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Complex Dynamics and Symbolic Dynamics

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ABSTRACT. In this paper we describe a remarkable connection between the group of automorphisms of the one-sided d-shift and the topology of the parameter plane for complex polynomial dynamical systems.

1. Introduction

Our goal in this paper is to present an interesting connection between two very different areas of dynamical systems, complex dynamics and symbolic dynamics. Usually, a dynamicist would regard symbolic dynamics as a tool to understand a particular (and usually very complicated) type of dynamical system. In this paper, however, we will take the opposite tack: we will use complex dynamics to explain an interesting topic in symbolic dynamics.

Specifically, we will consider the structure of the parameter space for complex polynomial maps. For a degree $d \geq 2$ polynomial, the parameter space is $\mathbb{C}^{d-1}$, i.e., one complex parameter for each finite critical point of the map. This parameter space divides into two distinct regions. One region corresponds to the set of parameters for which all critical points escape to ∞, the so-called escape locus. The complementary region consists of those parameters for which one or more critical points have bounded orbits. We call this (somewhat inaccurately) the boundedness locus. In the quadratic case, this set of parameters is the well-known Mandelbrot set. Note that, up to homotopy, we have only one non-trivial closed curve in the complement of the Mandelbrot set. In the degree d case, the escape locus is much more complicated from a topological point of view: there are infinitely many topologically distinct curves in this piece of parameter space. That is, the topology of the boundedness locus is much more complicated when the degree is 3 or more.

In complex dynamics, the important object from the dynamics point of view is the Julia set. It is on this set where all of the interesting chaotic dynamics resides. When we choose a parameter from the escape locus, then it is known that the Julia set is homeomorphic to a Cantor set and the dynamics on this set is equivalent

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to the one-sided shift map (described below). The shift map is without doubt the fundamental object in symbolic dynamics. Now when we follow this Cantor set as we traverse a closed curve in the escape locus, we induce a map on the Julia set, the monodromy map. If the loop is homotopically trivial, then the induced map on the Cantor set is the identity map. But if the loop is non-trivial, then so is the induced map on the Cantor set. More importantly, this induced map commutes with the shift map. Thus we get what is called an automorphism of the shift. An important problem in symbolic dynamics was to describe the set (actually, group) of all automorphisms of the shift. This was resolved in 1990, thanks to work of Boyle, Franks, and Kitchens [**BFK**] and Ashley [**Ash**]. Our goal in this paper is to show that all automorphisms of the shift arise from such monodromies in the escape locus.

This leads to a much more difficult unsolved question: what about automorphisms of the two-sided (invertible) shift map? It is known that the group structure here is much more complicated than in the one-sided case. At the end of this paper we discuss some musings about this situation, and suggest that it is the Hénon mapping that may shed some dynamical light on the structure of this group.

2. Automorphisms of the Shift

In this section we describe some preliminary material regarding automorphisms of the one-sided d-shift. Let Σ_d denote the space of sequences on the d symbols $0, 1, \ldots, d-1$. We endow Σ_d with the usual topology, i.e., two sequences are close if their first n entries are the same. Let $\sigma : \Sigma_d \to \Sigma_d$ denote the shift map given by

$$\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots).$$

As is well known, σ is a continuous, d-to-1 map onto Σ_d.

An *automorphism* of the d-shift is a homeomorphism $\phi : \Sigma_d \to \Sigma_d$ that commutes with the shift map, i.e., $\phi \circ \sigma = \sigma \circ \phi$. The collection of all such automorphisms, Aut_d, is clearly a group, and a natural question in symbolic dynamics is to describe the structure of this group. Our goal in this paper is to describe a set of generators for Aut_d.

EXAMPLE 1. Let $\phi : \Sigma_2 \to \Sigma_2$ be the map that interchanges each 0 and 1 in a sequence in Σ_2. Clearly, $\phi \in \mathrm{Aut}_2$. A result of Hedlund [**H**] shows that ϕ is the only non-trivial ($\neq$ identity) automorphism of Σ_2.

EXAMPLE 2. In similar fashion, the map generated by any permutation of the d symbols $0, 1, \ldots, d-1$ generates an automorphism of Σ_d. As mentioned above, Aut_d consists of many other, very different automorphisms when $d > 2$.

EXAMPLE 3. Consider the map ϕ_0 defined on Σ_3 as follows. Given a sequence $s_0 s_1 s_2 \ldots$, if $s_{j+1} = 0$, then the map interchanges 1 and 2 if these digits appear in the s_j slot. If $s_j = 0$, nothing happens. For example

$$\phi_0(10120012100\ldots) = (20110012200\ldots)$$

ϕ_0 is easily seen to be an automorphism of Σ_3. We call the digit 0 a *marker* and the map ϕ_0 a *marker automorphism.*

EXAMPLE 4. Again in Σ_3, we let $\overline{0}$ stand for the digit "not 0," i.e., $\overline{0}$ represents either 1 or 2. Let $\phi_{\overline{0}}$ denote the map of Σ_3 that interchanges a 1 and 2 whenever a

1 or 2 is followed by $\overline{0}$. For example

$$\phi_{\overline{0}}(11200102100\ldots) = (22200101100\ldots)$$

As before, it is easy to check that $\phi_{\overline{0}}$ is a marker automorphism of Σ_3.

EXAMPLE 5. The map ϕ_1 defined on Σ_3 by "interchange 1 and 2" whenever these digits are followed by a 1 is **not** an automorphism of Σ_3 since

$$\phi_1(111\ldots) = (222\ldots) = \phi(222\ldots),$$

so ϕ_1 is not one-to-one.

EXAMPLE 6. There are many different strings of symbols that can serve as markers and thereby generate marker automorphisms. For example, the string 220 generates a marker automorphism ϕ_{220} which interchanges 1 and 2 whenever these digits are followed by 220. For example

$$\phi_{220}(1\,220\,22021\,220\ldots) = (2\,220\,22022\,220\ldots)$$

EXAMPLE 7. We leave it to the reader to check that the strings 100 and $10\overline{0}$ both generate marker automorphisms in Aut_3. However, 101 does not generate an element of Aut_3. Of course, we can generate other marker automorphisms by interchanging the symbols 0 and 2, or the symbols 0 and 1, when they are followed by an appropriate marker.

How can we tell if a given string can serve to generate a marker automorphism? The general principle is that markers cannot "interfere" with previous markers in a string. We will make this more precise below when we give an algorithm to generate all markers for Aut_3.

3. Dynamics of Quadratic Polynomials

We now shift (pardon the pun) gears. In this section we recall some of the elementary properties of complex quadratic polynomials, including a description of their Julia sets and the Mandelbrot set. For more details on this material we refer to [**B1**], [**DK**] [**Mi**].

Let $P_c(z) = z^2 + c$ with $z, c \in \mathbb{C}$. It suffices to deal with quadratic maps in this special form as any quadratic polynomial is conjugate to some P_c. That is, if Q is any quadratic polynomial, we may find an affine map h of $\mathbb{C}$ and a particular c value for which we have $h \circ Q(z) = (h(z))^2 + c$. It follows easily that Q and P_c have the same dynamical structure as h carries Q-orbits to P_c-orbits. For example, the well known "logistic" quadratic polynomial $Q(z) = 4z(1-z)$ is conjugate to $P_{-2}(z) = z^2 - 2$ via the map $h(z) = -4z + 2$, as the reader may easily verify.

DEFINITION 1. The Julia set of P_c, denoted by J_c, is the boundary of the set of points whose orbit escapes to ∞.

Remark. J_c has several other equivalent definitions. For example, J_c is also the closure of the set of repelling periodic points as well as the set of points at which the family of iterates of P_c, $\{P_c^n\}$, fails to be a normal family. As a consequence of this final fact, if $z \in J_c$ and U is any neighborhood of z, then $\cup_n P_c^n(U)$ fills all of $\mathbb{C}$ missing at most one point. This is the famous theorem of Montel.

EXAMPLE 8. $P_0(z) = z^2$ has Julia set equal to the unit circle. Indeed, $P_c^n(z) \to \infty$ if and only if $|z| > 1$.

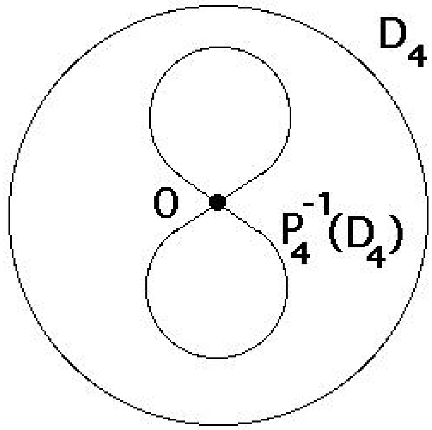

FIGURE 1. The case $c = 4$.

EXAMPLE 9. This second example will be crucial to what follows. Let $c = 4$. We claim that J_4 is a Cantor set. To see this, consider the closed disk D_4 of radius 4 centered at the origin. If $|z| \geq 4$, then $P_4^n(z) \to \infty$. This follows since

$$\begin{aligned} |z^2 + 4| &\geq |z|^2 - 4 \\ &\geq |z|^2 - |z| \\ &\geq (|z| - 1)|z| \\ &\geq 3|z| \end{aligned}$$

Hence J_4 is contained in the interior of D_4. To obtain J_4, we pull this disk back via preimages of P_4.

We claim that $P_4^{-1}(D_4)$ is a figure 8 region contained in the interior of D_4. Indeed, to obtain this preimage, we first subtract 4, obtaining a disk of radius 4 centered at -4. Then we take the square root, obtaining the result shown in Fig. 1. In particular, note that the maximum magnitude of any point in $P_4^{-1}(D_4)$ is $2\sqrt{2}$. Thus the preimage of the open disk $|z| < 4$ is contained in the two interior lobes of this figure eight. Hence J_4 lies in these two lobes which we denote by V_0 and V_1.

Note also that J_4 misses the closed disk $D_{1/2}$ of radius $1/2$ centered at 0, for this disk is mapped to a region that lies completely outside $V_0 \cup V_1$. Indeed, the image is a disk about 4 of radius only $1/4$. Since $|P_4'(z)| > 1$ in the exterior of $D_{1/2}$ it follows that $|P_4'| > 1$ on J_4.

So consider the two simply connected regions, $I_0 = V_0 - D_{1/2}$ and $I_1 = V_1 - D_{1/2}$. We have that $J_4 \subset I_0 \cup I_1$ and that either branch of P_4^{-1} restricted to I_j is a contraction. Let Q_j denote the branch of P_4^{-1} on D_4 taking values in I_j. $Q_j(I_0 \cup I_1)$ is a pair of small simply connected regions in I_j.

Now let $(s_0 s_1 s_2 \ldots) \in \Sigma_2$ and consider the subsets

$$I_{s_0 \ldots s_n} = Q_{s_0} \circ \ldots \circ Q_{s_{n-1}}(I_{s_n})$$

One checks easily that

(1) $I_{s_0 \ldots s_{n+1}} \subset I_{s_0 \ldots s_n}$.

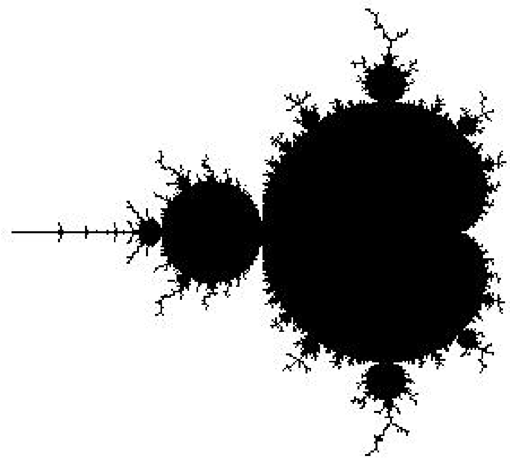

FIGURE 2. The Mandelbrot set.

(2) $h(s_0s_1s_2\ldots) = \bigcap_{n=0}^{\infty} I_{s_0\ldots s_n}$ is a point (this uses the fact that the Q_j are contractions).
(3) h is a homeomorphism of Σ_2 onto J_4.
(4) $h(s_0s_1s_2\ldots) \in J_4$ has itinerary $(s_0s_1\ldots)$ relative to I_0 and I_1.

Remark. The crucial ingredient in example 9 is the fact that the orbit of the critical point tends to ∞. It can be shown that, whenever this occurs, J_c is a Cantor set. See [**B1**], [**Mi**]. On the other hand, if the orbit of 0 does not escape to ∞, then J_c is a connected set. Roughly speaking, this occurs since closed curves outside the Julia set never contain the critical value, and therefore their preimage is never a figure eight. Thus we have the *fundamental dichotomy* of quadratic dynamics: either J_c is a Cantor set, in which case $P_c^n(0) \to \infty$, or J_c is connected, in which case $P_c^n(c)$ is bounded. It is the well-known Mandelbrot set M (see Fig. 2) that gives a picture of this dichotomy: the parameter value c lies in M if and only if J_c is connected.

We now show how automorphisms of the shift arise naturally in complex dynamics. Given our extensive work with P_4 above, the construction is easy.

Consider $c = c(\theta) = 4e^{2\pi i\theta}$ so that c lies on the circle $|z| = 4$. For each such c, the corresponding Julia set is again a Cantor set, since $P_c^{-1}(D_4)$ is again a figure eight region well inside D_4. So we may invoke symbolic dynamics exactly as before. However, suppose we try to let I_0 and I_1 depend continuously on c. Since the I_j are essentially given via branches of the square root, it follows that, as θ moves completely around the circle, I_0 and I_1 only move in half-circles, i.e., they are interchanged. Thus, if we follow a path around this circle and consider a point with initial itinerary $s = (s_0s_1s_2\ldots)$, when we return to the initial c-value, all of the digits in s will have been changed. That is, we induce the only nontrivial automorphism in Aut_2 via this process. We call the map on the Julia set of P_4 induced by this process the *monodromy map.*

4. Polar Coordinates

We pause to introduce the first of the tools necessary for the construction of automorphisms, the Douady-Hubbard theory of external rays.

In a neighborhood of ∞, the value of c is more or less irrelevant and so P_c acts dynamically like z^2. More precisely, in a neighborhood of ∞, P_c is analytically conjugate to the squaring map $P_0(z) = z^2$. It is straightforward to check that the function

$$\phi_c(z) = \lim_{n\to\infty} (P_c^n(z))^{1/2^n}$$

is analytic and satisfies $\phi_c(P_c(z)) = (\phi_c(z))^2$ [**B1**], [**DK**]. Here we choose the principal branch of the square root. This enables us to view the dynamics of P_c near ∞ in "polar coordinates." Following Douady and Hubbard [**DH1**], we define the potential, or rate of escape function, by

$$h(z) = \lim_{n\to\infty} \frac{\log_+ |P_c^n(z)|}{2^n}.$$

The map h is continuous. Via the conjugacy, we see that

$$h(P_c(z)) = 2h(z).$$

Define

$$\Gamma_c = \{z \in \overline{\mathbb{C}} \,|\, h(z) > h(0)\}.$$

One checks easily that ϕ_c extends to a homeomorphism of Γ_c onto the complement of D_R where $R = \exp(h(0)) \geq 1$. If $r > R$, ϕ_c maps the level curve

$$h(z) = \log(r)$$

to the circle $|z| = r$. This gives us the radial component of our polar coordinate system.

For fixed t, the preimage under ϕ_c of the ray $\rho \mapsto \rho \exp(2\pi i t)$, $\rho > R$, is called the external ray of argument t for P_c. We denote this external ray by θ_t. The external rays give the angular coordinate on Γ_c.

We call the value $\log |\phi_c(c)|$ the *escape rate* of the critical value and $\operatorname{Arg} \phi_c(c)$ its external angle.

Note that, if the orbit of 0 escapes under P_c, then we cannot extend the conjugacy ϕ_c further. Indeed, each point in Γ_c has two preimages under P_c with one exception, namely the critical value c. Thus the level curve $h(z) = h(0)$ is a figure 8 curve in essentially the same manner as in the previous section. See Fig. 3

On the other hand, if the orbit of 0 does not escape to ∞, then $h(0) = 0$. Hence we can define ϕ_c everywhere in the exterior of the Julia set. This is the basic idea behind the proof that the Julia set is connected when the orbit of 0 remains bounded. More detail on all of this construction may be found in [**DH1**] or [**B1**], or the AMS Short Course lecture notes [**DK**], [**D**].

Note that the critical value c always lies in Γ_c. Hence $\phi_c(c)$ is always defined. This value plays a special role in complex dynamics. For one thing, $\phi_c(c)$ determines the quadratic polynomial up to affine conjugacy. That is, if two polynomials of the form $z^2 + c$ have the same escape rate and external angle, then these maps are identical. In addition, the mapping $\Phi(c) = \phi_c(c)$ is the external Riemann map that takes the complement of the Mandelbrot set onto the exterior of the unit disk in $\mathbb{C}$. For more details, see [**DH1**].

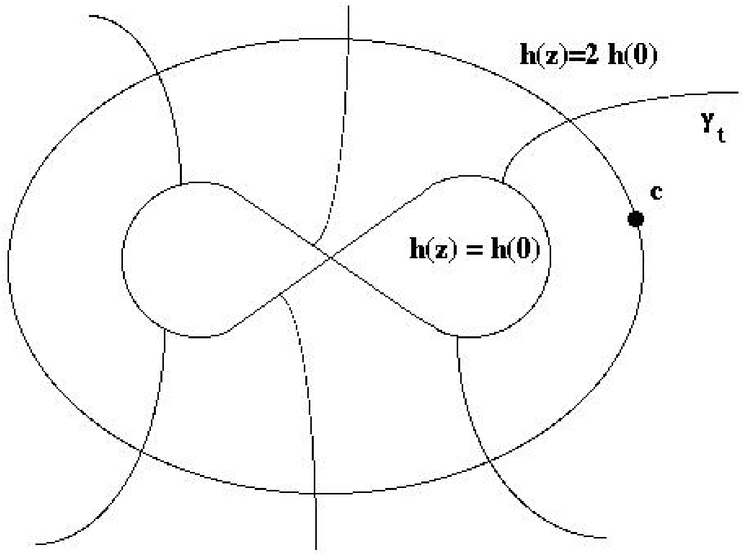

FIGURE 3. Polar coordinates.

5. The Measurable Riemann Mapping Theorem

Now we turn to our second tool, the Measurable Riemann Mapping Theorem (MRMT) of Ahlfors-Bers. This is a difficult and highly technical subject. Our treatment will be somewhat sketchy and without rigor, and will involve only the ideas we need to generate automorphisms of the shift. The idea is to convey just the flavor and power of this important result. Much more detail can be found in [**B1**], [**McM**].

Although this theory is quite general, we will restrict here to maps of the extended plane that are differentiable almost everywhere.

DEFINITION 2. An *ellipse field* or *complex structure* μ on $\mathbb{C}$ is a family of concentric ellipses that fills the tangent plane at each $z \in \mathbb{C}$.

Geometrically, we think of $\mu(z)$ as being a single (non-trivial) ellipse in the tangent plane at z, but technically $\mu(z)$ is the entire family given by multiplying the single ellipse by $t \in \mathbb{R}^+$. In our case, the ellipse fields will always vary smoothly a.e.

EXAMPLE 10. The *standard structure* μ_* on $\mathbb{C}$ is the field for which every ellipse is an actual circle. Note that a holomorphic map preserves μ_* in the sense that the derivative of this map sends circles in the tangent plane to circles in the image tangent plane. Indeed, a map is holomorphic if and only if it preserves μ_*.

EXAMPLE 11. A t-horizontal ellipse field is given by an ellipse at each z whose major axis lies parallel to the x-axis, whose minor axis lies parallel to the y-axis, and for which the ratio of the length of the major axis to the minor axis is $t > 1$.

EXAMPLE 12. Consider an annulus of the form

$$A = \{z \in \mathbb{C} \,|\, 0 < r_1 \leq |z| \leq r_2\}$$

with $r_1 < r_2$. The *Dehn twist map* T on A is given by

$$T(re^{2\pi it}) = r\exp(2\pi i(t + \frac{r - r_1}{r_2 - r_1})).$$

Note that T is the identity on the boundary of A and that T rotates each circle $|z| = r$ with $r_1 < r < r_2$ in the counterclockwise direction by an angle that increases from 0 to 2π as r increases from r_1 to r_2. We define a *twisted* complex structure on $\mathbb{C}$ to be the standard structure outside A and the image of μ_* under the derivative DT on A.

DEFINITION 3. The *dilatation* $d_\mu(z)$ of an ellipse field is the ratio of the length of the major axis to the minor axis.

So $d_\mu(z) \geq 1$ and $d_\mu(z) \equiv 1$ if and only if $\mu = \mu_*$. For the complex structures we consider, d_μ will always be bounded on $\mathbb{C}$.

DEFINITION 4. A homeomorphism $F\colon \mathbb{C} \to \mathbb{C}$ that is smooth almost everywhere is said to *straighten* an ellipse field μ if $DF(\mu) = \mu_*$ almost everywhere.

That is, F straightens μ if the map takes the given complex structure to the standard structure a.e. (It always seems strange that you "straighten" something by making it into a circle, but such is life in conformal geometry.) In the t-horizontal ellipse field above, we can straighten μ by applying the map $F(x, y) = (x/t, y)$. Similarly, we can straighten the twisted complex structure via a map that is the identity outside A and equal to the inverse of T on A. We do not worry about the structure on the boundary of A since we only require straightening a.e. We remark that such a straightening homeomorphism is called a *quasiconformal mapping* or a qc map for short.

In our framework, we can now state the celebrated Measurable Riemann Mapping Theorem of Ahlfors and Bers:

THEOREM 1. *(MRMT)*. Suppose μ is an ellipse field on $\mathbb{C}$ with bounded dilatation. Then there is a homeomorphism F of $\mathbb{C}$ that straightens μ. Moreover, if we specify two values of F, then F is unique.

We mention that our restriction to smooth a.e. ellipse fields and straightening maps is a real restriction; the actual MRMT only requires that the ellipse field be measurable! It is amazing that you can straighten virtually anything with a qc map.

Here is the way that we will make use of the MRMT. Suppose $Q\colon \mathbb{C} \to \mathbb{C}$ is a map that preserves the complex structure μ. And suppose that F straightens μ. Then the mapping $P = F \circ Q \circ F^{-1}$ is a mapping of the complex plane that preserves the standard structure, by the chain rule. Thus P is a holomorphic map. In particular, if we know that Q has finite degree, then P is a polynomial. In cases such as this, we say that Q is *quasiconformally conjugate* to the polynomial P.

6. Spinning the Critical Value

Now let's return to the quadratic example discussed above and use both the polar coordinates and the MRMT to generate the same result. The technique we will use is called *spinning the critical value* and will be the basic construction in the more difficult case of higher degree polynomials.

Throughout this section, we will fix $c \notin M$. We will construct a map that we will later show to be quasiconformally conjugate to P_c. Choose ρ_1, ρ_2 such that

$$h(c) < \rho_1 < \rho_2 < 2h(c)$$

and consider the annular region

$$A = \{z| \ \rho_1 \le h(z) \le \rho_2\}.$$

Let τ be the modified Dehn twist on A defined as follows. Let $r_i = e^{\rho_i}$ for $i = 1, 2$. Let

$$A' = \{z| \ r_1 \le |z| \le r_2\}.$$

Note that ϕ_c maps A onto A'. Define the usual Dehn twist T on the annulus A' as in the previous section. Then set

$$\tau(z) = \phi_c^{-1} \circ T \circ \phi_c(z).$$

Since the annular region A lies between the level sets of the potential containing c and $P_c(c)$, it follows that the inverse image $P_c^{-1}(A)$ is an annular region lying between the level sets of the potential containing 0 and c. Now define

$$F(z) = \begin{cases} P_c(z) & \text{if } z \notin P_c^{-1}(A) \\ \tau \circ P_c(z) & \text{if } z \in P_c^{-1}(A) \end{cases}$$

Clearly, F is a degree two branched cover which differs from P_c only on $P_c^{-1}(A)$. Although F is not a polynomial, the MRMT guarantees that it is quasiconformally conjugate to a polynomial — in fact, to P_c.

Proposition 1. The map F is quasiconformally conjugate to P_c.

Proof. We define a new conformal structure μ on $\overline{\mathbb{C}}$ which is preserved by F as follows. If $h(z) \ge h(c)$, we set $\mu(z) = \mu_*(z)$. We then use F to pull back μ to $\overline{\mathbb{C}} - J_c$. That is, if $h(0) \le h(z) < h(c)$, we define $\mu(z) = DF^{-1}(\mu(F(z)))$. Note that there is no ambiguity here since the region $h(0) \le h(z) < h(c)$ is disjoint from the region $h(z) \ge h(c)$. Note that $\mu = \mu_*$ everywhere in this region except in $P_c^{-1}(A)$. Continuing in this fashion we may define μ for all $z \in \overline{\mathbb{C}} - J_c$. By construction, F preserves μ. Since μ is the pullback of a complex structure with bounded dilatation by a map which is analytic except on $P_c^{-1}(A)$, it follows that μ has bounded dilatation on $\overline{\mathbb{C}} - J(P_c)$.

We extend μ to all of $\overline{\mathbb{C}}$ by setting $\mu = \mu_*$ on J_c. We may now apply the MRMT to obtain a quasiconformal homeomorphism f of $\overline{\mathbb{C}}$ which straightens μ a.e. Consequently, the map

$$Q = f \circ F \circ f^{-1}$$

preserves μ_* a.e. and so is analytic on $\overline{\mathbb{C}}$.

We claim that Q is a quadratic polynomial which is affine conjugate to P_c. To see this, we may normalize f so that $f(\infty) = \infty$ and $f(0) = 0$. Thus, Q is a degree two map which fixes ∞ and which has branch points at 0 and ∞. Since Q preserves μ_*, Q is analytic. It follows that Q is a polynomial of degree two. Now $F = P_c$ on the region $h(z) > h(c)$, and f maps the critical orbit of F to that of Q. Hence the external angles and escape rates of the critical values of P_c and Q agree. Therefore, P_c and Q are equal. $\square$

We now use the same idea to construct a one-parameter family of maps F_t, $0 \le t \le 1$; we "spin the critical value around a level curve of the potential." We show

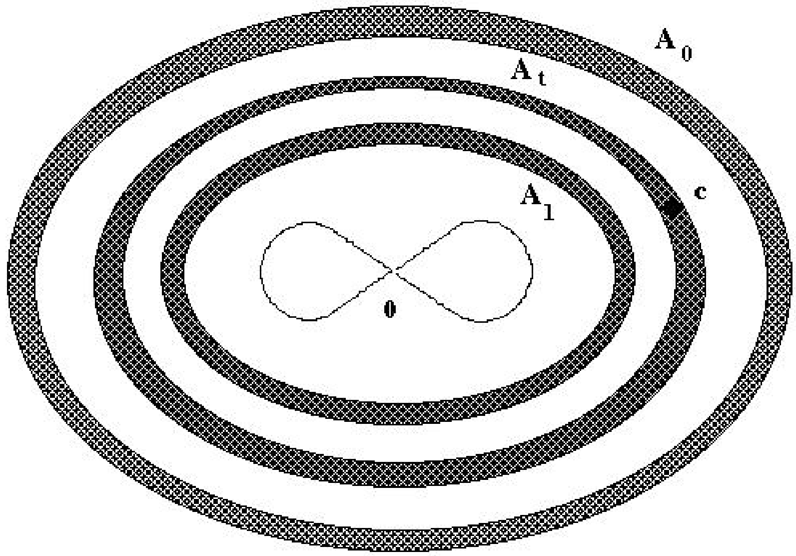

FIGURE 4. The annuli A_t.

that each member of this family is quasiconformally conjugate to a polynomial of the form $z^2 + c(t)$ where $c(t)$ is continuous and winds once around the Mandelbrot set for $0 \leq t \leq 1$. Hence this family induces the nontrivial automorphism of Σ_2 as in the special case $c = 4$ above.

Define a one-parameter family of annuli A_t for $0 \leq t \leq 1$ where $A_0 = \{z \mid \rho_1 \leq h(z) \leq \rho_2\}$ as above. For $0 \leq t \leq 1$, set $\eta_1(t) = (1 - \frac{1}{2}t)\rho_1$ and $\eta_0(t) = (1 - \frac{1}{2}t)\rho_2$. Let A_t denote the annular region

$$A_t = \{z \mid \eta_1(t) \leq h(z) \leq \eta_0(t)\}.$$

That is, A_t is the annular region whose inner and outer boundary curves are the h-level curves $\eta_1(t)$ and $\eta_0(t)$ respectively. Note that $A_0 = A$ and that A_1 is an annular region contained between the level sets $h(0) = \frac{1}{2}h(c)$ and $h(c)$. See Fig. 4. For each t we define a Dehn twist τ_t along the level curves of the potential in A_t exactly as we defined τ on A. Then we set

$$F_t(z) = \begin{cases} P_c(z) & \text{if } z \notin P_c^{-1}(A_t) \\ \tau_t \circ P_c(z) & \text{if } z \in P_c^{-1}(A_t) \end{cases}$$

Note that $F_0 = F$. Note also that there is an interval of t-values for which $c = P_c(0) \in A_t$. As t increases, it follows that the critical value $F_t(0) = \tau_t \circ P_c(0)$ is spun once around the level curve $h(z) = h(c)$. When $t = 1$, the critical value returns to its original location, i.e., $F_1(0) = c$.

PROPOSITION 2. *The map F_1 is also quasiconformally conjugate to P_c.*

Proof. The proof is the same as before: First define a new F_1-invariant complex structure as in the previous proposition. Use the MRMT with the same normalizations to straighten this structure via a quasiconformal homeomorphism f_1. Then $f_1 \circ F_1 \circ f_1^{-1}$ is a polynomial of degree two. Note that f_1 is analytic on

$\{z \mid h(z) > h(c)\}$ and preserves the critical orbit. Hence $f_1 \circ F_1 \circ f_1^{-1}$ is quasiconformally conjugate to P_c as before. □

PROPOSITION 3. *There is a continuous function $c(t)$, $0 \leq t \leq 1$, such that:*

(1) *F_t is quasiconformally conjugate to $z^2 + c(t)$ for each t;*
(2) *$h(c(t))$ is constant; and*
(3) *the external argument of $c(t)$ increases monotonically from 0 to 1 as t increases from 0 to 1.*

Proof. The crucial observation here is that the external rays for P_c are identical to those of F_t in the exterior of $P_c^{-1}(A_t)$. However, the location of the critical value $\tau_t \circ P_c(0)$ relative to these rays changes. Indeed, $\tau_t \circ P_c(0)$ passes through each ray exactly once as t increases. Thus we may invoke the preceding arguments to show that F_t is quasiconformally conjugate to $z^2 + c(t)$ where $c(t)$ satisfies 2 and 3. □

Consequently, the polynomials which are conjugate to F_t lie on a loop that winds once around M. As in the previous section, the monodromy map around this loop induces the non-trivial automorphism of Σ_2.

7. Higher Degree Polynomials

Let P be a polynomial of degree d. The Julia set, J_P, is defined exactly as in the quadratic case: J_P is the boundary of the set of points whose orbits tend to ∞. If the orbits of all critical points tend to ∞, then the Julia set of P is totally disconnected and $P|J_P$ is topologically conjugate to the d-shift. The proof of this is analogous to the proof in the quadratic case; see [**B1**], [**Mi**] for details.

The fundamental dichotomy for quadratic polynomials no longer holds when $d \geq 3$. Nevertheless, there is an intimate relationship between the connectivity properties of the Julia set and the orbits of the finite critical points. For example, if all of these orbits are bounded, then the Julia set is connected. If at least one critical point iterates to infinity, then the Julia set is disconnected, but it may or may not be totally disconnected.

For example, consider the situation when $d = 3$. A generic cubic has two distinct critical points. There are three cases:

(1) Both critical orbits are bounded. Then J_P is connected.
(2) Both critical points tend to ∞. Then J_P is a Cantor set, and the dynamics of the polynomial on J_P is conjugate to the 3-shift.
(3) One critical point tends to ∞ and the other has a bounded orbit. When this happens, the Julia set is disconnected, and it may even be totally-disconnected.

Let X_d be the set of all monic, centered polynomials of degree d. That is, all polynomials of the form

$$z^d + a_{d-2}z^{d-2} + a_{d-3}z^{d-3} + \ldots + a_1 z + a_0$$

where the $a_j \in \mathbb{C}$. As in the quadratic case, any degree-d polynomial is conjugate to a polynomial of this form. Hence the parameter space for X_d has complex dimension $d - 1$, the same as the number of critical points for a "typical" polynomial in X_d.

Let S_d be the escape locus, i.e., the subset of X_d consisting of polynomials whose critical points all escape to infinity. If $P \in S_d$, the Julia set of P is a Cantor set and P is conjugate to the d-shift on $J(P)$. S_d is open in X_d. We will think of the boundedness locus $X_d - S_d$ as the analogue of the Mandelbrot set for degree d

polynomials, although one might argue that this analogue should really be the set of polynomials whose critical orbits are all bounded. Nonetheless, we will consider loops that lie in S_d and circle around various arms of $X_d - S_d$ that extend off to ∞.

To accomplish this, we generalize the spinning construction of the previous section. The ideas here are basically the same as in the quadratic case; the presence of additional critical points adds only some minor technicalities. As in the quadratic case, if $P \in X_d$, there is a neighborhood U of infinity and an analytic homeomorphism $\phi : U \to \overline{\mathbb{C}} - D_r$ such that $\phi \circ P(z) = (\phi(z))^d$. That is, any polynomial in X_d is conjugate near ∞ to the map $z \mapsto z^d$. As for quadratics, define the potential function

$$h_P(z) = \lim_{n\to\infty} \frac{\log_+ |P^n(z)|}{d^n}$$

For $z \in U$, let $h_P(z) = \log|\phi(z)|$. Let $c_1, \ldots, c_{d-1}$ be the finite critical points of P and $v_1, v_2, \ldots, v_{d-1}$ be the corresponding critical values. Then S_d can be characterized by

$$S_d = \{P \in X_d \,|\, h_P(c_i) = \frac{1}{d} h_P(v_i) > 0 \text{ for all } i = 1, \ldots, d-1\}.$$

We will always assume that c_1 is the critical point with the slowest escape rate, i.e., $h_P(c_1) \le h_P(c_j)$ for $j = 2, \ldots, d-1$.

Let $P_i \in S_d$ for $i = 1, 2$, and define

$$U_i = \{z \,|\, h_i(z) \ge h_i(c_1)$$

where h_i, c_1 are the corresponding potentials and lowest critical points for P_i. Here we have suppressed the dependence of c_1 on i.

The next Proposition is proved more or less just as in the quadratic case. See [**BDK**] for more details.

PROPOSITION 4. Let P_i, for $i = 1, 2$, be two polynomials in S_d. If $f : U_1 \to U_2$ is a quasiconformal conjugacy, then f can be extended to a quasiconformal conjugacy defined on all of $\overline{\mathbb{C}}$.

Now we move to the spinning construction for higher degree polynomials. Let $P \in S_d$ and let h be the associated potential. Suppose that $h(c_1) < h(c_j)$ for $j = 2, \ldots, d-1$. Thus there exists $\epsilon > 0$ such that c_1 is the only critical point in the region $\{z \,|\, h(z) \le h(c_1) + \epsilon\}$. We also assume that $d^n h(c_1) \ne h(c_j)$ for all $j > 1$ and n. These two assumptions serve to isolate the level sets of the potential corresponding to the orbit of c_1 from all of the other critical levels. Since these level sets are dynamical invariants, this in turn allows us to guarantee the existence of quasiconformal conjugacies during the spinning construction.

Let γ denote the component of the level set $h^{-1}(h(v_1))$ which contains v_1. There are precisely $d-1$ components of $P^{-1}(\gamma)$. We denote them by $\alpha_1, \ldots, \alpha_{d-1}$ and assume that α_1 is the component containing c_1. Hence α_1 is a figure eight, while the remaining α_j are simple closed curves. We may choose $\epsilon > 0$ small enough so that the region

$$N = \{z | h(v_1) - \epsilon \le h(z) \le h(v_1) + \epsilon\}$$

is disjoint from all of the other critical level curves

$$\{z \,|\, h(z) = \frac{1}{d^m} h(v_k)\} \; m = 1, 2, \ldots.$$

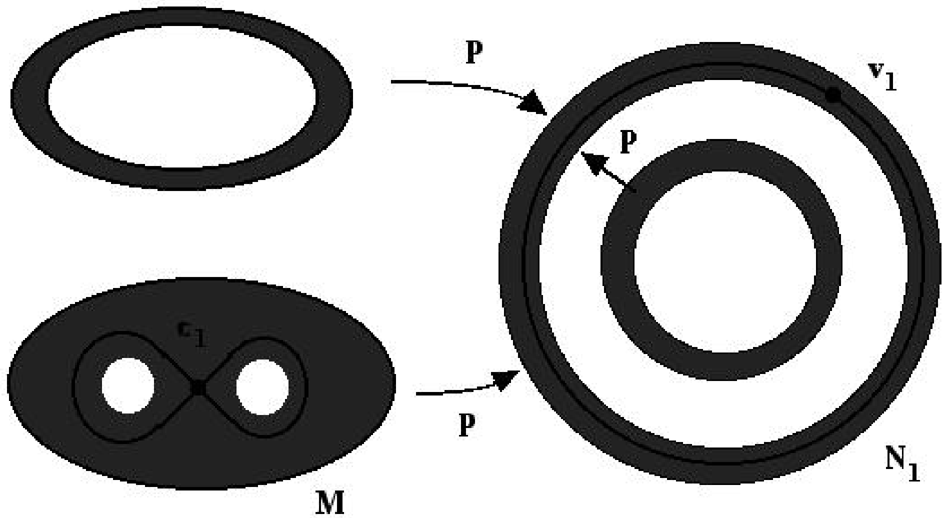

FIGURE 5. The components of $P^{-1}(N_1), M, N_1$.

Let N_1 denote the component of N containing v_1 (and hence γ). It follows that N_1 is an annular region. Let M denote the component of $P^{-1}(N_1)$ that contains c_1. See Fig. 5.

In analogy with the construction in the quadratic case, we will modify P on M. For each $t \in [0,1]$, let A_t be the annular region

$$A_t = \{z \in N_1 \,|\, h(v_1) - \epsilon t \leq h(z) \leq h(v_1) + \epsilon(1-t)\}$$

Note that A_1 is an annular region bounded by γ on the outside while A_0 is bounded by γ on the inside.

As in Section 6, there is a conformal map

$$\phi_t : A_t \to A'_t = \{\xi \,|\, 1 \leq |\xi| \leq r_0(t)\}$$

and we may choose ϕ_t so that these maps depend continuously on t. Let

$$T_t : A'_t \to A'_t$$

be a Dehn twist, i.e.,

$$T_t(re^{2\pi i s}) = r\exp(2\pi i(s + \frac{r-1}{r_0 - 1})).$$

Let $\tau_t = \phi_t^{-1} \circ T_t \circ \phi_t$. Now define

$$F_t(z) = \begin{cases} P(z) & z \notin M \cap P^{-1}(A_t) \\ \tau_t \circ P(z) & z \in M \cap P^{-1}(A_t) \end{cases}$$

PROPOSITION 5. *For all $t \in [0,1]$, there is a polynomial Q_t such that Q_t is quasiconformally conjugate to F_t.*

Proof. As above, we define a new conformal structure μ_t which is preserved by F_t. Let μ_* be the standard conformal structure. We set $\mu_t = \mu_*$ in the region

$$\{z \,|\, h(z) > h(v_1) - \epsilon(1-t)\}.$$

Let μ_t be the pullback of μ_* by F_t. This defines μ_t everywhere except on J_P. By definition, F_t preserves μ_t.

We define μ_t on J_P by setting $\mu_t = \mu_*$. Since μ_t is given by pulling back a complex structure with bounded dilatation by a polynomial, it follows that μ_t is a complex structure with bounded dilatation as well.

Now apply the MRMT. This gives a quasiconformal homeomorphism f_t which straightens μ_t. We may normalize so that $f_t(\infty) = \infty$. Then we have $f_t \circ Q_t \circ f_t^{-1} = Q_t$ is an analytic map of degree d which has a super attracting fixed point at ∞. Therefore, Q_t is a polynomial. If we further normalize so that $f_t'(\infty) = 1$ and $f_t(0) = 0$, then it follows that Q_t is monic and centered. □

PROPOSITION 6. *In the above construction, both* $Q_0 = Q_1 = P$.

Proof. We will prove this by writing down an explicit conjugacy g_i between F_i and P for $i = 0, 1$. These conjugacies will be quasiconformal. Moreover, they will vanish at 0 and be equal to the identity on a neighborhood of ∞. Hence they are the conjugacies that we obtained by the MRMT in the above argument.

We begin with F_0. Define $\Gamma_r = \{z \,|\, h(z) > r\}$. Define g_0 to be the identity map on $\Gamma_{h(c_1)+\epsilon}$. Clearly, we have $g_0 \circ F_0 = P \circ g_0$ on this region.

Let $B \subset M$ be the component of $P^{-1}(A_0)$ which contains c_1. The interior of B is an open annulus. There are two smooth maps $\zeta_i : B \to B$, $i = 1, 2$, that satisfy $P \circ \zeta_i = \tau_0 \circ P$. Each of the ζ_i fixes one of the boundary curves of B and rotates the other by a half twist. We choose τ_0' to be the ζ_i which fixes the outer boundary of B. Set $g_0 = \tau_0'$ on B. Also define g_0 to be the identity on $\Gamma_{h(c_1)} - B$. It follows that $P \circ g_0 = g_0 \circ F_0$. We now extend g_0 to lower h-levels in the natural way. If z satisfies

$$\frac{1}{d} h(c_1) \leq h(z) < h(c_1)$$

we set $g_0(z) = P^{-1} \circ g_0 \circ F_0(z)$ where we choose the appropriate branch of the inverse of P^{-1} to make g_0 continuous.

It is important to note that $P^{-1} \circ F_0$ is not the identity on the two components of the interior of the figure eight component of α_1. Indeed, g_0 interchanges these two components while preserving the interiors of all other components.

Now continue as in the quadratic case. This defines a quasiconformal homeomorphism of $\mathbb{C} - J_P$ which extends quasiconformally to all of $\overline{\mathbb{C}}$ as in Proposition 4. Hence P is quasiconformally conjugate to F_0 via a conjugacy that is the identity on a neighborhood of ∞. Furthermore, g_0 fixes the critical points of P. Hence $P = Q_0$.

We now turn to Q_1. We remark that Q_1 is affine conjugate to P by Proposition 5 since f_1 is conformal on $\Gamma_{h(c_1)}$. We prefer, for later purposes, to construct f_1 directly. To do this, define g_1 = identity on $\Gamma_{h(c_1)}$. Let B be the component of $P^{-1}(A_1)$ which contains c_1. Note that, unlike the previous case, the interior of B consists of two disjoint annuli each mapped isomorphically onto B by P. We can choose a map τ_1' on each of these components so that $P \circ \tau_1' = \tau_1 \circ P$ since P is an isomorphism on each component. Hence we set $g_1 = \tau_1'$ on B. Define g_1 to be the identity on $\Gamma_{h(c_1)-\epsilon} - B$ and then continue as before. It follows again that g_1 is a quasiconformal homeomorphism conjugating P and Q_1. Note that, unlike the previous case, g_1 preserves all components of $\mathbb{C} - \Gamma_{h(c_1)-\epsilon}$. □

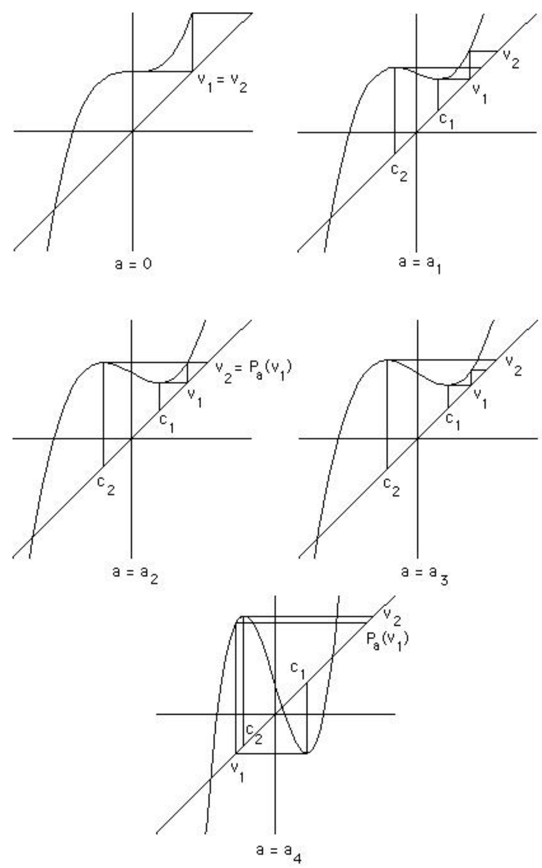

FIGURE 6. Graphs of the P_a.

COROLLARY 1. *As t decreases from 1 to 0, the components of the interior of $h_{Q_t}^{-1}(h(c_1))$ containing c_1 are interchanged.*

8. The Cubic Case: Four Examples

Rather than give the general construction of automorphisms in the cubic case, we will restrict our attention to a number of special but nonetheless illustrative examples. All are contained in the one-parameter family

$$P_a(z) = z^3 - 3a^2 z + 5$$

where $a \in \mathbb{R}$. The critical points of P_a are $\pm a$ and the value of the constant term is of little significance other than its being large enough to display all of the desired cases. The five examples that we discuss are defined by the graphs in Fig. 6. In particular, the a-values are determined by these graphs. Let $h_a(z)$ denote the corresponding potential function.

We show how to generate specific automorphisms of the 3-shift by applying the spinning construction to each of these examples.

Case 1: $a = 0$**.** Note that $P_0(z) = z^3 + 5$ has a unique critical point at 0. The level sets for the potential for P_0 containing the critical point and value are shown in Fig. 7. As in the quadratic case, the critical value lies on a simple closed curve. The preimage of this curve is mapped in three-to-one fashion onto this curve, except, of course, at 0, which is the only preimage of 5. Hence the preimage is a curve which has three lobes pinched at 0. We denote the 3 components of $h_0(z) < h_0(0)$ by 0, 1 and 2 respectively. This is analogous to the quadratic case, where we labeled the two components I_0 and I_1. Note that there is an arbitrary choice involved in this selection. If we now use the spinning construction of the previous section to spin the critical value around the level set $\gamma = h_0^{-1}(h_0(5))$, then we induce a one-third turn on the lobes of $\alpha = h_0^{-1}(h_0(0))$, and this yields an automorphism that cyclically permutes the symbols. This, of course, follows just as in the quadratic case. Now, however, by changing a, we can separate the critical points and thereby change the configuration of the level sets.

Case 2: $a = a_1$**.** According to Fig. 6, the orbit of c_1 escapes a little more more slowly than the orbit of c_2 when $a = a_1$. More precisely, we see that

$$h_a(c_1) < h_a(c_2) < h_a(v_1).$$

The corresponding critical levels of the potential are shown in Fig. 8. The critical values now lie on distinct level sets, each of which is a simple closed curve. Preimages of level sets for which $h_a(z) > h_a(v_2)$ are simple closed curves. When the value of the potential reaches $h_a(v_2)$, the preimage becomes pinched at the critical point c_2. One of the lobes inside this level curve is mapped in two-to-one fashion onto $h_a(z) \le h_a(v_2$. This corresponds to a region containing two of the symbols from the previous case, say 1 and 2. Thus we denote this curve by $\overline{0}$. This is the lobe containing c_1. The other lobe is mapped in one-to-one fashion onto $h_a(z) \le h_a(v_2)$; we denote this lobe by 0.

Now the preimage of the level curve containing v_1 consists of two distinct curves. One is a figure eight curve that contains c_1 and lies in $\overline{0}$; the other is a simple closed curve in 0. The region $\{z \,|\, h_a(z) < h_a(c_1)\}$ may again be used to describe the symbolic dynamics on the Julia set. In analogy with the quadratic case, this labeling indicates the existence of a homeomorphism between the Julia set J and Σ_3 where each point in J contained in the disk labeled 0 (or 1 or 2 respectively) is labeled with a sequence starting with the symbol 0 (or 1 or 2 respectively).

If we now spin the critical value v_1 around the curve $\gamma = h_a^{-1}(h_a(v_1))$, the resulting automorphism simply interchanges every 1 and 2; 0 is left fixed.

Case 3: $a = a_2$**.** This is the special case where $v_2 = P_a(v_1)$ and

$$h_a(c_1) < h_a(c_2) = h_a(v_1) < h_a(v_2).$$

As above, the level set $h_a(z) = h_a(c_2)$ is a figure eight curve with one lobe (previously called $\overline{0}$) mapped in two-to-one fashion onto the image, and the other mapped in one-to-one fashion. By assumption, v_1 lies on this curve, but not at c_2. Let us assume that v_1 lies on the boundary of the lobe containing c_1. (Actually, given the graph of P_a, this must be the case.) This forces some additional pinching in the level set $h_a(z) = h_a(c_1)$; we get pinching at c_1 as well as at the two preimages of c_2. That is, the region $h_a(z) \le h_a(c_1)$ now consists of four lobes in $\overline{0}$. Two of these lobes are mapped to the lobe called 0; these we denote by 10 and 20. The other two lobes map onto $\overline{0}$; these we denote by $1\overline{0}$ and $2\overline{0}$. These level curves are depicted in

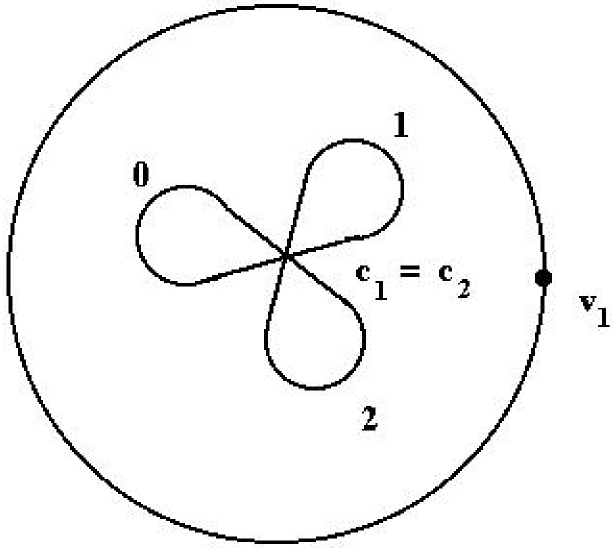

FIGURE 7. Level curves for $a = 0$.

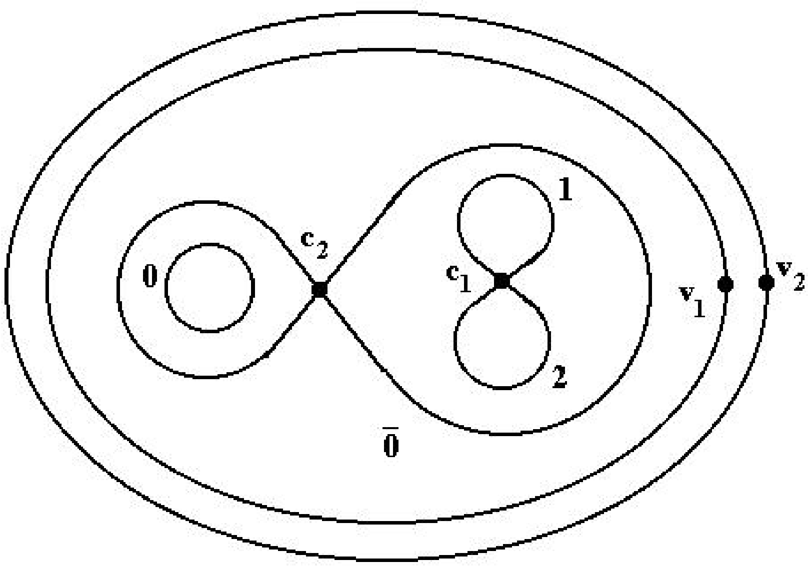

FIGURE 8. Level curves for $a = a_1$.

Fig. 9. In the region 0, the preimage of $h_a(z) = h_a(c_2)$ is a figure eight curve which is mapped in one-to-one fashion onto $h_a(z) = h_a(c_2)$. Thus one of these lobes is denoted 00, while the other (the lobe mapped onto $\overline{0}$) is denoted $0\overline{0}$.

We will not spin the critical value v_1 at this level as the critical value does not lie on a simple closed level curve.

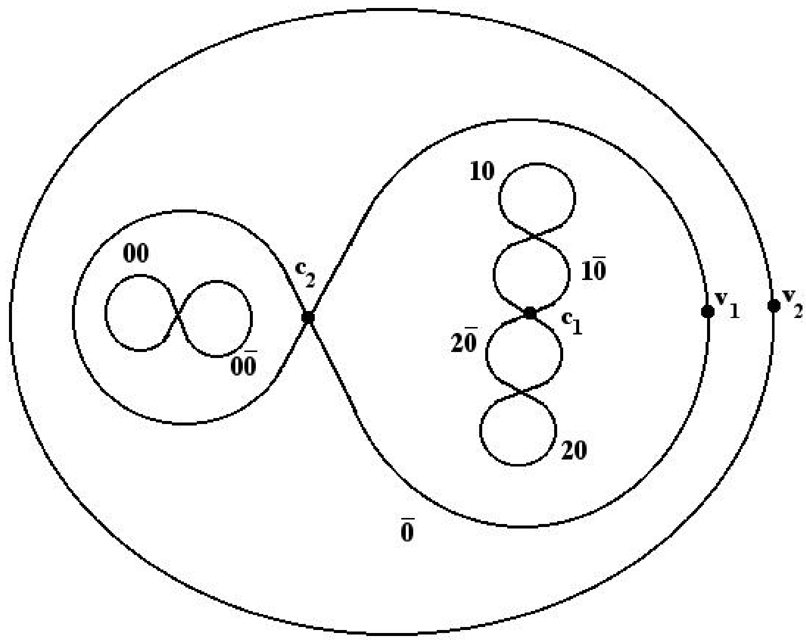

FIGURE 9. Level curves for $a = a_2$.

Case 4: $a = a_3$**.** Now suppose we change the parameter a so that

$$h_a(v_1) < h_a(c_2) < h_a(P_a(v_1)) < h_a(v_2).$$

From the graph, it follows that v_1 now lies slightly inside the level curve containing c_2. The level set $h_a(z) = h_a(v_1)$ now consists of two simple closed curves, one inside each lobe of $h_a(z) = h_a(c_2)$. Then the multiply pinched critical level curves of $h_a(z)$ then break apart as shown in Fig. 10. Inside $\overline{0}$, we have a figure eight curve containing c_1 and bounding the regions $1\overline{0}$ and $2\overline{0}$. This region is mapped onto the level set containing v_1. There are two other preimages inside $\overline{0}$, namely 10 and 20; these are mapped onto the other component of $h_a(z) = h_a(v_1)$, specifically the one in the region 0.

If we now spin the critical value v_1 around its level curve, then only the components of the Julia set marked $1\overline{0}$ and $2\overline{0}$ are interchanged; the components 10 and 20 are left fixed. Thus, spinning induces the automorphism given by the marker set $\overline{0}$: interchange 1 and 2 whenever followed by 1 or 2.

At this point we note that we have reached the polynomial P_{a_4} by simply "pushing" the critical value v_1 down the real axis. Note that $h_a(v_1)$ decreases during this process. This will be the essential ingredient in the next section when we show how to perform this "pushing deformation" in general.

9. The pushing deformation.

In passing from $a = a_1$ to $a = a_3$ in the previous examples, we pushed the critical value v_1 down the real axis across the level set $h_a(z) = h_a(c_2)$. This is

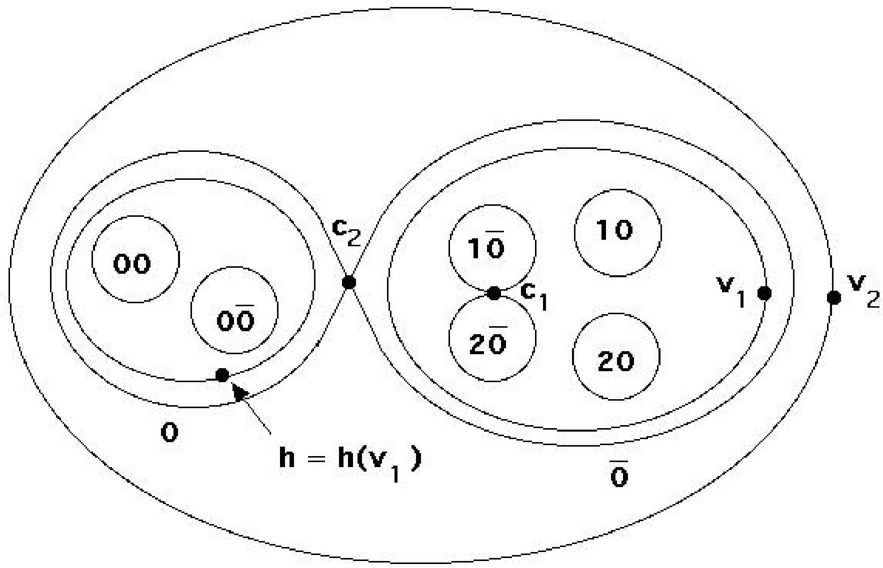

FIGURE 10. Level curves for $a = a_3$.

the second construction involving the MRMT. In the general case we will push the critical value v_1 through the level set $h(z) = 3^{-k}h(c_2)$ for each $k \geq 0$.

To be precise, let r satisfy $h_a(c_2) < h_a(v_1) < 3r < h_a(v_2) < 9r$, or, equivalently, $h_a(c_1) < r < h_a(c_2) < h_a(v_1) < 3r$. Consider the region

$$S = \{z \,|\, r \leq h_a(z) \leq 3r\}.$$

Note that both v_1 and c_2 lie in S. See Fig. 11. We can construct a homeomorphism f of S that takes v_1 to a point that lies inside one of the components defined by the $\overline{0}$ lobe of the level curve $h_a(z) = h_a(c_2)$. Let S' denote the preimage of S that contains c_1

Then define

$$F(z) = \begin{cases} P_a(z) & \text{if } z \in \overline{\mathbb{C}} - S' \\ f \circ P_a(z) & \text{if } z \in S'. \end{cases}$$

Note that $F(z)$ is conformal outside of S'. Next we define (in the usual manner) an F-invariant quasiconformal structure on $\overline{\mathbb{C}} - S'$. We pull this structure back to $\overline{\mathbb{C}} - J_{P_a}$ using F. Finally, we extend the structure to all of $\overline{\mathbb{C}}$ by setting it equal to the standard structure on J_{P_a}. This structure has bounded dilatation, since F is conformal except on S' and the pullback from S to S' introduces only a finite amount of distortion. By applying the MRMT to this new F-invariant structure, we obtain a quasiconformal homeomorphism $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $f^{-1} \circ F \circ f$ is a polynomial P_f. Now, given P_f, we can perform the spinning construction as above to make v_1 spin as in Case 4.

It is important to note that the above construction is analytic in its parameters (see [AB]). For example, if we apply it to a one-parameter family f_t which varies analytically in t, then the resulting polynomials P_{f_t} also vary analytically in t. In

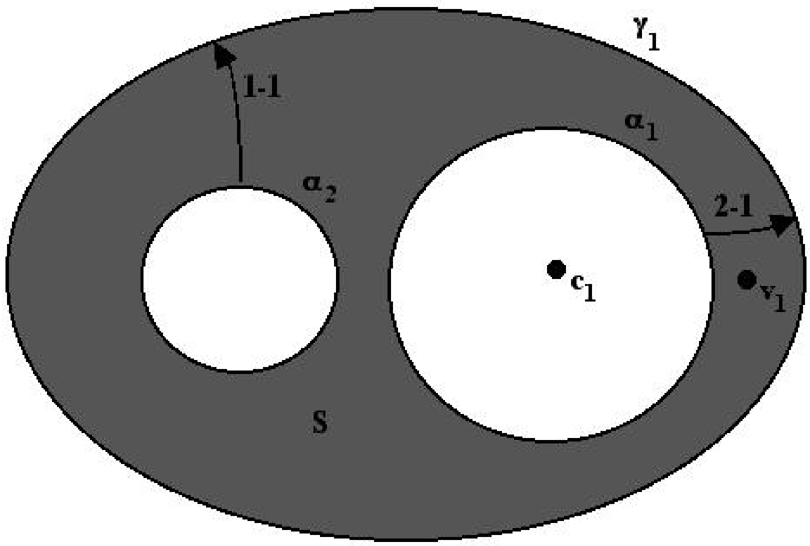

FIGURE 11. The region S.

essence, the pushing construction allows us to deform the polynomial continuously using a quasiconformal deformation of the Riemann surface S.

Now, using the pushing deformation, we could have equally well moved v_1 into the other component defined by $h(z) = h(v_1)$. This brings us to our last example.
Case 5: $a = a_4$. The levels for this a-value satisfy

$$h_a(v_1) < h_a(c_2) < h_a(P_a(v_1)) < h_a(v_2)$$

as before, but note that the critical value v_1 is now in the other component of $h(z) = h(v_1)$. The level sets for h_a inherit a different marking in this case. See Fig. 12.

Spinning the critical value here interchanges the components marked 10 and 20, i.e., it induces the automorphism with marker 0.

Exercise. In cases 4 and 5 above, continue pushing the critical value deeper and identify the corresponding automorphisms that are generated. Specifically, first consider the case where $h_a(P_a(v_1)) = h_a(c_2)$ and sketch the level sets of the potential. Then investigate what happens when the critical value descends further.

If we continue inductively by pushing and spinning, we create a tree whose edges correspond to the pushing deformation and whose vertices correspond to polynomials in the escape locus from which we begin the spinning process. This tree thus generates a collection of automorphisms. Using results of Ashley [**Ash**], we have proved in [**BDK**]:

THEOREM 2. The tree generated by following all possible topologically distinct paths in the cubic escape locus yields a complete set of generators for Aut_3. A similar statement holds for Aut_d.

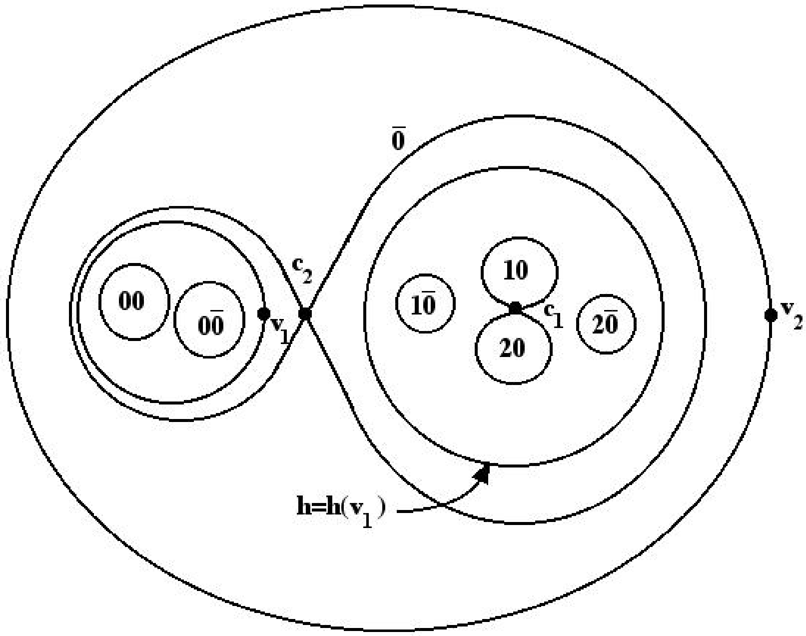

FIGURE 12. Level curves for $a = a_4$.

Table 1 lists the minimal markers and marker sets of length j for $j = 1, 2, 3, 4$. These correspond to the vertices of T at depth j

Length 1:	0	$\overline{0}$				
Length 2:	00	$0\overline{0}$	10	20	$\overline{00}$	
Length 3:	000	$00\overline{0}$	$0\overline{0}0$	$01\overline{0}$	$02\overline{0}$	100
	$10\overline{0}$	200	$20\overline{0}$	110	120	210
	220	$\overline{000}$				
Length 4:	0000	$000\overline{0}$	$00\overline{0}0$	$002\overline{0}$	$001\overline{0}$	0100
	0200	$0\overline{00}\,\overline{0}$	$01\overline{0}0$	$011\overline{0}$	$012\overline{0}$	$02\overline{0}0$
	$021\overline{0}$	$022\overline{0}$	1000	$100\overline{0}$	1010	1020
	$10\overline{00}$	2000	$200\overline{0}$	2010	2020	$20\overline{00}$
	1100	$110\overline{0}$	1200	$120\overline{0}$	2100	$210\overline{0}$
	2200	$220\overline{0}$	1110	1210	2110	2210
	1120	2120	1220	2220	$\overline{0000}$	

Table 1

10. Conclusion

The natural question that arises involves the two-sided d-shift map. In this case the sequence space is the set of two sided sequences $(\ldots s_{-2}s_{-1}.s_0s_1s_2\ldots)$ where $0 \leq s_j \leq d-1$, and the shift map σ really is a shift in this case, as σ simply shifts the decimal point one spot to the right:

$$\sigma(\ldots s_{-2}s_{-1}.s_0s_1s_2\ldots) = (\ldots s_{-2}s_{-1}s_0.s_1s_2\ldots)$$

The question here is: What is the structure of the group of automorphisms of this shift? It is known that, even in the case $d = 2$, this is a very large group.

We conjecture that the answer to this question is intimately related (in the case $d = 2$) to the family of maps known as the Hénon map. This family of maps is given by

$$\begin{aligned} x_1 &= A - By_0 - x^2 \\ y_1 &= x. \end{aligned}$$

Note that the Jacobian of this map is identically equal to B; this is what makes this map special. In the plane, it is known that, provided A is large enough, this map has an invariant set homeomorphic to a Cantor set on which the dynamics are equivalent to the two-sided shift map. See [**DN**]. Indeed, this is a simple nonlinear version of the classic Smale horseshoe map. If we move to $\mathbb{C}^2$ (making the two parameters A and B as well as x_0 and y_0 complex, then Hubbard and Oberste-Vorth [**HOV**] have shown that similar results apply. Moreover, we can now consider the set of parameters for which we have such a Cantor set (the analogue of the escape locus, although there are no critical points around). As before, moving around loops in this locus also induces non-trivial automorphisms of the two-sided shift. The big question then is: Can we generate all automorphisms of the two-sided shift in this way?

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