Simple Mandelpinski Necklaces for $z^2 + \lambda/z^2$ *

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Abstract

For the family of maps $F_\lambda(z) = z^n + \lambda/z^n$ where $n \geq 3$, it is known that there is a McMullen domain surrounding the origin in the parameter plane. This domain is then surrounded by infinitely many "Mandelpinski" necklaces $\mathcal{S}_k$ for $k = 0, 1, 2, \ldots$. These are simple closed curves surrounding the McMullen domain and passing through exactly $(n-2)n^k + 1$ centers of baby Mandelbrot sets and the same number of centers of Sierpinski holes. When $n = 2$ there is no such McMullen domain in the parameter plane. However, we show in this paper that there do exist Mandelpinski necklaces $\mathcal{S}_k$ in this case. Now these necklaces converge down to the origin. And, consistent with the formula for higher values of n, each $\mathcal{S}_k$ passes through the centers of only one Mandelbrot set and one Sierpinski hole.

In recent years a number of papers have appeared that deal with the dynamics of singularly perturbed maps of the form $F_\lambda(z) = z^n + \lambda/z^n$ where $n \geq 2$. It turns out that the case $n = 2$ is very different from the case $n > 2$. One reason for this is that, when $n = 2$, as $\lambda \to 0$, the Julia sets of F_λ converge to the closed unit disk (the filled Julia set of z^2), but this does not occur when $n > 2$ [6]. A second difference is that there is a *McMullen domain* in the parameter plane when $n > 2$. This is a punctured open disk surrounding the origin that consists of parameters for which the Julia sets of the corresponding maps are Cantor sets of simple closed curves, all of which are dynamically and topologically the same [10]. There is no such region when $n = 2$. Rather, in any neighborhood of 0 in the parameter plane, there are infinitely many different topological types of Julia sets [4]. And a third difference is that, when $n > 2$, the second images of the free critical points all tend to ∞ as $\lambda \to 0$ (this is what generates the McMullen domain), whereas when $n = 2$, the second images of the free critical points tend to $1/4$ as $\lambda \to 0$ (and $1/4$ is not in the basin of ∞ when λ is small).

In the case $n > 2$, there is an interesting structure that surrounds the McMullen domain. In [2], [3], and [8] it is shown that this domain in the parameter plane is surrounded by infinitely many disjoint simple closed curves $\mathcal{S}_k$ for $k = 0, 1, 2, \ldots$ called Mandelpinski necklaces. Each $\mathcal{S}_k$ passes alternately through the centers of $(n-2)n^k + 1$ Mandelbrot sets with base period $k+1$ (with one slight exception when $k = 1$) and the same number of centers of Sierpinski holes with escape time $k+3$. See Figure 1. A center of a Mandelbrot set of base period ℓ is a parameter in the main cardioid for which a critical point is periodic with prime period ℓ. A Sierpinski hole with escape time ℓ is a collection of parameters for which the critical orbits all land in the immediate basin of ∞ at iteration ℓ. A center of a Sierpinski hole is a parameter for which the critical orbits actually land at ∞.

These Mandelpinski necklaces provide a great deal of structure around the McMullen domain when $n > 2$. For example, when $n = 3$, the necklace $\mathcal{S}_{16}$ passes through exactly 43,046,722 Mandelbrot sets and Sierpinski holes. When the parameter lies in one of these Mandelbrot sets there are infinitely many small copies of quadratic Julia sets embedded in the much larger Julia set of F_λ. And when the parameter lies in a Sierpinski hole, the Julia set of F_λ is a Sierpinski curve, i.e., a set that is homeomorphic to the Sierpinski carpet fractal.

Because of the different behaviors of the critical orbits as $\lambda \to 0$ and the lack of a McMullen domain when $n = 2$, it was always assumed that there were no such Mandelpinski necklaces when $n = 2$. However, note that the above formula says that each necklace $\mathcal{S}_k$ should pass through exactly $(2-2)2^k + 1 = 1$ Mandelbrot set and 1 Sierpinski hole when $n = 2$ for each k. In fact, as we show in this paper, this does indeed happen. So we do have some simplified Mandelpinski necklaces in this case. These necklaces no longer surround a McMullen domain; rather, they converge to the origin as $k \to \infty$. So the structure of the parameter plane around the origin when $n = 2$ is very different from the case $n > 2$. See Figure 2. We conjecture that

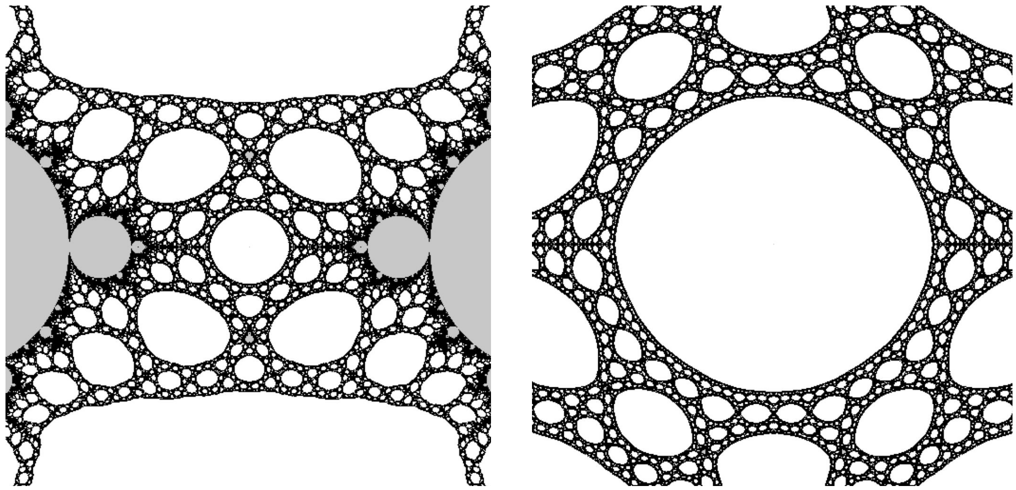

Figure 1: Magnifications of the parameter plane for the family $z^3 + \lambda/z^3$ around the McMullen domain (the central white disk).

the existence of these simple necklaces will allow us to begin to understand the very complicated structure of the parameter plane for $n = 2$ around the origin, just as the Mandelpinski necklaces did in the case $n > 2$.

1 Preliminaries

In this paper we shall concentrate on the family of complex rational maps given by

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2}$$

where $\lambda \in \mathbb{C}$. One checks easily that the point at ∞ is fixed in the Riemann sphere and $F'_\lambda(\infty) = 0$ so ∞ is a superattracting fixed point. We denote the immediate basin of attraction at ∞ by B_λ. Since 0 is a pole there is an open set about 0 that is mapped to B_λ. This set may or may not be disjoint from B_λ, but in the cases we consider in this paper, these two sets will be disjoint. We then call the preimage of B_λ surrounding 0 the *trap door* and denote this

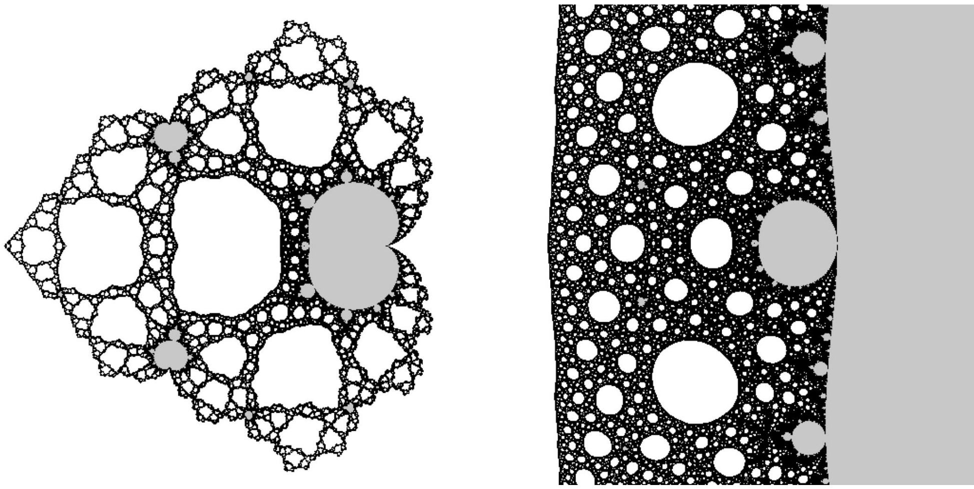

Figure 2: The parameter plane and a magnification around the origin for the family $z^2+\lambda/z^2$. The large central disk is a Sierpinski hole, not the McMullen domain. The origin is located at the "tip of the tail" of the Mandelbrot set that appears to straddle the positive real axis.

set by T_λ.

It is well known that the *Julia set* of F_λ, denoted by $J(F_\lambda)$, has several equivalent definitions [11]. One definition is that $J(F_\lambda)$ consists of all points at which the family of iterates of F_λ fails to be a normal family in the sense of Montel. A second definition is that the Julia set is the closure of the set of repelling periodic points of F_λ. And a third definition is that the Julia set is the boundary of the set of all points whose orbits tend to ∞ (not just those in B_λ). These definitions then imply that $J(F_\lambda)$ is the chaotic regime since arbitrarily close to any point in the Julia set there are points whose orbits tend to ∞ and other points whose orbits are periodic. More importantly, Montel's Theorem implies that any neighborhood of a point in $J(F_\lambda)$ is mapped over the entire Riemann sphere by the set of all iterates of maps in this family. So, on the Julia set, F_λ exhibits extreme sensitive dependence on initial conditions.

There are several symmetries in the dynamical and parameter planes for these maps. We have $F_\lambda(-z) = F_\lambda(z)$ and $F_\lambda(iz) = -F_\lambda(z)$. Therefore the orbits of z and iz are the same after two iterations. As a consequence, the Julia set is symmetric under the map $z \to iz$, i.e., $J(F_\lambda)$ has fourfold symmetry. Also, let $H_\lambda(z) = \lambda^{1/2}/z$. Then $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, so the Julia set is also symmetric under the involution H_λ. We also have that F_λ is conjugate to $F_{\overline{\lambda}}$ via the map $z \to \overline{z}$, so the parameter plane is symmetric under complex conjugation.

A straightforward computation shows that there are four free critical points for F_λ that are given by $\lambda^{1/4}$. We call these critical points "free" since there are two other critical points at ∞ and 0, but ∞ is fixed and 0 maps directly to ∞ for each λ. However, there are only two critical values given by $\pm 2\sqrt{\lambda}$ since two of the free critical points are mapped to $+2\sqrt{\lambda}$ and the other two are mapped to $-2\sqrt{\lambda}$. In fact, just like the quadratic polynomial family $z^2 + c$, there really is only one free critical orbit as both critical values are then mapped to $4\lambda + 1/4$, so all of the critical points end up on the same orbit after two iterations.

There are also four prepoles for F_λ given by $(-\lambda)^{1/4}$. So the prepoles and critical points all lie on the circle of radius $|\lambda|^{1/4}$ centered at the origin. We call this circle the *critical circle* and denote it by C_0^λ. Another easy computation shows that F_λ maps the critical circle 4-to-1 onto the line segment connecting the two critical values $\pm 2\sqrt{\lambda}$ and passing through the origin. We call this line the *critical segment.* Any other circle centered at the origin is then mapped as a 2-to-1 covering onto an ellipse whose foci are the critical values. In particular, the region in the exterior of the critical circle is then mapped as a 2-to-1 covering onto the complement of the critical segment in the Riemann sphere and so too is the interior of the critical circle.

We shall assume for the remainder of this paper that the critical values both lie on or inside the critical circle, so the critical segment will always lie in the the disk bounded by the critical circle. It is known [1], [9] that, in this case, $J(F_\lambda)$ is connected and that ∂B_λ is a simple closed curve lying in the exterior of C_0^λ. Since $H_\lambda(B_\lambda) = T_\lambda$, we have that ∂T_λ is also a simple closed curve that lies inside C_0^λ.

Let $\mathcal{O}$ be the punctured disk in the parameter plane that consists of all nonzero parameters for which the critical segment lies strictly inside the critical circle. When λ lies on the boundary of $\mathcal{O}$, we must have $2|\sqrt{\lambda}| = |\lambda|^{1/4}$, so it follows that $|\lambda| = 1/16$. Therefore the boundary of $\mathcal{O}$ is the circle of radius $1/16$ centered at the origin in the parameter plane. For $\lambda \in \mathcal{O}$, F_λ maps the exterior of the critical circle as a 2-to-1 covering onto the exterior of the critical segment. Thus there is a simple closed curve in the exterior of C_0^λ that is mapped 2-to-1 onto C_0^λ. Call this curve C_1^λ. Since C_0^λ contains four critical points and four prepoles, C_1^λ contains eight pre-critical points and eight pre-prepoles. Since the exterior of C_1^λ is then mapped onto the exterior of C_0^λ as a 2-to-1 covering, there is another simple closed curve C_2^λ that lies outside C_1^λ and is mapped 2-to-1 onto C_1^λ. Continuing in this fashion, we find an infinite collection of simple closed curves C_k^λ for $k > 0$ satisfying $F_\lambda(C_k^\lambda) = C_{k-1}^\lambda$ and hence $F_\lambda^k(C_k^\lambda) = C_0^\lambda$. Note that the C_k^λ are all disjoint and these curves converge outward toward ∂B_λ as $k \to \infty$. This follows since, if this were not the case, the limiting set of the C_k^λ would be a closed, invariant set, say Λ_λ. If $\Lambda_\lambda \neq \partial B_\lambda$, then the region bounded by ∂B_λ and Λ_λ would also be invariant. But this cannot happen since there would then be points in ∂B_λ that have neighborhoods on which the family of functions $\{F_\lambda^k\}$ would be normal, which cannot happen since $\partial B_\lambda \subset J(F_\lambda)$. In addition, C_k^λ contains 2^{k+2} points that are mapped by F_λ^k to critical points and the same number of points that are mapped to the prepoles on C_0^λ. The points that map to critical points and to prepoles are arranged alternately around C_λ^k.

Since the interior of the critical circle is also mapped as a 2-to-1 covering of the exterior of the critical segment, there are other simple closed curves C_{-k}^λ for $k = 1, 2, \ldots$ such that F_λ maps C_{-k}^λ as a 2-to-1 covering of C_{k-1}^λ just as above. We have $H_\lambda(C_{-k}^\lambda) = C_k^\lambda$. The C_{-k}^λ now converge down to ∂T_λ as $k \to \infty$. And, just as above, C_{-k}^λ contains exactly 2^{k+2} points that are mapped to critical points and the same number to prepoles by F_λ^k.

2 Rings in the Parameter Plane

In this section, we prove that the origin in the parameter plane is surrounded by infinitely many disjoint simple closed curves $\mathcal{S}_k$ with the $\mathcal{S}_k$ converging to 0 as $k \to \infty$. The curve $\mathcal{S}_k$ will consist of parameters for which the critical orbit lands on the critical circle after exactly $k+1$ iterations in a manner specifed below. We shall show that $\mathcal{S}_k$ contains exactly one parameter for which one of the critical points is periodic with period $k+1$. Results in [5] shows that this parameter is a center of the main cardioid of a Mandelbrot set in the parameter plane (with two exceptions noted at the end of this section). And we shall show that there is one other parameter in $\mathcal{S}_k$ for which the critical orbits all land on ∞ at iteration $k+3$. It is known [12] that this parameter is then the center of a Sierpinski hole with escape time $k+3$. The parameters that are centers of a main cardioid of a Mandelbrot set will lie in $\mathbb{R}^+$ while the parameters that are centers of a Sierpinski hole will lie in $\mathbb{R}^-$.

We first describe the ring $\mathcal{S}_0$ in the parameter plane. This curve consists of λ-values for which the critical values lie on the critical circle C_0^λ in the dynamical plane. So, on this set, we must have $|\lambda|^{1/4} = 2|\sqrt{\lambda}|$. Solving this equation shows that $\mathcal{S}_0$ is the circle of radius 1/16 centered at the origin in the parameter plane, i.e., the boundary of $\mathcal{O}$. When $\lambda \in \mathcal{S}_0$, the critical circle C_0^λ is the circle of radius $1/2$ centered at the origin. Note that, as λ rotates around $\mathcal{S}_0$, the critical points and prepoles each rotate around C_0^λ by a quarter of a turn while the critical values rotate by half a turn. It then follows that there is exactly one parameter in $\mathcal{S}_0$ for which the critical values land on a critical point, namely $\lambda = 1/16$, and one other parameter for which they land on a prepole, namely $\lambda = -1/16$. So, for $\lambda = 1/16$, F_λ has a superattracting fixed point while, for $\lambda = -16$, the critical orbit escapes at iteration 3. This gives the result for $\mathcal{S}_0$.

For $\lambda \in \mathcal{O}$ with $0 \leq \operatorname{Arg} \lambda < 2\pi$, let $c_0^\lambda = \lambda^{1/4}$ denote the critical point satisfying $0 \leq \operatorname{Arg} c_0^\lambda < \pi/2$ and let c_j^λ, $j = 1, 2, 3$ denote the other three critical points where the c_j^λ are arranged in the counterclockwise direction around the origin. Let I_0^λ denote the closed sector in $\mathbb{C}$ bounded by the two critical point rays that are given by tc_0^λ and tc_3^λ with $t \geq 0$. Let I_j^λ denote the similar sector bounded by tc_{j-1}^λ and tc_j^λ. Note that the interior of each I_j^λ is mapped one-to-one onto $\mathbb{C}$ minus the two critical value rays given by $\pm tv_\lambda$, $t \geq 1$. One of the critical point rays that bounds each I_j^λ is mapped onto one of these critical value rays while the other critical point ray is mapped to the other critical value ray. Note also that, when $\lambda \in \mathbb{R}^+$, the critical value rays lie in $I_0^\lambda \cap I_1^\lambda = \mathbb{R}^+$ and $I_2^\lambda \cap I_3^\lambda = \mathbb{R}^-$. For all other λ-values, one of the critical value rays lies in the interior of I_1^λ while the other lies in the interior of I_3^λ. See Figure 3.

Since C_0^λ is an actual circle, we may define a natural parametrization $C_0^\lambda(\theta)$ of this curve by setting $C_0^\lambda(0) = c_0^\lambda$. We choose this parameterization so that $C_0^\lambda(\theta)$ rotates in the clockwise direction as θ increases. Here we again assume that $0 \leq \operatorname{Arg} \lambda < 2\pi$. Let γ_0^λ be the portion of C_0^λ that lies inside I_0^λ, i.e., $\gamma_0^\lambda(\theta) = C_0^\lambda(\theta)$ where $0 \leq \theta \leq \pi/2$. Now the sector I_0^λ is mapped over itself univalently (except when $\lambda \in \mathbb{R}^+$ in which case one boundary curve is

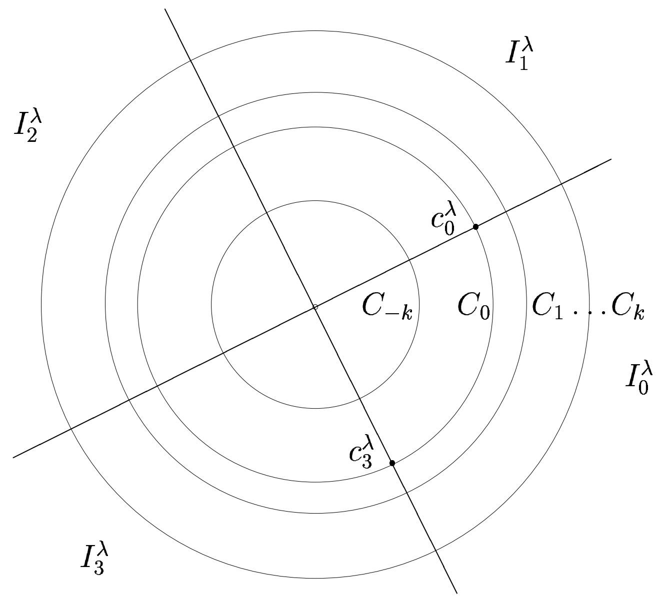

Figure 3: The critical circle and its preimages together with the sectors I_j.

mapped 2-to-1 to a portion of $\mathbb{R}^+$). In all cases there is then a smooth curve γ_1^λ lying in $C_1^\lambda \cap I_0^\lambda$ that is mapped univalently onto γ_0^λ. We define $\gamma_1^\lambda(\theta)$ to be the point on this portion of C_1^λ that is mapped to $\gamma_0^\lambda(\theta)$. Inductively, we then define $\gamma_k^\lambda(\theta)$ to be the point in $C_k^\lambda \cap I_0^\lambda$ for which $F_\lambda(\gamma_k^\lambda(\theta)) = \gamma_{k-1}^\lambda(\theta)$ for each $k \geq 1$. Then we let $\gamma_{-k}^\lambda(\theta) = H_\lambda(\gamma_k^\lambda(\theta))$ where H_λ is the involution $z \to \lambda^{1/2}/z$ that fixes the critical points $\pm c_0^\lambda$. So $F_\lambda(\gamma_{-k}^\lambda(\theta)) = \gamma_{k-1}^\lambda(\theta)$. One checks easily that H_λ interchanges I_0^λ and I_1^λ, so $\gamma_{-k}^\lambda(\theta)$ lies in $I_1^\lambda \cap C_{-k}^\lambda$ for each $k > 0$.

Lemma. *Given $k > 0$, there exists $\lambda^* > 0$ such that, if $|\lambda| \leq \lambda^*$, then both critical values of F_λ lie strictly inside the curve C_{-k}^λ.*

Proof: For $|\lambda|$ sufficiently small, the critical circle C_0^λ has magnitude that is very small. Since $F_\lambda \approx z^2$ away from the origin when $|\lambda|$ is small, we may choose λ_1 so that, if $|\lambda| < \lambda_1$, then the closed curve C_{k-1}^λ lies strictly inside the circle of radius 1/8 surrounding the origin. Also, since $F_\lambda(v_\lambda) = 1/4 + 4\lambda$, we may choose λ_2 so that, if $|\lambda| < \lambda_2$, then $|F_\lambda(v_\lambda)| > 1/8$. Let $\lambda^* = \min(\lambda_1, \lambda_2)$. Then we have that, for each λ inside the circle of radius λ^*, the image of the critical value lies outside the circle of radius $1/8$ and hence outside C^λ_{k-1}. Therefore $\pm v_\lambda$ lies strictly inside the closed curve C^λ_{-k}.

□

We now define the rings $\mathcal{S}_k$ for $k \geq 1$ in the parameter plane. Recall that $\mathcal{O}$ is the set of nonzero parameters for which v_λ lies inside the critical circle.

Proposition. *Suppose $\lambda \in \mathcal{O}$. Fix $k \geq 1$ and θ in the interval $[0, \pi/2]$. Then there is a unique parameter $\lambda = \lambda^k_\theta$ in $\mathcal{O}$ for which a critical value lies at the point $\gamma^\lambda_{-k}(\theta)$. Moreover, λ^k_θ varies continuously with θ and $\lambda^k_0 = \lambda^k_{\pi/2}$ is a parameter in $\mathbb{R}^+$.*

Proof: Since $\mathcal{O}$ is the open disk of radius $1/16$ with the origin removed, we have the universal covering half-plane $\tilde{\mathcal{O}}$ given by $\operatorname{Re} z < \log(1/16)$. We then have two maps defined on $\tilde{\mathcal{O}}$.

The first is a map that we shall denote by $\tilde{V}(\lambda)$. To define this map, let v_λ be the critical value that lies in the upper half plane when $0 < \operatorname{Arg} \lambda < 2\pi$. Clearly, the map $\lambda \mapsto v_\lambda$ is not well-defined on $\mathcal{O}$ since v_λ moves to $-v_\lambda$ as $\operatorname{Arg} \lambda$ rotates from 0 to 2π. However, we can lift this map to a new map $V : \tilde{\mathcal{O}} \to \mathcal{X}$ where $\mathcal{X}$ is the annulus $0 < |z| < 1/2$ so that V agrees with the map $\lambda \mapsto v_\lambda$ when $0 < \operatorname{Arg} \lambda < 2\pi$. Let $\tilde{\mathcal{X}}$ be the universal covering of $\mathcal{X}$. Then we can lift V to a map $\tilde{V} : \tilde{\mathcal{O}} \to \tilde{\mathcal{X}}$. Note that $\tilde{V}$ is an analytic, invertible map on $\tilde{\mathcal{O}}$, and, as the argument of λ increases by 2π in $\mathcal{O}$, the imaginary part of $\tilde{V}$ in $\tilde{\mathcal{X}}$ increases by exactly π.

For fixed values of $k \geq 1$ and $\theta \in [0, \pi/2]$, we also have the map $\lambda \mapsto \gamma^\lambda_{-k}(\theta)$ defined when $0 \leq \operatorname{Arg} \lambda < 2\pi$. Again this map is not well-defined on $\mathcal{O}$, but we can lift it to a new map $L : \tilde{\mathcal{O}} \to \mathbb{C}$ as above. By construction, $L(\lambda)$ is strictly contained inside the annulus $\mathcal{X}$. So we may lift L to a map $\tilde{L} : \tilde{\mathcal{O}} \to \tilde{\mathcal{X}}$.

As we have shown, the point $\gamma_{-k}^{\lambda}(\theta)$ is contained in the sector I_1^{λ} as long as $\theta \in [0, \pi/2]$. And this sector rotates by exactly $\pi/2$ radians as $\operatorname{Arg} \lambda$ increases from 0 to 2π. Moreover, the argument of $\gamma_{-k}^{\lambda}(\theta)$ never increases by π as λ rotates around the origin, since this would imply that this point visited both the positive and negative real axis enroute. Hence the argument of $\tilde{L}(\lambda)$ increases by an amount strictly less than π as the argument of λ increases by 2π.

Now, since $\tilde{V}$ is invertible, we may consider the composition $\Phi = \tilde{V}^{-1} \circ \tilde{L} : \tilde{\mathcal{O}} \to \tilde{\mathcal{O}}$. We claim that there is a unique fixed point for Φ in $\tilde{\mathcal{O}}$. To see this, first note that we may extend both $\tilde{V}$ and $\tilde{L}$ to the boundary $\operatorname{Re} z = 1/16$ of $\tilde{\mathcal{O}}$ and Φ maps this boundary strictly inside $\tilde{\mathcal{O}}$. This follows since γ_{-k}^{λ} always lies strictly inside the critical circle for F_λ, which then lies inside the critical circle for F_μ when μ lies on the boundary of $\mathcal{O}$. But this is the circle $r = 1/2$. Hence there are no fixed points on the boundary of $\tilde{\mathcal{O}}$. Next note that there are no fixed points in the far left half-plane in $\tilde{\mathcal{O}}$. This follows immediately from the previous Lemma. Finally, since the argument of $\tilde{V}$ increases by more than the argument of $\tilde{L}$ as λ rotates around the origin, it follows that Φ must have a fixed point in $\tilde{\mathcal{O}}$ and, by the Schwarz Lemma, this fixed point must be unique. Then the projection of this point into $\mathcal{O}$ is a parameter λ for which a critical value lands on the point $\gamma_{-k}^{\lambda}(\theta)$. This is the parameter λ_θ^k. Since all of the above varies continuously with θ, it follows that $\theta \mapsto \lambda_\theta^k$ traces out a continuous curve in $\mathcal{O}$.

Now when $\lambda \in \mathbb{R}^+$, elementary arguments using real dynamics shows that there is a superstable parameter value for which

$$v_\lambda < c_0^\lambda = F_\lambda^{k+1}(c_0^\lambda) < F_\lambda^k(c_0^\lambda) < \ldots < F_\lambda^2(c_0^\lambda).$$

This is then the parameter λ_0^k. Similarly, when $\operatorname{Arg} \lambda = 2\pi$, the above result shows that we have a similar unique parameter for which v_λ lies in $\mathbb{R}^-$ and then maps onto the same superattracting cycle. This is now the parameter $\lambda^k_{\pi/2}$. But this then implies that $\lambda^k_0 = \lambda^k_{\pi/2}$. Note that these are the only two θ-values for which two "different" λ^k_θ's coincide. Therefore the parameters λ^k_θ lie along a simple closed curve surrounding the origin in the parameter plane.

□

Thus we may define the ring $\mathcal{S}_k$ to be the simple closed curve parametrized by $\theta \mapsto \lambda^k_\theta$.

Corollary. *There is a unique parameter in $\mathcal{S}_k$ for which a critical point lies on a superattracting cycle of period $k+1$ and another unique parameter for which the critical orbits escape at iteration $k+3$.*

Proof: As shown above, the parameter $\lambda^k_0 = \lambda^k_{\pi/2}$ is the unique parameter in $\mathcal{S}_k$ for which $F^k_\lambda(v_\lambda)$ lands on a critical point in the curve γ^λ_0. There is also a unique parameter for which $F^k_\lambda(v_\lambda)$ lands on the prepole in γ^λ_0 and hence the critical orbit escapes at iteration $k+3$. The graph of the real function F_λ shows that this parameter lies in $\mathbb{R}^-$.

□

Remarks:

1. Note that the parameter λ^0_0 is the parameter for which c^λ_0 is a superattracting fixed point and this parameter appears to lie at the center of the main cardioid of a Mandelbrot set that straddles the positive real axis. However, this is not quite a "full" Mandelbrot set, as the tip of the tail (i.e., the parameter corresponding to $c = -2$ for the quadratic Mandelbrot set) lies at the origin, so the dynamics associated to this parameter do not correspond to those for the parameter $c = -2$. We conjecture that this is the only portion of the Mandelbrot set that is missing.

2. Also, the parameter value λ^1_0 does not lie at the center of a main cardioid of a baby Mandelbrot set; rather, this parameter lies at the center of the

period 2 bulb of the above Mandelbrot set.
3. All other parameters λ_0^k do lie at the center of a baby Mandelbrot set that lies inside the main Mandelbrot set on the real axis. This follows from a polynomial-like map construction. See [5] for details.

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