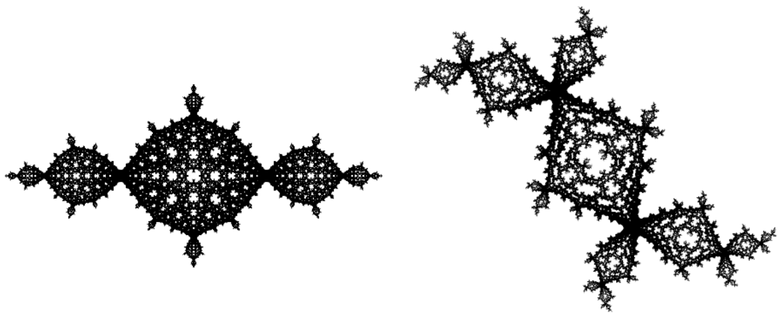

Figure 1.2. Perturbations of the basilica ($\lambda = -0.001$) and the Douady rabbit ($\lambda = 0.0013 - 0.002i$) Julia sets.

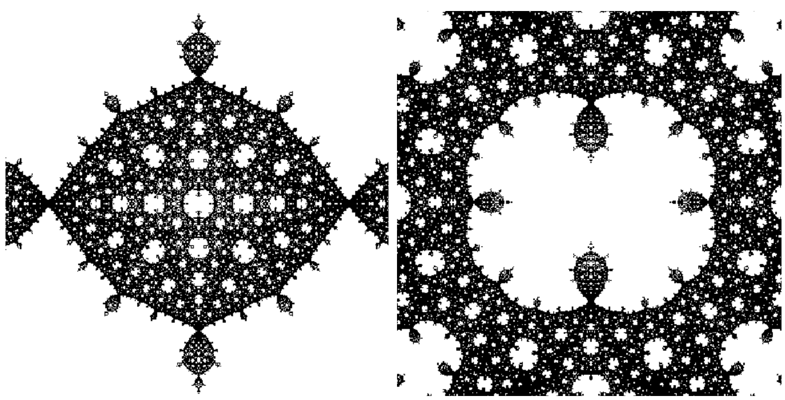

Figure 1.3. Several magnifications of the perturbed basilica Julia set.

double inversion of the rabbit, for example, we mean the following. Choose one of the C_j and translate the rabbit linearly so that the periodic point inside C_j moves to the origin. Then invert the set via the two-to-one map $z \mapsto 1/z^2$. This map moves all the components of the filled Julia set that lie in the exterior of C_j so that they now lie inside the image of the boundary of C_j, and the external boundary of this set is now a simple closed curve that is the image of the boundary of C_j. A homeomorphic copy of this set is what we called a doubly inverted rabbit. See Figure 1.4.

As a consequence of Theorem 1, there is a region $C_j(\lambda)$ that corresponds

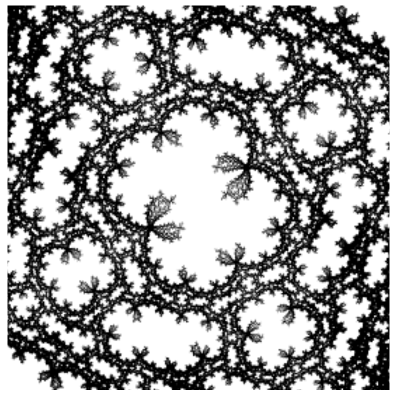

Figure 1.4. A magnification of the doubly inverted rabbit. Note that there are some "quadruply" inverted copies of the rabbit surrounding this set. These bound regions that contain critical points of F_λ or their preimages.

to the region C_j for F_0. Consider the set of points in $C_j(\lambda)$ whose orbits travel through the $\cup C_i(\lambda)$ in the exact order that the point c_j travels through the $\cup C_i$ under F_0. Call this set Λ_λ^j. Note that Λ_λ^j is contained in the disk $C_j(\lambda)$ and is invariant under F_λ^k. We shall prove:

Theorem 2. *Suppose that $|\lambda|$ is sufficiently small and that all of the free critical orbits of F_λ escape to ∞ but the critical points themselves do not lie in B_λ. Then, for $j = 0, \ldots, k-1$, the set Λ_λ^j is a Sierpinski curve.*

Corollary. *Inside every component of the interior of $\mathbb{C} - \partial B_\lambda$ that corresponds to an eventually periodic component of the interior of $K(F_0)$ there is a similar copy of a Sierpinski curve that eventually maps to the Sierpinski curves inside the $C_j(\lambda)$. Furthermore, each interior complementary domain of all of these Sierpinski curves contains an inverted copy of the Julia set of F_0, and then each interior component of this set also contains a Sierpinski curve, and so forth.*

Thus, when λ becomes nonzero and the critical orbits eventually escape to ∞, we see a similar phenomenon as in the case of z^2: suddenly each component of filled Julia set inherits the structure of a Sierpinski curve while ∂B_λ remains homeomorphic to ∂B_0. However, there is actually much more to the structure of the full Julia set of F_λ than that described in the above Theorems.

By Theorems 1 and 2, when $|\lambda|$ is sufficiently small, we have two types of invariant subsets of the Julia set of F_λ: ∂B_λ and the Λ_λ^j. Moreover, we completely understand both the topology of and the dynamics on these sets. However, each of these sets has infinitely many preimages and each of these preimages lies in the Julia set but contains no periodic points. So there are many other points in $J(F_\lambda)$.

To describe these points, we shall assign in Section 5 an itinerary to each such point in the Julia set (excluding those in the various preimages of ∂B_λ). This itinerary will be an infinite sequence of non-negative integers that specifies how the orbit of the given point moves through the various preimages of the $C_j(\lambda)$. For example, the itinerary of any point in Λ_λ^0 will be $\overline{012\ldots n-1}$ and the itinerary of any point in Λ_λ^j will be the j-fold shift of this sequence. Similarly, any point in a preimage of any of the sets Λ_λ^j will be a sequence that terminates in such a sequence. The itinerary of any other point in the Julia set will not have this property. Then we shall prove:

Theorem 3. *Let $\Gamma_s(\lambda)$ denote the set of points whose itinerary is the sequence of non-negative integers $s = (s_0 s_1 s_2 \ldots)$. Then, if s ends in a repeating sequence of the form $\overline{012\ldots n-1}$, $\Gamma_s(\lambda)$ is a Sierpinski curve. Otherwise, $\Gamma_s(\lambda)$ is a Cantor set.*

2 Preliminaries

Let c be a center of a hyperbolic component of the Mandelbrot set with period greater than 1. Let

$$F_\lambda(z) = z^2 + c + \frac{\lambda}{z^2}$$

where $\lambda \in \mathbb{C}$. When $\lambda \neq 0$, these maps have critical points at 0, ∞, and the four points $\lambda^{1/4}$. Since 0 maps to ∞, which is a superattracting fixed point, we call the remaining four critical points the *free critical points*. There are really only two free critical orbits for this family, since $F_\lambda(-z) = F_\lambda(z)$, so $\pm\lambda^{1/4}$ both map onto the same orbit after one iteration. Thus we have only two critical values for F_λ, namely $c \pm 2\lambda^{1/2}$.

One checks easily that the circle of radius $|\lambda|^{1/4}$ centered at the origin is mapped four-to-one onto the straight line segment connecting the two critical values $c \pm 2\lambda^{1/2}$. We call this circle the *critical circle* and its image the *critical segment*. The points $(-\lambda)^{1/4}$ on the critical circle are all mapped to c. Also, the straight lines from the origin to ∞ passing through each of the four critical points are mapped two-to-one onto straight line segments extending from one of the two the critical values to ∞ and extending the critical segment so that these lines together with the critical segment form a single straight line in the plane. One also checks easily that any circle centered at the origin (except the critical circle) is mapped by F_λ two-to-one onto an ellipse whose foci are the two critical values. As these circles tend to the critical circle, the image ellipses tend to the critical segment. Thus the exterior (resp., interior) of the critical circle in $\mathbb{C}$ is mapped as a two-to-one covering of the complement of the critical segment.

For these maps, recall that we always have an immediate basin of attraction of ∞ denoted by B_λ. For each $\lambda \neq 0$, there is a neighborhood of 0 that is mapped into B_λ. If this neighborhood is disjoint from B_λ, we call the component of the full basin of ∞ that contains the origin the *trap door* and denote it by T_λ. We will be primarily concerned with the case where B_λ and T_λ are disjoint in this paper.

The *Julia set* of F_λ, denoted by $J(F_\lambda)$, is the set of points at which the family of iterates of F_λ fails to be a normal family in the sense of Montel. The complement of the Julia set is the *Fatou set*. It is known that $J(F_\lambda)$ is the closure of the set of repelling periodic points of F_λ. The Julia set is also the boundary of the full basin of attraction of ∞. When T_λ and B_λ are disjoint, there are infinitely many distinct components of the entire basin of ∞, so the Julia set surrounds infinitely many disjoint open sets in which orbits eventually escape into B_λ. These holes all lie in the Fatou set.

Since $F_\lambda(-z) = F_\lambda(z)$, it follows that $J(F_\lambda)$ is symmetric under $z \mapsto -z$. There is a second symmetry for these maps: let $H_\lambda(z) = \sqrt{\lambda}/z$. Then we have $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, so $J(F_\lambda)$ is also symmetric under each of the involutions H_λ.

Recall that we have assumed that 0 lies on a cycle of period $k > 1$ for F_0. Let $c_j = F_0^j(0)$ for $j = 1, \ldots, k-1$. The set C_j is the closure of the component of the interior of $\mathbb{C} - J(F_0)$ that contains c_j. As is well known, the interior of C_j is the immediate basin of attraction of F_0^k surrounding c_j. Also, F_0^k maps C_j to itself as a two-to-one branched covering with c_j acting as the only branch point. On the boundary of C_j, F_0^k is conjugate to the map $z \mapsto z^2$ on the unit circle. All other components of $\mathbb{C} - J(F_0)$ eventually map to the C_j (with the exception of the basin of attraction of ∞, which is mapped to itself).

3 The Boundary of the Basin of ∞

Our goal in this section is to prove Theorem 1.

Theorem 1. *There exists $\delta > 0$ such that, if $|\lambda| < \delta$, the boundary of B_λ is homeomorphic to $\partial B_0 = J(F_0)$ and F_λ restricted to ∂B_λ is conjugate to F_0 on $J(F_0)$.*

Proof: We shall use quasiconformal surgery to modify each of the maps F_λ so that the resulting maps are all conjugate to F_0 via a conjugacy h_λ, at least for $|\lambda|$ small enough. Then h_λ will be shown to be a homeomorphism taking ∂B_λ to $\partial B_0 = J(F_0)$.

Let $\mathcal{O}_0$ be the closed disk of radius r about the origin. We choose r small enough so that $\mathcal{O}_0$ lies in the interior of the Fatou component of $K(F_0)$ that contains the origin. For $i = 1, \ldots, k$, let $\mathcal{O}_i = F_0^i(\mathcal{O}_0)$. Note that $\mathcal{O}_k$ is strictly contained in the interior of $\mathcal{O}_0$. Let β_i denote the boundary of $\mathcal{O}_i$. There is a simple closed curve γ_0 that lies outside of β_0 in the component of $K(F_0)$ containing the origin and that is mapped two-to-one onto β_0 by F_0^k. We may then choose $\delta > 0$ small enough so that, if $|\lambda| < \delta$, there is a similar curve $\gamma_0(\lambda)$ lying outside β_0 that is mapped two-to-one onto β_0 by F_λ^k. This follows since, for $|\lambda|$ small enough, $F_\lambda \approx F_0$ outside of $\mathcal{O}_0$. Let $\gamma_i(\lambda) = F_\lambda^i(\gamma_0(\lambda))$ for $i = 1, \ldots, k$ so that $\gamma_k(\lambda) = \beta_0$. Let $A_i(\lambda)$ denote the closed annulus bounded by β_i and $\gamma_i(\lambda)$ for each $i \leq k$.

For $|\lambda| < \delta$, we define a new map G_λ on $\overline{\mathbb{C}}$ as follows. We first set $G_\lambda = F_0$ on each of the $\mathcal{O}_i$. Then we set $G_\lambda = F_\lambda$ on the region outside the union of all the $\mathcal{O}_i$ and $A_i(\lambda)$. We now only need to define G_λ on the $A_i(\lambda)$ for $i = 0, \ldots, k-1$. To do this, recall that F_0 maps β_i to β_{i+1} while F_λ maps $\gamma_i(\lambda)$ to $\gamma_{i+1}(\lambda)$. For $i = 0, \ldots, k-1$, we then define $G_\lambda : A_i(\lambda) \to A_{i+1}(\lambda)$ to be a smooth map that:

1. G_λ agrees with F_0 on β_i and with F_λ on $\gamma_i(\lambda)$;
2. $G_0 = F_0$ on each $A_i(\lambda)$;
3. G_λ is a two-to-one covering map on $A_0(\lambda)$ and one-to-one on $A_i(\lambda)$ for $1 \leq \lambda \leq k-1$;
4. G_λ varies continuously with λ.

According to this definition, we have that $G_0 = F_0$ everywhere on $\overline{\mathbb{C}}$. Furthermore, G_λ is holomorphic at all points outside of the $A_i(\lambda)$ and G_λ has a superattracting cycle of period k at 0. Finally, $G_\lambda = F_\lambda$ on B_λ and its boundary, so the immediate basin of ∞ for G_λ is just B_λ.

We now construct a measurable ellipse field ξ_λ that is invariant under G_λ. Define ξ_λ to be the standard complex structure on the union of the $\mathcal{O}_i$, i.e., the circular ellipse field. Now we begin pulling back this structure by successive preimages of G_λ. The first k preimages defines ξ_λ on the union of the $A_i(\lambda)$ (and elsewhere). Each of these pullbacks yields an ellipse field on the $A_i(\lambda)$ since we are pulling back by a map that is not necessarily holomorphic on these annuli. However, since G_λ is a smooth map on these annuli, this new portion of the ellipse field has bounded dilatation. Then all subsequent pullbacks of the ellipse field are done by holomorphic maps since $G_\lambda = F_\lambda$ outside of the $A_i(\lambda)$. This defines ξ_λ on the union of all of forward and backward images of the $\mathcal{O}_i$. As defined so far, ξ_0 is just the standard complex structure on the union of all the bounded components of

$K(F_0)$. Furthermore, each G_λ preserves ξ_λ. To complete the definition of ξ_λ, we set ξ_λ to be the standard complex structure on all remaining points in $\overline{\mathbb{C}}$. Since $G_\lambda = F_\lambda$ on this set of points, it follows that G_λ preserves ξ_λ everywhere and, in particular, ξ_λ has bounded dilatation on the entire Riemann sphere.

By the Measurable Riemann Mapping Theorem, there is then a quasiconformal homeomorphism h_λ that converts ξ_λ to the standard complex structure on $\overline{\mathbb{C}}$. We may normalize h_λ so that $h_\lambda(\infty) = \infty$, $h_\lambda(c) = c$, and $h_\lambda(0) = 0$. Since ξ_λ depends continuously on λ, so too does h_λ. Moreover, h_0 is the identity map. Thus h_λ conjugates each G_λ to a holomorphic map that is a polynomial of degree two with a superattracting cycle of period k. This polynomial must therefore be F_0 for each λ. We therefore have that h_λ is a homeomorphism that takes ∂B_λ to $J(F_0)$. This completes the proof.

□

4 Sierpinski Carpets

Our goal in this section is to prove Theorem 2. For the rest of this section, we fix a λ value with $|\lambda| \le \delta$ so that, by Theorem 1, ∂B_λ is homeomorphic to $J(F_0)$. Hence we have the k regions $C_j(\lambda)$ for F_λ that correspond to the periodic regions C_j for F_0. By Theorem 1, each of the $C_j(\lambda)$ is a closed disk that is bounded by a simple closed curve. We shall prove that there is an F_λ^k-invariant set $\Lambda_\lambda \subset J(F_\lambda)$ that is contained in $C_0(\lambda)$, is homeomorphic to the Sierpinski carpet, and has the property that all points in this set have orbits that remain for all iterations in $\cup C_i(\lambda)$ and travel through these sets in the same order as the orbit of 0 does under F_0. The other parts of Theorem 2 and its Corollary then follow immediately from this result by taking appropriate preimages of Λ_λ.

So consider the region $C_0(\lambda)$ and its boundary curve $\nu_0(\lambda)$. Similarly, let $\nu_j(\lambda)$ denote the boundary of $C_j(\lambda)$. Since $|\lambda| \le \delta$, the critical segment lies inside $C_1(\lambda)$, so the critical circle lies in the interior of $C_0(\lambda)$. Now recall that F_λ maps the interior of the critical circle as a two-to-one covering onto the exterior of the critical segment in $\overline{\mathbb{C}}$. It follows that there is another simple closed curve in $C_0(\lambda)$ that lies inside the critical circle (and hence inside $\nu_0(\lambda)$), and, like $\nu_0(\lambda)$, this curve is mapped two-to-one onto $\nu_1(\lambda)$. Call this curve $\xi_0(\lambda)$. The region between $\xi_0(\lambda)$ and $\nu_0(\lambda)$ is therefore an annulus that is mapped by F_λ as a four-to-one branched covering onto the interior of the disk $C_1(\lambda)$. Call this annulus $\mathcal{A}_\lambda$. Note that all four of the free critical points of F_λ lie in $\mathcal{A}_\lambda$ since the critical values reside in the interior of $C_1(\lambda)$.

The complement of $\mathcal{A}_\lambda$ in $C_0(\lambda)$ is therefore a closed disk that is mapped by F_λ two-to-one to the complement of the interior of $C_1(\lambda)$ in $\overline{\mathbb{C}}$. Hence there is a subset of this disk that is mapped two-to-one onto ∂B_λ. This subset includes the boundary curve $\xi_0(\lambda)$ that is mapped two-to-one to $\nu_1(\lambda)$ in ∂B_λ and the preimages of all of the other points in ∂B_λ lie strictly inside the curve $\xi_0(\lambda)$. Since F_λ is two-to-one inside $\xi_0(\lambda)$, the preimage of ∂B_λ is thus a doubly inverted copy of ∂B_λ. Note that there is a component of the complement of this inverted copy of ∂B_λ that is an open set containing the origin that is mapped two-to-one onto B_λ. This set is the trap door, T_λ.

To prove that the set Λ_λ in $C_0(\lambda)$ is homeomorphic to the Sierpinski carpet, we use quasiconformal surgery. We shall construct a quasiconformal map $L_\lambda : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that agrees with F_λ^k in $\mathcal{A}_\lambda$. The set of points whose orbits under L_λ are bounded will be exactly the set Λ_λ. So we then show that L_λ is conjugate to a rational map of the form

$$Q_{\lambda,\alpha}(z) = z^2 + \frac{\lambda}{z^2} + \alpha$$

where α is a complex parameter and that, with the given assumptions on the critical orbits of F_λ, the Julia set of $Q_{\lambda,\alpha}$ is a Sierpinski curve. This will show that Λ_λ is a Sierpinski curve.

To construct L_λ, first recall that F_λ^k maps $\nu_0(\lambda)$ two-to-one onto itself and is hyperbolic in a neighborhood of this set. Also, $\nu_0(\lambda)$ is symmetric under $z \mapsto -z$. Hence we may choose a simple closed curve $\zeta_1(\lambda)$ having the following properties:

1. $\zeta_1(\lambda)$ lies close to but strictly outside $\nu_0(\lambda)$ and surrounds $\nu_0(\lambda)$;
2. $\zeta_1(\lambda)$ is symmetric under $z \mapsto -z$;
3. there is a preimage of $\zeta_1(\lambda)$ under F_λ^k, namely $\zeta_0(\lambda)$, that lies between $\nu_0(\lambda)$ and $\zeta_1(\lambda)$ and F_λ^k maps $\zeta_0(\lambda)$ to $\zeta_1(\lambda)$ as a two-to-one covering, so $\zeta_0(\lambda)$ is also symmetric under $z \mapsto -z$;
4. all points in the open annulus betwen $\nu_0(\lambda)$ and $\zeta_1(\lambda)$ eventually leave this annulus under iteration of F_λ^k.

We remark that the curve $\zeta_1(\lambda)$ does not lie in B_λ; indeed, $\zeta_1(\lambda)$ passes through portions of $J(F_\lambda)$ close to but outside the curve $\nu_0(\lambda)$.

We now define L_λ in stages. We first define $L_\lambda(z) = F_\lambda^k(z)$ if z is in the closed annulus bounded on the outside by $\zeta_0(\lambda)$ and on the inside by $\xi_0(\lambda)$. To define L_λ outside $\zeta_0(\lambda)$, we proceed in two stages. First consider the region V_λ outside $\zeta_1(\lambda)$ in the Riemann sphere. In this region we "glue in" the map $z \mapsto z^2$. More precisely, since $\zeta_1(\lambda)$ is invariant under $z \mapsto -z$, it follows that V_λ also has this property. Let ϕ_λ be the exterior Riemann map taking V_λ onto the disk

$$D_2 = \{z \in \overline{\mathbb{C}} \,|\, |z| \geq 2\}$$

in $\overline{\mathbb{C}}$ and fixing ∞ with $\phi'_\lambda(\infty) > 0$. Because of the $z \mapsto -z$ symmetry in V_λ, we have that $\phi_\lambda(-z) = -\phi_\lambda(z)$. Let $f(z) = z^2$, so f takes D_2 to

$$D_4 = \{z \in \overline{\mathbb{C}} \,|\, |z| \geq 4\}.$$

We then define L_λ on V_λ by

$$L_\lambda(z) = \phi_\lambda^{-1}(\phi_\lambda(z))^2.$$

Note that $L_\lambda(z) = L_\lambda(-z)$ since $\phi_\lambda(-z) = -\phi_\lambda(z)$.

So we now have L_λ defined on the region inside $\zeta_0(\lambda)$ (but outside $\xi_0(\lambda)$) and also outside $\zeta_1(\lambda)$. We next need to define L_λ on the open annulus U_λ between $\zeta_0(\lambda)$ and $\zeta_1(\lambda)$. On the boundary curve $\zeta_0(\lambda)$ of U_λ, we have that L_λ is the two-to-one covering map F_λ^k and $L_\lambda(\zeta_0(\lambda)) = \zeta_1(\lambda)$; on the other boundary curve $\zeta_1(\lambda)$, L_λ is the map above that is conjugate to $z \mapsto z^2$. So we define a smooth map q_λ on U_λ such that:

1. q_λ takes U_λ to the annulus bounded by $\zeta_1(\lambda)$ and $L_\lambda(\zeta_1(\lambda))$ as a two-to-one covering;
2. q_λ agrees with L_λ on both boundaries of U_λ;
3. $q_\lambda(-z) = q_\lambda(z)$;
4. q_λ varies smoothly with λ.

We now have L_λ defined everywhere outside $\xi_0(\lambda)$. Inside $\xi_0(\lambda)$, we then set $L_\lambda(z) = L_\lambda(H_\lambda(z))$. Since H_λ maps the disk bounded on the outside by $\xi_0(\lambda)$ to the exterior of $\nu_0(\lambda)$ and then L_λ maps this region to itself, it follows that L_λ takes the disk bounded by $\xi_0(\lambda)$ onto the exterior of $\nu_0(\lambda)$ in two-to-one fashion. Note that L_λ is continuous along $\xi_0(\lambda)$, since, on this curve, the "exterior" definition of L_λ, namely F_λ^k, and the interior definition, $L_\lambda \circ H_\lambda = F_\lambda^k \circ H_\lambda$ agree. Also, we have that $L_\lambda(-z) = L_\lambda(z)$. See Figure 4.5.

Proposition. *The set of points whose entire orbits under L_λ lie in $\mathcal{A}_\lambda$ is precisely the set Λ_λ.*

Proof: Suppose $z \in \mathcal{A}_\lambda$. Then we have that $F_\lambda^j(z) \in C_j(\lambda)$ for $j = 0, \ldots, k-1$ since $F_\lambda^j(\mathcal{A}_\lambda) = C_j(\lambda)$ for each such j. If $L_\lambda(z)$ also lies in $\mathcal{A}_\lambda$, then we have that $F_\lambda^k(z)$ also lies in $C_0(\lambda)$ and the first k points on the orbit

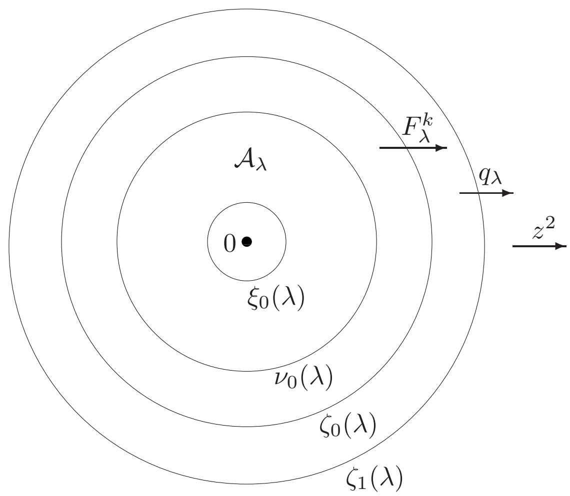

Figure 4.5. Construction of L_λ.

of z under F_λ travel around the $C_j(\lambda)$ in the correct fashion. Similarly, if $L_\lambda^j(z)$ lies in $\mathcal{A}_\lambda$ for all j, then the entire orbit of z visits the $C_j(\lambda)$ in the correct order and we have that $z \in \Lambda_\lambda$.

Conversely, if $L_\lambda^i(z) \in \mathcal{A}_\lambda$ for $0 \le i < j$ but $L_\lambda^j(z) \notin \mathcal{A}_\lambda$, then $L_\lambda^j(z)$ must lie inside the disk bounded by $\xi_0(\lambda)$. But F_λ takes this disk to the exterior of $C_1(\lambda)$, so the orbit of z does not follow the orbit of 0 and so $z \notin \Lambda_\lambda$.

$\square$

Proposition. *The map L_λ is quasiconformally conjugate to a rational map of the form*

$$Q_{\lambda,\alpha}(z) = z^2 + \alpha + \frac{\lambda}{z^2}.$$

Proof: We first construct an L_λ-invariant ellipse field in $\overline{\mathbb{C}}$. First define this field to be the standard complex structure in the region outside $\zeta_1(\lambda)$. Then pull this structure back by q_λ to define the ellipse field in the annulus between $\zeta_0(\lambda)$ and $\zeta_1(\lambda)$. Since q_λ is a smooth map, the ellipse field in this region has bounded dilatation. To define the ellipse field in the annulus between $\nu_0(\lambda)$ and $\zeta_0(\lambda)$, we keep pulling the already defined ellipses back by the appropriate branch of L_λ, which, in this region, equals F_λ^k. Since

F_λ^k is holomorphic, the ellipse field continues to have bounded dilatation under these pull-backs.

We next define the ellipse field inside $\xi_0(\lambda)$ by pulling the given field back by L_λ. This is possible since L_λ maps the region inside $\xi_0(\lambda)$ as a two-to-one covering of the exterior of $\nu_0(\lambda)$.

Finally, we extend the ellipse field to the annulus between $\xi_0(\lambda)$ and $\nu_0(\lambda)$ as follows. We first use the iterates of the map F_λ^k to define the ellipses at any point whose orbit eventually enters the disk bounded by $\xi_0(\lambda)$. If a point never enters this region, then we put the standard structure at this point. This defines the ellipse field almost everywhere. Note that this field is preserved by L_λ, has bounded dilatation, is symmetric under $z \mapsto -z$, and is also preserved by H_λ.

By the Measurable Riemann Mapping Theorem, there exists a quasiconformal homeomorphism ψ_λ that straightens this ellipse field. We may normalize ψ_λ so that $\psi_\lambda(0) = 0$ and $\psi_\lambda(\infty) = \infty$. Because of the symmetries in the ellipse field, we have that $\psi_\lambda(-z) = -\psi_\lambda(z)$ and $\psi_\lambda(H_\lambda(z)) = H_\lambda(\psi_\lambda(z))$. Therefore the map $\psi_\lambda \circ L_\lambda \circ \psi_\lambda^{-1}$ is a rational map of degree 4 that fixes ∞ and has a pole of order 2 at the origin. So we have

$$\psi_\lambda \circ L_\lambda \circ \psi_\lambda^{-1}(z) = \frac{a_4 z^4 + \ldots + a_1 z + a_0}{z^2}.$$

Since $\psi_\lambda(-z) = -\psi_\lambda(z)$ and $L_\lambda(-z) = L_\lambda(z)$, we must have $a_3 = a_1 = 0$ and this map simplifies to

$$\psi_\lambda \circ L_\lambda \circ \psi_\lambda^{-1}(z) = \frac{a_4 z^4 + a_2 z^2 + a_0}{z^2} = a_4 z^2 + a_2 + \frac{a_0}{z^2}.$$

we may scale this map so that $a_4 = 1$ and then the H_λ-symmetry shows that $a_0 = \lambda$. Therefore, L_λ is conjugate to the rational map

$$Q_{\lambda,\alpha}(z) = z^2 + \frac{\lambda}{z^2} + \alpha.$$

□

We now complete the proof that Λ_λ is a Sierpinski curve. It suffices to show that Λ_λ is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by simple closed curves that are pairwise disjoint.

By the previous Propositions, we know Λ_λ is the set of points whose orbits are bounded under iteration of L_λ. Also, the set Λ_λ contains no critical points by assumption. Moreover, this set is homeomorphic to the filled Julia set of $Q_{\lambda,\alpha}$. Thus all of the critical points of $Q_{\lambda,\alpha}$ must escape to ∞ as well and so the Fatou set of $Q_{\lambda,\alpha}$ is the union of all of the preimages of the basin of ∞. Standard facts about the Julia sets of rational maps then yields the fact that $J(Q_{\lambda,\alpha})$ is compact and nowhere dense. In particular, the filled Julia set of this map is equal to $J(Q_{\lambda,\alpha})$. By a result of Yin [15], since all the critical orbits of $Q_{\lambda,\alpha}$ tend to ∞, the Julia set is locally connected.

Now we know that the set of L_λ-bounded orbits is bounded on the outside by the simple closed curve $\nu_0(\lambda)$ and on the inside by $\xi_0(\lambda)$, and all the other complementary domains are preimages of these sets. Therefore this set is connected and all of the complementary domains are bounded by simple closed curves. These curves must be pairwise disjoint for, otherwise, a point of intersection would necessarily be a critical point whose orbit would then be bounded. But this cannot happen since all of the critical orbits escape to ∞. This completes the proof.

5 Dynamics on the Rest of the Julia Set

In this section we turn our attention to the dynamical behavior of all other points in $J(F_\lambda)$. As in the previous sections, we continue to assume that $|\lambda|$ is sufficiently small and that all of the critical orbits of F_λ tend to ∞ (but the critical points themselves do not lie in B_λ).

By Theorem 2 there exists the invariant set ∂B_λ on which F_λ is conjugate to F_0 on $J(F_0)$. Let $\mathcal{B}_\lambda$ denote the set consisting of ∂B_λ together with all of its preimages under F_λ^j for each $j \geq 0$. Similarly, by Theorem 3, there exist Sierpinski curves Λ_λ^j for $j = 0, \ldots, k-1$ on each of which F_λ^k is conjugate to a map of the form $z^2 + \alpha(\lambda) + \lambda/z^2$. Let Ω_λ denote the union of the Λ_λ^j together with all of the preimages of these sets under all iterates of F_λ. By our earlier results, we understand the topology of and dynamics on each of these sets.

We therefore consider points in the set

$$\mathcal{O}_\lambda = J(F_\lambda) - (\mathcal{B}_\lambda \cup \Omega_\lambda).$$

Note that there must be infinitely many points in this set since none of the preimages of the Λ_λ^j or ∂B_λ contain periodic points and, as is well known, repelling periodic points must be dense in $J(F_\lambda)$.

To describe the structure of the set $\mathcal{O}_\lambda$, we first assign an itinerary to each point in this set. For $0 \leq j \leq k-1$, let I_j denote the interior of the disk C_j in $K(F_0)$ that contains $F_0^j(0)$. The interior of $K(F_0)$ consists of infinitely many other such open disks. So for each $j > k-1$, we let I_j denote a unique such disk. How these I_j are indexed is not important. Then let $I_j(\lambda)$ denote the corresponding open disk for F_λ.

Given $z \in \mathcal{O}_\lambda$, we define the *itinerary* of z to be the sequence of nonnegative integers $S(z) = (s_0 s_1 s_2 \ldots)$ where, as usual, $s_j = \ell$ if and only if $F_\lambda^j(z) \in I_\ell(\lambda)$. The itinerary is said to be *allowable* if it actually corresponds to a point in $\mathcal{O}_\lambda$. Note the following:

1. We do not assign an itinerary to any point in $\mathcal{B}_\lambda$ since the $I_j(\lambda)$ are disjoint from this set (and we already understand the dynamics on this set anyway).

2. The itinerary of any point in $\mathcal{O}_\lambda$ necessarily contains infinitely many zeroes. This follows immediately from the fact that each I_j with $j > 0$ must eventually be mapped to I_0 by some iterate of F_0 and so the same must be true for $I_j(\lambda)$ and F_λ.

3. The itinerary of $z \in \mathcal{O}_\lambda$ cannot end in an infinite string of the form $(\overline{0\,1\,\ldots\,k-1})$ since we have assumed that $z \notin \Omega_\lambda$.

Now suppose $z \in \mathcal{O}_\lambda$. If some entry of $S(z) = 0$, say $s_j = 0$, then either $s_{j+1} = 1$, in which case we have that

$$s_{j+2} = 2, s_{j+3} = 3, \ldots, s_{j+k} = 0,$$

or else $s_{j+1} \neq 1$. In the latter case, we call the index j a *departure index*, since these are the points on the orbit of z where this orbit "deviates" from a similar orbit for F_0. As above, there must be infinitely many departure indices for any orbit in $\mathcal{O}_\lambda$, since this itinerary cannot end in the repeating sequence $(\overline{0\,1\,\ldots\,k-1})$.

Before turning to the proof of Theorem 3, we give several illustrative examples of why the set of points in $\mathcal{O}_\lambda$ with a given itinerary is a Cantor set. For clarity, we restrict to the case where $F_\lambda(z) = z^2 - 1 + \lambda/z^2$, i.e., the case where c is drawn from the center of the period two bulb in the Mandelbrot set. We let $I_2(\lambda) = -I_1(\lambda)$, so $I_2(\lambda)$ is the other preimage of $I_0(\lambda)$.

Example 1: The itinerary $(\overline{\mathbf{0}})$. In this case, each index j is a departure index since $s_{j+1} \neq 1$. Let $V_n(\lambda)$ be the set of points in $I_0(\lambda)$ whose itinerary begins with $n+1$ consecutive zeroes. Then $V_0(\lambda) = I_0(\lambda)$ and $V_1(\lambda)$ is a pair of disjoint open disks in $I_0(\lambda)$, each of which lies inside the curve $\xi_0(\lambda)$ that is mapped two-to-one to the boundary of $I_1(\lambda)$. Each of these disks is mapped univalently over $I_0(\lambda)$ since the critical points of F_λ lie outside the curve $\xi_0(\lambda)$ and are mapped to $I_1(\lambda)$. But then $V_2(\lambda)$ consists of 4 disjoint open disks, two in each component of $V_1(\lambda)$ that are mapped onto the two components of $V_1(\lambda)$. Continuing in this fashion, we see that $V_n(\lambda)$ consists of 2^n disjoint open disks, and $V_n(\lambda) \subset V_{n-1}(\lambda)$ for each n. Since F_λ maps $V_n(\lambda)$ to $V_{n-1}(\lambda)$ as above, standard arguments from complex dynamics then show that the set of points in $\mathcal{O}_\lambda$ whose itinerary is $(\overline{0})$ is a Cantor set.

From now on, we let $W_{s_0 s_1 \ldots s_n}(\lambda)$ denote the set of points in $\mathcal{O}_\lambda$ whose itinerary begins with $s_0 s_1 \ldots s_n$.

Example 2: The itinerary ($\overline{\mathbf{02}}$). In this case we again have that the critical points are mapped into $I_1(\lambda)$, so, as above, $W_{02}(\lambda)$ is a pair of open disks in $I_0(\lambda)$, each of which is mapped univalently onto $I_2(\lambda)$. Thus $W_{202}(\lambda)$ is also pair of disks lying in $I_2(\lambda)$. But then, since F_λ maps each disk in $W_{02}(\lambda)$ univalently onto $I_2(\lambda)$, we have that $W_{0202}(\lambda)$ consists of four disks, two in each of the disks comprising $W_{02}(\lambda)$. Continuing, we see that every second iterate produces double the number of disks contained in the previous $W(\lambda)$, so again we see that the set of points on $\mathcal{O}_\lambda$ with itinerary ($\overline{02}$) is a Cantor set.

Example 3: The itinerary ($\overline{\mathbf{0102}}$). As in the previous example, $W_{02}(\lambda)$ is a pair of disks, since each of the critical points in $I_0(\lambda)$ is mapped into $I_1(\lambda)$ for small $|\lambda|$. Then $W_{102}(\lambda)$ is a pair of disks in $I_1(\lambda)$ since $I_1(\lambda)$ is mapped univalently onto $I_0(\lambda)$. But now $W_{0102}(\lambda)$ consists of at least 4 and at most 8 disjoint disks in $I_0(\lambda)$. To see this, note that the preimage of $I_1(\lambda)$ in $I_0(\lambda)$ is the annulus bounded by the curves $\xi_0(\lambda)$ and $\nu_0(\lambda)$ defined earlier, and F_λ takes this annulus four-to-one onto $I_1(\lambda)$. There are four critical points in this annulus, and it could be the case that one of the critical points map into one of the two disks in $W_{102}(\lambda)$. If that happens, then the negative of this critical point (also a critical point) maps to the same disk. So the preimage of this particular disk has either one or two components since the map is four-to-one. But, if this preimage has only one component, by the $z \mapsto -z$ symmetry, this component would necessarily surround the origin. Now this preimage must be disjoint from the Sierpinski curve invariant set in $\overline{I}_0(\lambda)$ and also separate $\xi_0(\lambda)$ from $\nu_0(\lambda)$. This then gives a contradiction to the connectedness of the Sierpinski curve. Hence each of these disks would have at least two preimages for a total of at least four and at most eight preimages of $W_{102}(\lambda)$ in $I_0(\lambda)$. But then $W_{20102}(\lambda)$ also consists of at least 4 and at most 8 disks, while $W_{020102}(\lambda)$ now consists of double this number of disks, since F_λ maps $I_0(\lambda)$ two-to-one onto $I_2(\lambda)$, but the critical points map into $I_1(\lambda)$. That is, each of the two original disks in $W_{02}(\lambda)$ acquires from 4 to 8 preimages when we pull them back by the four appropriate inverses of F_λ. Then, continuing in this fashion, each time we pull back each of these disks, again by the four appropriate preimages, we find at least 4 preimages for each one. Again, the set of points with this itinerary is a Cantor set.

We now complete the proof of Theorem 3. Consider the allowable itinerary $(s_0 s_1 s_2 \ldots)$. We may asume at the outset that $s_0 = 0$ and that 0 is a departure index. So say that the itinerary is given by $(0 s_1 \ldots s_i 0 \ldots)$ where $s_j \neq 0$ for $1 \leq j \leq i$. Then the set of points whose itinerary begins this way is a pair of disks in $I_0(\lambda)$, and each of these disks is mapped univalently onto $I_0(\lambda)$ by F_λ^{i+1}. Then there are two cases: either $i+1$ is a departure index or it is not. In the former case, the itinerary may be continued $(0s_1 \ldots s_i 0 s_{i+2} \ldots s_{i+\ell} 0 \ldots)$ where again $s_j \neq 0$ for $i+2 \leq j \leq i+\ell$. Just as in Examples 1 and 2, the set of points in $I_0(\lambda)$ whose itinerary begins in this fashion now consists of 4 disks. In the other case, we have that $s_{i+1} = 1, \ldots, s_{i+k-1} = k-1, s_{i+k} = 0$. Arguing as in Example 3, we have that the set of points whose itinerary now begins in this fashion consists of between four and eight disjoint open disks, all contained in the original pair of disks, and each mapped onto $I_0(\lambda)$ (at most two-to-one) by $F_\lambda^{i+\ell+1}$. In any event, the number of disks that correspond to this initial itinerary has at least doubled. Continuing in this fashion, we see that at each index for which $s_j = 0$, we find at least double the number of disjoint open disks in $I_0(\lambda)$ that begin with this itinerary. These disks are nested and converge to points. Hence the set of points with the given itinerary is a Cantor set.

6 Several Examples

For completeness, we give several examples of parameters in the family

$$F_\lambda(z) = z^2 - 1 + \frac{\lambda}{z^2}$$

for which both critical orbits eventually escape to ∞ and the above theorems hold. We shall choose λ real and negative. Note that, in this case, we have $F_\lambda(\overline{z}) = \overline{F_\lambda(z)}$, so the Julia sets are symmetric under $z \mapsto \overline{z}$. More importantly, the two critical values are symmetric under this map, so if one critical value eventually escapes, then so does the other one.

The first example occurs when $\lambda = -.0025$ so that one of the critical values is $v_\lambda = -1 + 0.1i$. We then compute

$$\begin{aligned}
F_\lambda(v_\lambda) &= -.0124262 - .20049i \\
F_\lambda^2(v_\lambda) &= -.978559 + .0126334i \\
F_\lambda^3(v_\lambda) &= -.0451908 - .0247924i \\
F_\lambda^4(v_\lambda) &= -1.50415 + .795835i \\
F_\lambda^5(v_\lambda) &= .628637 - 2.39483i
\end{aligned}$$

We have that v_λ and $F_\lambda^2(v_\lambda)$ belong to $I_1(\lambda)$ whereas $F_\lambda(v_\lambda)$ and $F_\lambda^3(v_\lambda)$ belong to $I_0(\lambda)$. One checks easily that $F_\lambda^4(v_\lambda)$ then lies outside ∂B_λ. Hence both critical orbits escape at this iteration. See Figure 6.6 for a picture of this Julia set.

The second example occurs when $\lambda = -.0001$ so that one of the critical

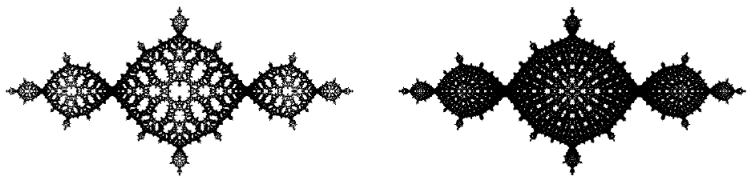

Figure 6.6. The Julia sets for $z^2 - 1 - \lambda/z^2$ when $\lambda = -.0025$ and also $\lambda = -.0001$.

values is $-1 + .02i$. We then compute

$$
\begin{aligned}
F_\lambda(v_\lambda) &= .00049988 - .040004i \\
F_\lambda^2(v_\lambda) &= -.939142 + .00160116i \\
F_\lambda^3(v_\lambda) &= -.118129 - .00300783i \\
F_\lambda^4(v_\lambda) &= -.993207 + .00107508 \\
F_\lambda^5(v_\lambda) &= -.0136424 - .00213578i \\
F_\lambda^6(v_\lambda) &= -1.449917 + .160339i \\
F_\lambda^7(v_\lambda) &= 1.22177 - .48076i \\
F_\lambda^8(v_\lambda) &= .261538 - 1.17479i \\
F_\lambda^9(v_\lambda) &= -2.31167 - .614536i
\end{aligned}
$$

so that $F_\lambda^6(v_\lambda)$ now is the first point on the critical orbit to lie outside ∂B_λ. See Figure 6.6.

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Bibliography

[1] Blanchard, P., Devaney, R. L., Look, D. M., Seal, P., and Shapiro, Y. Sierpinski Curve Julia Sets and Singular Perturbations of Complex Polynomials. *Ergodic Theory and Dynamical Systems* **25** (2005), 1047-1055.

[2] Devaney, R. L. Structure of the McMullen Domain in the Parameter Planes for Rational Maps. *Fundamenta Mathematicae* **185** (2005), 267-285.

[3] Devaney, R. L. and Garijo Real, A. Julia Sets Converging to the Unit Disk. To appear in *Proc. Amer. Math. Soc.*

[4] Devaney, R. L. and Look, D. M, A Criterion for Sierpinski Curve Julia Sets. *Topology Proceedings* **30** (2006), 163-179.

[5] Devaney, R. L., Look, D. M, and Uminsky, D. The Escape Trichotomy for Singularly Perturbed Rational Maps. *Indiana University Mathematics Journal* **54** (2005), 1621-1634.

[6] Devaney, R. L., Marotta, S. M. Evolution of the McMullen Domain for Singularly Perturbed Rational Maps. June 2006. To appear in Topology Proceedings.

[7] Douady, A. and Hubbard, J. Etude Dynamique des Polynômes Complexes. *Publ. Math. D'Orsay* (1984).

[8] McMullen, C. Automorphisms of Rational Maps. *Holomorphic Functions and Moduli*. Vol. 1. Math. Sci. Res. Inst. Publ. **10**. Springer, New York, 1988.

[9] McMullen, C. The Classification of Conformal Dynamical Systems. *Current Developments in Mathematics*. Internat. Press, Cambridge, MA, (1995) 323-360.

[10] Milnor, J. Dynamics in One Complex Variable. Third Edition. Princeton University Press.

[11] Milnor, J. and Tan Lei. A "Sierpinski Carpet" as Julia Set. Appendix F in Geometry and Dynamics of Quadratic Rational Maps. *Experiment. Math.* **2** (1993), 37-83.

[12] Petersen, C. and Ryd, G. *Convergence of Rational Rays in Parameter Spaces*, The Mandelbrot set: Theme and Variations, London Mathematical Society, Lecture Note Series 274, Cambridge University Press, 161-172, 2000.

[13] Roesch, P. On Capture Zones for the Family $f_\lambda(z) = z^2 + \lambda/z^2$. In *Dynamics on the Riemann Sphere*, European Mathematical Society, (2006), 121-130.

[14] Whyburn, G. T. Topological Characterization of the Sierpinski Curve. *Fundamenta Mathematicae* **45** (1958), 320-324.

[15] Yin, Y. Julia Sets of Semi-Hyperbolic Rational Maps. *Chinese Ann. Math. Ser. A* **20** (1999), 559-566. Translation in *Chinese J. Contemp. Math.* **20** (2000), 469-476.