Cantor Sets of Circles of Sierpinski Curve Julia Sets

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Abstract

Our goal in this paper is to give an example of a one-parameter family of rational maps for which, in the parameter plane, there is a Cantor set of simple closed curves consisting of parameters for which the corresponding Julia set is a Sierpinski curve. Hence the Julia sets for each of these parameters are homeomorphic. However, each of the maps in this set is dynamically distinct from (i.e., not topologically conjugate to) any other map in this set (with only finitely many exceptions). We also show that, in the dynamical plane for any map drawn from a large open set the connectedness locus in this family, there is a Cantor set of invariant simple closed curves on which the map is conjugate to the product of certain subshift of finite type with the maps $z \mapsto \pm z^n$ on the unit circle.

1 Introduction

One of the most interesting planar sets from a topological point of view is the *Sierpinski curve.* A Sierpinski curve is any planar set that is homeomorphic to the well-known Sierpinski carpet fractal. There is an equivalent topological characterization of these sets due to Whyburn [15]: any planar set that is compact, connected, locally connected, nowhere dense, and has the property that each pair of complementary domains is bounded by disjoint simple closed curves is known to be a Sierpinski curve. From a topological point of view these sets are important since they are universal planar sets in the sense that they contain a homeomorphic copy of any compact, connected, one-dimensional set.

Sierpinski curves have been shown to occur as the Julia sets of many different types of rational maps [4], [8], [11], [16]. We describe some of these possibilities in the next section. In particular, for most of these examples, the Sierpinski curve Julia sets occur in a structurally stable setting. That is, these sets arise in a specific family of rational maps and any small enough perturbation of such a map within this family yields a Julia set that is also a Sierpinski curve and, moreover, the two maps are topologically conjugate on their Julia sets.

In this paper, we give a very different example of a family of rational maps that has Sierpinski curve Julia sets. The family is $F_\lambda(z) = z^3 + \lambda/z^3$. Our main result is:

Theorem. *There is a Cantor set $\mathcal{C}$ of simple closed curves in the λ-plane for the family F_λ having the following properties:*

1. *For any $\lambda \in \mathcal{C}$, the Julia set of F_λ is a Sierpinski curve;*
2. *Given any $\lambda \in \mathcal{C}$, there are at most finitely many $\mu \in \mathcal{C}$ such that F_λ and F_μ are topologically conjugate;*
3. *If λ and μ lie on different curves in $\mathcal{C}$, then F_λ is not topologically conjugate to F_μ.*

In particular, it follows that none of the maps corresponding to parameters in $\mathcal{C}$ are structurally stable within the family F_λ, for small perturbations change the dynamics on these Sierpinski curve Julia sets.

The motivation for this paper comes from a paper of McMullen [9] in which he shows that, for any map of the form $G_\lambda(z) = z^n + \lambda/z^d$, provided that λ is small enough and $1/n + 1/d < 1$, the Julia set of G_λ is a Cantor set of simple closed curves. In particular, for families of the form $F_\lambda(z) = z^n + \lambda/z^n$ (i.e., $n = d > 2$), it is known [3] that the set of parameters with this type of Julia set lies in a simply connected open disk that surrounds the origin in the λ-plane and that is bounded by a simple closed curve. This region in the parameter plane is called the McMullen domain and is denoted by $\mathcal{M}$. If $\lambda \in \mathcal{M}$, it is known that the restriction of F_λ to the Julia set is topologically conjugate to the product of the full one-sided shift map on two symbols with the map $\theta \mapsto \pm n\theta$ on the unit circle. See Figure 1 for a picture of the parameter plane in the case $n = 3$ together with several magnifications of the McMullen domain. In this Figure, there are infinitely many closed curves surrounding $\mathcal{M}$ that pass through the centers of a collection of disks; these are the Sierpinski holes described later. Along these curves also lie the centers of many small copies of the Mandelbrot set. This structure was explained in [5]. The Cantor set of simple closed curves in the above Theorem lies in the regions between between these circles.

To prove the result about the parameter plane in the case $n = 3$, we first prove the following:

Theorem. *There is an open neighborhood $\mathcal{O}$ in the parameter plane for $F_\lambda(z) = z^n + \lambda/z^n$ where $n \geq 3$ that strictly contains the closure of $\mathcal{M}$ and has the property that, if $\lambda \in \mathcal{O}$, then there is an invariant subset of the Julia set that is homeomorphic to a Cantor set of simple closed curves. On this*

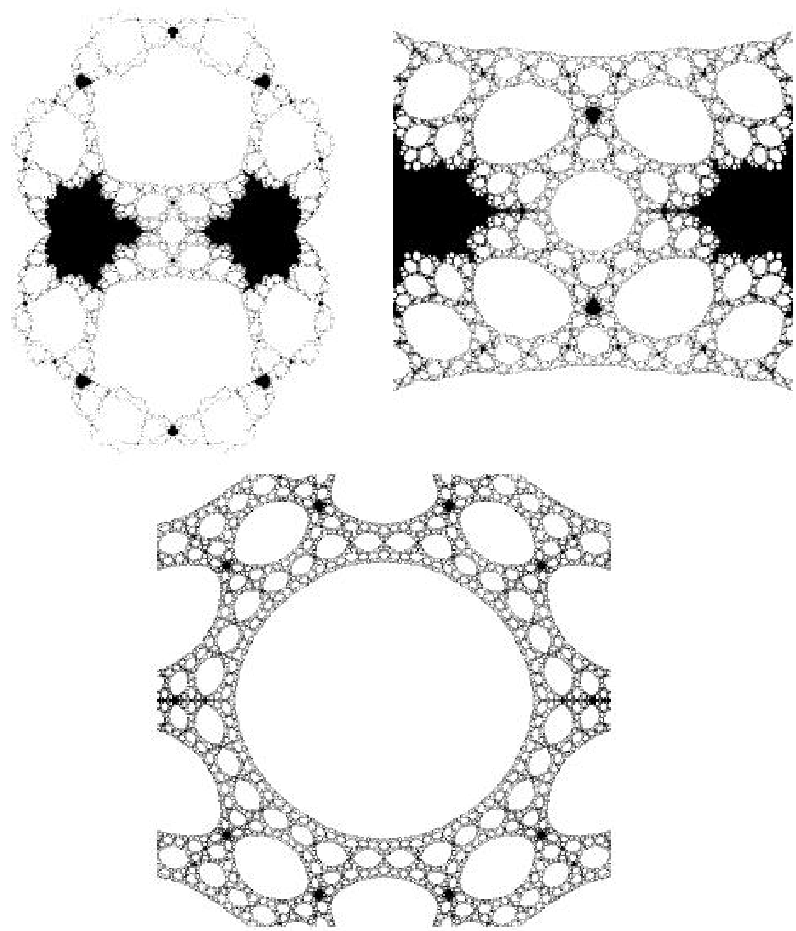

Figure 1: The parameter plane for the family $z^3 + \lambda/z^3$ and several magnifications. The central white disk is the McMullen domain $\mathcal{M}$. The simple closed curves in the Theorem accumulate on the boundary of $\mathcal{M}$.

invariant subset, the map is now conjugate to the product of a certain subshift of finite type on the space of one-sided sequences of two symbols and the map $\theta \mapsto \pm n\theta$ on the unit circle.

That is, within the McMullen domain, the entire Julia set consists of simple closed curves that are interchanged by F_λ in a manner prescribed by the full shift on two symbols. Just outside of $\mathcal{M}$, some of these closed curves disappear, but an uncountable collection of them persist (and other structures arise). On this set of closed curves, we can identify specifically the subshift matrix that governs how the curves are interchanged by F_λ. Then we can use this information to show how, at least in the case $n = 3$, this entire scenario may be translated over to the parameter plane.

We remark that, because of the above result in the dynamical plane, it should be relatively straightforward to extend the parameter plane result in the case where $n > 3$. Certain technical estimates giving the location of the Cantor set of simple closed curves in the dynamical plane will have to be modified in the general case.

2 Preliminaries

Let $F_\lambda(z) = z^n + \lambda/z^n$ where $\lambda \in \mathbb{C}$ is a parameter and $n \geq 3$. When $|z|$ is large, $F_\lambda(z) \approx z^n$, so F_λ has an immediate basin of attraction at ∞ that we denote by B_λ. Each F_λ also has a pole of order n at the origin. Hence there is an open neighborhood of 0 that is mapped into B_λ. Now either this neighborhood is disjoint from the immediate basin B_λ or else this neighborhood is contained in B_λ. In the former case, we denote the entire preimage of B_λ that contains the origin by T_λ. We call this region the *trap door* since any point z that does not lie in B_λ but for which $F_\lambda^k(z)$ does lie in B_λ for some $k > 0$ has the property that there is a unique point on this orbit that lies in T_λ.

Besides 0 and ∞, F_λ has $2n$ additional "free" critical points given by $c_\lambda = \lambda^{1/2n}$. However, F_λ has only two critical values given by $v_\lambda = \pm 2\sqrt{\lambda}$. In fact, there is only one free critical orbit for F_λ up to symmetry. For, if n is even, we have $F_\lambda(2\sqrt{\lambda}) = F_\lambda(-2\sqrt{\lambda})$, so each of the critical points land on the same orbit after two iterations. If n is odd, then we have $F_\lambda(-z) = -F_\lambda(z)$, so the orbits of $\pm 2\sqrt{\lambda}$ are always symmetric under $z \mapsto -z$.

F_λ also has $2n$ prepoles given by $p_\lambda = (-\lambda)^{1/2n}$. Note that the free critical points and the prepoles all lie on the circle $|z| = |\lambda|^{1/2n}$. We call this circle the *critical circle* and denote it by C_λ. An easy computation shows that the C_λ is mapped $2n$ to 1 onto the straight line segment connecting $\pm v_\lambda$. We call this segment the *critical segment*. Similarly, any other circle centered at the origin is mapped n to 1 onto an ellipse whose major axis contains the critical segment.

We call the straight rays given by tc_λ with $t > 0$ the *critical point rays*. Note that

$$F_\lambda(tc_\lambda) = \lambda^{1/2}\left(t^n + \frac{1}{t^n}\right),$$

so it follows that each critical point ray is mapped two to one onto the straight ray that extends from one of the critical values v_λ to ∞. We call these two rays the *critical value rays*. The rays tp_λ with $t > 0$ are called *prepole rays*, and these rays are mapped one to one onto the entire line segment passing through $\pm iv_\lambda$ and extending to ∞ in both directions. Note that these lines lie perpendicular to line formed by the critical segment and the critical value rays.

Recall that the *Julia set* $J(F_\lambda)$ for the rational map F_λ has several equivalent definitions. It is known that the Julia set is the closure of the set of repelling periodic points as well as the boundary of the set of points whose orbits tend to ∞ [10]. The complement of the Julia set is called the *Fatou set*.

There are several symmetries in the dynamical plane. First, let $\nu = \exp(\pi i/n)$. Then we have $F_\lambda(\nu z) = -F_\lambda(z)$, so, as above, either the orbits of z and νz coincide after two iterations (when n is even), or else they behave symmetrically under $z \mapsto -z$ (when n is odd). In either event, the dynamical plane and the Julia set both possess $2n$-fold symmetry. Second, let $H_\lambda(z)$ be one of the n involutions given by $\lambda^{1/n}/z$. Then $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, so the dynamical plane and Julia set are also symmetric under each H_λ. Note that $H_\lambda(B_\lambda) = T_\lambda$.

The following result gives one instance of how Sierpinski curve Julia sets occur in the family F_λ [7].

Theorem *(The Escape Trichotomy). Let $F_\lambda(z) = z^n + \lambda/z^n$ and consider the orbit of v_λ.*

1. *If v_λ lies in B_λ, then $J(F_\lambda)$ is a Cantor set;*

2. *If v_λ lies in T_λ, then $J(F_\lambda)$ is a Cantor set of simple closed curves, each of which surrounds the origin;*

3. If $F_\lambda^k(v_\lambda)$ lies in T_λ where $k \geq 1$, then $J(F_\lambda)$ is a Sierpinski curve.

Finally, if the orbit of v_λ does not escape to ∞, then $J(F_\lambda)$ is a connected set.

We remark that case 2 of the above result was proved by McMullen [9].

Because of the Escape Trichotomy, the parameter plane for F_λ (the λ-plane) divides into three distinct regions. Let $\mathcal{L}$ be the set of parameters for which $v_\lambda \in B_\lambda$ so, by the Escape Trichotomy, $J(F_\lambda)$ is a Cantor set. We call $\mathcal{L}$ the *Cantor set locus*. As mentioned earlier, the McMullen domain $\mathcal{M}$ is the set of parameters for which $v_\lambda \in T_\lambda$; $\mathcal{M}$ is the central open region in Figure 2. Let $\mathcal{N}$ denote the complement of $\mathcal{L} \cup \mathcal{M}$. $\mathcal{N}$ is called the connectedness locus. It is known that $\mathcal{N}$ contains precisely $(2n)^{k-3}(n-1)$ *Sierpinski holes* with escape time $k \geq 3$ [3], [13]. These are open disks in $\mathcal{N}$ in which each corresponding map has the property that the critical point lands in B_λ at iteration k or, equivalently, the critical value lands in T_λ at iteration $k-2$. See Figure 2.

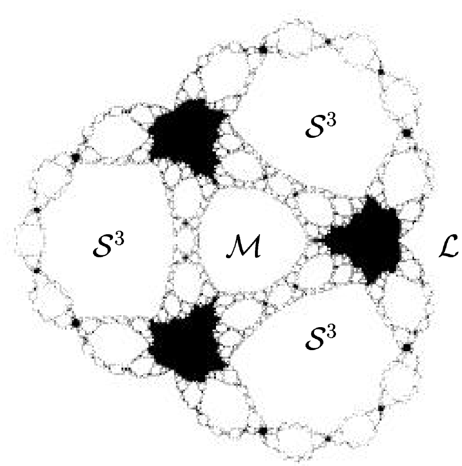

Figure 2: The parameter plane when $n = 4$. The open disks marked $\mathcal{S}^3$ are the Sierpinski holes with escape time 3.

It is known [12] that there is a Böttcher coordinate on B_λ. That is, assuming that $v_\lambda \notin B_\lambda$, $F_\lambda \,|\, B_\lambda$ is topologically conjugate to $z \mapsto z^n$ outside the unit disk in $\mathbb{C}$. As a consequence, the usual theory of external rays goes over immediately to the family F_λ [10].

In Figure 2, there are three clearly visible copies of the Mandelbrot Set. In fact, there are infinitely many other copies of the Mandelbrot set in $\mathcal{N}$ [3]. Many of these Mandelbrot sets have cusps that lie on the boundary of the Cantor set locus, but there appear to be infinitely many other Mandelbrot sets that do not meet this boundary. This leads to another way that Sierpinski curves can occur as Julia sets in these families. In [6] it was shown:

Theorem. *Given $k \geq 3$, there are open disks in the parameter plane for which F_λ has an attracting cycle of prime period k and the Julia set of F_λ is a Sierpinski curve. If λ and μ are parameters drawn from these disks with different periods, then F_λ and F_μ are not topologically conjugate.*

The open disks in this Theorem are the main cardioids of some of the "buried" Mandelbrot sets in the parameter plane. Note that maps with Sierpinski curve Julia sets that arise when the critical orbits escape cannot be topologically conjugate to those in the above result.

3 Cantor Sets of Circles in the Dynamical Plane

In this section we consider the family of functions $F_\lambda(z) = z^n + \lambda/z^n$ where $n \geq 3$ and $\lambda \in \mathbb{C}$. Our aim is to show that there is an open set $\mathcal{O}$ in the parameter plane for this family that contains the McMullen domain and has the property that, if $\lambda \in \mathcal{O}$, there is an invariant Cantor set Λ_λ of simple closed curves lying the $J(F_\lambda)$. We call these curves "circles," though technically they are only quasicircles. Moreover, F_λ on Λ_λ is conjugate to the product of a certain subshift of finite type and one of the two circle maps given by $\theta \mapsto \pm n\theta$.

To define this conjugacy, let Σ denote the subset of the space of one-sided sequences of 0's and 1's generated by the subshift of finite type whose transition matrix is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

That is, Σ consists of all sequences $s = (s_0 s_1 s_2 \ldots)$ where $s_j = 0$ or 1 subject to the restriction that 1 cannot follow 1. Let $\sigma : \Sigma \to \Sigma$ be the shift map on this space of sequences. It is straightforward to check that periodic points of σ are dense in Σ. Also let $\phi_0 : S^1 \to S^1$ be the map $\phi_0(\theta) = -n\theta$ mod 1 and let $\phi_1 : S^1 \to S^1$ be the map $\phi_1(\theta) = n\theta \bmod 1$. Then we shall show that F_λ is topologically conjugate on Λ_λ to the map

$$\Phi : \Sigma \times S^1 \to \Sigma \times S^1$$

given by $\Phi(s, \theta) = (\sigma(s), \phi_{s_0}(\theta))$. That is, for a given $s = (s_0 s_1 s_2 \ldots) \in \Sigma$, the map Φ takes the circle $(s, S^1) \subset \Sigma \times S^1$ as an n to 1 covering of the circle $(\sigma(s), S^1)$. If $s_0 = 0$, the map on the circle is $\theta \mapsto -n\theta$ and so is orientation reversing; otherwise, the map is $\theta \mapsto n\theta$ and so is orientation preserving.

Suppose that

$$|\lambda| < \left(\frac{1}{4}\right)^{\frac{n}{n-1}}.$$

Then we have that

$$|F_\lambda(v_\lambda)| = |2\sqrt{\lambda}| < |\lambda^{1/2n}| = |c_\lambda|$$

so, for these parameters, F_λ takes the critical circle C_λ to the critical segment which therefore lies strictly inside C_λ. F_λ is an n to 1 covering map on the region outside C_λ and also on the region inside C_λ. F_λ takes each of these regions onto the complement of the critical segment in $\mathbb{C}$. In the exterior region, F_λ takes circles centered at the origin to ellipses that surround the critical segment in an orientation preserving manner; in the interior region, F_λ reverses the orientation on such circles.

In what follows we sometimes denote the critical circle C_λ by γ_0. Of course, γ_0 depends on λ, but we drop the parameter dependence for convenience of notation. Note that there is a preimage of γ_0 that lies strictly outside γ_0. Call this preimage γ_1. There is also a similar preimage γ_{-1} inside γ_0. F_λ maps both γ_1 and γ_{-1} as an n to 1 covering onto γ_0. So each of these curves is a simple closed curve. We again call them "circles," though, unlike γ_0, they are not actual circles. Then there is a preimage γ_2 of γ_1 lying outside γ_1 and another preimage γ_{-2} of γ_1 lying inside γ_{-1}. Inductively, for each $k \geq 1$, there is a pair of preimages of γ_k, one called γ_{k+1} lying outside γ_k, and another called $\gamma_{-(k+1)}$ lying inside γ_{-k}, and each of these circles is mapped n to 1 onto γ_k. Note that F_λ^k therefore maps $\gamma_{\pm k}$ as an n^k to 1 covering onto γ_0. Also, $\gamma_k \to \partial B_\lambda$ as $k \to \infty$, whereas $\gamma_{-k} \to \partial T_\lambda$. See Figure 3.

When $\lambda \in \mathcal{M}$, we have that $v_\lambda \in T_\lambda$, so v_λ lies inside each circle γ_k. But when λ lies outside $\mathcal{M}$, $F_\lambda(v_\lambda)$ no longer lies in B_λ so v_λ now lies outside certain of the γ_k. There is an open set of parameters for which v_λ lies inside the

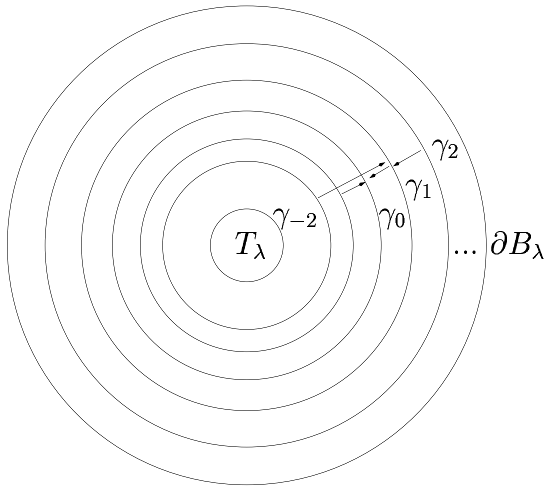

Figure 3: The critical circle and it preimages γ_k.

open disk in the dynamical plane bounded by γ_{-2}. Let $\mathcal{O}$ be the component of this open set that contains the McMullen domain. So for each $\lambda \in \mathcal{O}$, since v_λ lies inside γ_{-2}, we have that $F_\lambda(v_\lambda)$ lies outside the circle γ_1.

Theorem. *Suppose $\lambda \in \mathcal{O}$. Then there is an invariant subset Λ_λ of $J(F_\lambda)$ that is homeomorphic to the Cantor set of circles $\Sigma \times S^1$. Moreover, $F_\lambda \mid \Lambda_\lambda$ is topologically conjugate to the map Φ on $\Sigma \times S^1$.*

Proof: Consider the annular region bounded by γ_1 and γ_{-2}. By assumption, v_λ lies in the exterior of this annulus. Let B_1 be the outer annulus in this region that is bounded by γ_0 and γ_1. Let B_0 be the inner annulus bounded by γ_0 and γ_{-2}. Then F_λ takes $B_1 - \gamma_0$ as an n to 1 covering over the disk bounded by γ_0 minus the critical segment. Hence there is a preimage of B_0 lying in B_1. Call this preimage A_1. Since F_λ is an n to 1 covering map on B_1, it follows that A_1 is also an annulus. In similar fashion, F_λ takes $B_0 - \gamma_0$ in n to 1 fashion over the disk bounded by γ_1 minus the critical segment. Hence there is another annulus $A_0 \subset B_0$ that is mapped as an n to 1 covering onto $B_0 \cup B_1$. In particular, $F_\lambda(A_0)$ properly contains $A_0 \cup A_1$ while $F_\lambda(A_1)$ properly contains A_0. Standard arguments from complex dynamics then say that

$$\Lambda_\lambda = \bigcap_{j=0}^{\infty} F_\lambda^j(A_0 \cup A_1)$$

is a Cantor set of quasicircles and F_λ restricted to this set is conjugate to Φ. □

In the following section, we shall construct this conjugacy more explicitly (in the case $n = 3$) and show that the conjugacy varies analytically with λ. Note that the exterior annulus between γ_1 and γ_2 is mapped as an n to 1 covering of the annulus between γ_0 and γ_1. Hence there is a preimage of a portion of Λ_λ in this annulus. This preimage is also a Cantor set of circles. In similar fashion, any annulus bounded by γ_k and γ_{k+1} for any $k \in \mathbb{Z}$ contains a Cantor set of circles which either lies in Λ_λ or else eventually maps into Λ_λ.

We remark that, in the above proof, we could just as well have chosen B_0 to be the annulus between γ_0 and γ_{-1} and then B_{-1} to be the annulus between γ_{-2} and γ_{-1}. This would give a subshift on the three symbols $-1, 0,$ and 1 with the following pairs as the only allowed followers in the sequence space: 00, $0(-1)$, 10, and $(-1)1$.

For later use, we can now give a more dynamical construction of Λ_λ. We have shown that there is a circle in Λ_λ corresponding to the sequence $\overline{001}$ in Σ; let μ_0 denote this circle. Let $\mu_1 = F_\lambda(\mu_0)$ and $\mu_2 = F_\lambda(\mu_1)$. So μ_1 corresponds to the sequence $\overline{010}$ and μ_2 to $\overline{100}$. So the μ_j form a collection of circles that is invariant under F_λ^3. We call this collection a cycle of circles of period 3. Let ν_0 denote the preimage of μ_1 that is not equal to μ_0, so ν_0 corresponds to the sequence $1\overline{010}$. As before, the μ_j and ν_0 all depend on λ.

Since μ_0 and μ_1 have itineraries in Σ that begin with 0, both of these circles lie in the annulus A_0 constructed in the above proof. Similarly, μ_2 and ν_0 lie in the annulus A_1. Let U_0 be the annular region bounded by μ_0 and μ_1 and U_1 the annular region bounded by μ_2 and ν_0. Since F_λ takes μ_1 outside of A_0, it follows that μ_1 lies inside the circle μ_0. Similarly, ν_0 lies inside μ_2. We therefore have that F_λ takes U_1 onto U_0 since the bounding circles are mapped to bounding circles. Similarly, F_λ takes U_0 onto the entire larger annulus bounded by μ_1 and μ_2 (and therefore containing μ_0 and ν_0). As a consequence, we have the exact situation as in the above proof, with U_0 and U_1 playing the roles of A_0 and A_1. So the Cantor set of circles Λ_λ lie in the annuli U_0 and U_1 that are bounded by specific circles in Λ_λ. See Figure 4. We have shown:

Corollary. *The invariant set of circles Λ_λ is contained in the pair of annuli*

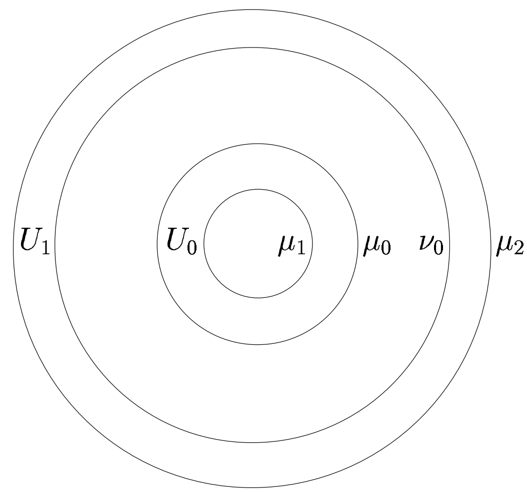

Figure 4: The annuli U_0 and U_1. $F_\lambda(U_1) = U_0$ while $F_\lambda(U_0) \supset U_0 \cup U_1$.

bounded by the specific circles μ_0, μ_1, μ_2, *and* ν_0, *where each of these circles lies in* Λ_λ. *The* μ_j *are circles that lie on a cycle of circles of period* 3 *while* ν_0 *is mapped onto the circle* μ_1 *by* F_λ.

Using this result, we can now show that the set Λ_λ is a dynamical invariant.

Proposition. *Suppose that* $\lambda, \mu \in \mathcal{O}$ *and that* F_λ *is topologically conjugate to* F_μ *via a conjugacy* h. *Then* h *maps* Λ_λ *to* Λ_μ *and therefore gives a conjugacy between* F_λ *and* F_μ *on these invariant sets.*

Proof: First note that there is a unique invariant circle surrounding the origin for both F_λ and F_μ. Indeed, only the annulus between the curves γ_0 and γ_{-1} is mapped over itself by F_λ and F_μ; the annulus between γ_k and γ_{k-1} is mapped to a region disjoint from this annulus whenever $k \neq 0$. Hence the annulus between γ_0 and γ_{-1} contains a unique invariant circle for both F_λ and F_μ, and so h must take the invariant circle for F_λ to that for F_μ. Note that these circles correspond to the sequence $\overline{0}$ in Σ.

In similar fashion, there is a unique pair of circles for F_λ and F_μ that have period 2 under these maps; these are the circles corresponding to the sequences $\overline{01}$ and $\overline{10}$.

For the period three case, the situation can be different. Both F_λ and F_μ

have a cycle of circles lying in Λ_λ and corresponding to the sequence $\overline{001}$ as shown above. But there may be another such cycle. Before we had assumed that the critical values lie inside the curve γ_{-2}. But if the critical values lie inside γ_{-3}, then we can construct another cycle of circles of period 3 as follows. The annulus between γ_1 and γ_2 is mapped onto the annulus between γ_0 and γ_1. Then the annulus between γ_0 and γ_1 is mapped over all of the annuli bounded by curves of the form γ_k and γ_{k-1} where $k \leq 0$. In particular, the image of this annulus covers the annulus between γ_{-2} and γ_{-3}. But then this annulus is mapped over the original annulus between γ_1 and γ_2. So arguing as before, there is a cycle of circles of period three that visits these annuli in order under both F_λ and F_μ. (So these circles would correspond to an itinerary of the form $\overline{21(-2)}$ if we denote the annulus between γ_{-2} and γ_{-3} by A_{-2}.) One checks easily that this is the only other possibility for a cycle of circles of period three that is concentric about the origin.

We claim, however, that, even in the case where two such cycles of circles exist, the conjugacy takes the cycle corresponding to the sequence $\overline{001}$ for F_λ to the corresponding cycle for F_μ. Indeed, for this cycle of circles, F_λ reverses the orientation twice as it maps circle to circle, namely on each of the two circles that lie in A_0. On the other hand, for the other cycle of circles, there is only one reversal of orientation. So the conjugacy cannot map a cycle of circles corresponding to $\overline{001}$ to a cycle corresponding to the other sequence.

As a consequence, h must take the circle corresponding to the sequence $1\overline{010}$ for F_λ to the corresponding circle for F_μ. It follows that the conjugacy maps points in the Julia set for F_λ whose orbits remain in the annuli $U_0 \cup U_1$ for all iterations to the corresponding annuli for F_μ. That is, h maps Λ_λ to Λ_μ and is thus a conjugacy on these subsets of the Julia set.

□

4 Cantor Sets of Circles in the Parameter Plane

For the remainder of this paper, we consider only the family $F_\lambda(z) = z^3 + \lambda/z^3$. Our goal here is to prove that there is a Cantor set $\mathcal{C}$ of circles in the parameter plane for this family with the following properties:

1. If $\lambda \in \mathcal{C}$, then $J(F_\lambda)$ is a Sierpinski curve, so all of these Julia sets are homeomorphic;

2. If λ and μ lie on different circles in $\mathcal{C}$, then F_λ is not topologically conjugate to F_μ;

3. Given $\lambda \in \mathcal{C}$, there are only finitely many $\mu \in \mathcal{C}$ such that F_λ and F_μ are topologically conjugate. So there are uncountably many dynamically different maps corresponding to parameters on each circle in $\mathcal{C}$.

The way we will prove this result is by showing that there is a unique parameter such that $F_\lambda(v_\lambda)$ lands at a prescribed point in the invariant Cantor set of circles Λ_λ constructed in the previous section.

In order to prove this, we first need specific estimates on the size and location of both the McMullen domain and the invariant Cantor set of circles Λ_λ for parameters close to the McMullen domain, and secondly we must make precise what we mean by a "prescribed" point in Λ_λ. So, from now on we restrict attention to parameters λ in the annulus $\mathcal{S}$ given by $0.014 \leq |\lambda| \leq 1/49 = 0.0204\ldots$.

We first claim that $\mathcal{S} \subset \mathcal{O}$, so that for each $\lambda \in \mathcal{S}$, v_λ lies inside γ_{-2} so $F_\lambda(v_\lambda)$ lies outside the circle γ_1 in the dynamical plane and the invariant Cantor set of circle Λ_λ is well-defined.

Lemma. *Suppose $\lambda \in \mathcal{S}$. Then $F_\lambda(v_\lambda)$ lies outside the circle γ_1.*

Proof: First note that, if $\lambda \in \mathcal{S}$, then $|c_\lambda| = |\lambda|^{1/6} \leq (1/49)^{1/6} = 0.5227\ldots$. If $|z| = 0.85$, then

$$|F_\lambda(z)| \geq (0.85)^3 - \frac{1}{(49)(0.85)^3} \geq 0.58\ldots.$$

so $|F_\lambda(z)| > |c_\lambda|$. Therefore the circle $|z| = 0.85$ lies strictly outside $\gamma_1(\lambda)$ for each $\lambda \in \mathcal{S}$. We have

$$F_\lambda(v_\lambda) = \left(2\sqrt{\lambda}\right)^3 + \frac{\lambda}{\left(2\sqrt{\lambda}\right)^3}.$$

Therefore

$$|F_\lambda(v_\lambda)| \geq \frac{1}{8|\lambda|^{1/2}} - 8|\lambda|^{3/2} \geq \frac{7}{8} - \frac{8}{7^3} > 0.85.$$

Therefore $F_\lambda(v_\lambda)$ lies outside γ_1.

□

Let $\mathcal{A}$ denote the annulus in the dynamical plane given by

$$\mathcal{A} = \{z \in \mathbb{C} \,|\, 0.238 \leq |z| \leq 0.28\}.$$

Lemma. *For each $\lambda \in \mathcal{S}$, ∂T_λ is contained in the interior of the annulus $\mathcal{A}$.*

Proof: If $|z| = 0.28$ and $\lambda \in \mathcal{S}$, we have

$$|F_\lambda(z)| \leq (0.28)^3 + \frac{1}{(49)(0.28)^3} < 0.96.$$

But if $|w| = 0.96$, then

$$|F_\lambda(w)| \leq (0.96)^3 + \frac{1}{(49)(0.96)^3} < 0.91,$$

so B_λ lies outside the circle $|w| = 0.96$. Therefore points on the circle $|z| = 0.28$ do not lie in $\overline{T}_\lambda$. Hence ∂T_λ lies inside this circle.

On the other hand, if $|z| = 0.238$, then

$$|F_\lambda(z)| \geq \frac{|\lambda|}{(0.238)^3} - (0.238)^3 \geq \frac{0.014}{(0.238)^3} - (0.238)^3 > 1.02.$$

But if $|w| = 1.02$, then we have

$$|F_\lambda(w)| \geq (1.02)^3 - \frac{1}{49}\left(\frac{1}{1.02}\right)^3 > 1.04$$

so w lies in B_λ. Therefore, all points with $|z| = 0.238$ lie in the trap door, so ∂T_λ lies outside this circle. Hence ∂T_λ lies in the interior of $\mathcal{A}$.

□

We remark that, in order to extend the main Theorem of this section to the case $n > 3$, it is only necessary to provide similar estimates as in the previous two lemmas.

Note that if $|\lambda| = 0.014$, then $|2\sqrt{\lambda}| = 0.2366\ldots$ whereas if $|\lambda| = 1/49$, then $|2\sqrt{\lambda}| = 0.285\ldots$. So as λ moves around the annulus $\mathcal{S}$ in parameter plane, the critical values $\pm v_\lambda$ fill an annulus in the dynamical plane that properly contains $\mathcal{A}$ and hence ∂T_λ for each $\lambda \in \mathcal{S}$.

Since $\mathcal{S}$ is a compact subset of the parameter plane and the circles γ_{-j} accumulate on ∂T_λ as $j \to \infty$, we have:

Corollary. *There exists $K < 0$ such that, for each $\lambda \in \mathcal{S}$ and $k \leq K$, the curves $\gamma_k(\lambda)$ are contained in $\mathcal{A}$.*

As we described earlier, there is a different Cantor set of circles lying between γ_1 and γ_2 that is mapped by F_λ onto the portion of Λ_λ lying between γ_0 and γ_1. Similarly, for each $k > 1$, there are other Cantor sets of circles lying between γ_k and γ_{k+1} that are mapped to the same portion of Λ_λ by F_λ^k. Let Λ_λ^k denote these sets. For $k \leq -2$, we have similar sets between γ_k and γ_{k-1} that are mapped to $\Lambda_\lambda^{|k|-1}$ by F_λ. Call these sets Λ_λ^k as well. So when k is negative, we have that $F_\lambda^{|k|}$ maps Λ_λ^k to the same portion of Λ_λ as before. By the previous corollary, we may then find $K < 0$ such that, if $k \leq K$, Λ_λ^k is contained in the annulus $\mathcal{A}$. So we let Γ_λ^K denote the union of all the Cantor sets of circles Λ_λ^k in $\mathcal{A}$ for $k \leq K$. The "Cantor sets of circles" in the parameter plane will consist of parameters for which the critical values of F_λ lie on the circles in Γ_λ^K, though technically this is not a Cantor set since we do not include the boundary of the McMullen domain in this set to close it up.

To construct this set in the parameter plane, we must first produce a parametrization of each circle in Λ_λ that varies analytically with λ. In order to do this, we need to recall the construction of the internal rays (i.e., the spines of the Cantor necklaces) as defined in [2]. In that paper, the construction is given only in the case $n = 2$, but the same construction goes over immediately to the higher degree case. Hence we merely sketch this construction for $n = 3$ here.

Suppose first that $0 < \operatorname{Arg}\lambda < 2\pi$. Let $c_0 = c_0(\lambda)$ denote the unique critical point that lies in the sector $0 < \operatorname{Arg}\lambda < \pi/3$ for these parameter values. Let $c_j = c_j(\lambda)$ be the critical point given by $\exp(2\pi i j/6) \cdot c_0$. Let $P_j = P_j(\lambda)$ be the closed prepole sector in $\overline{\mathbb{C}}$ bounded by the critical point rays through c_j and c_{j-1} and containing both the origin and the point at ∞. So P_0 is bounded by the critical point rays through c_0 and c_5.

Since $0 < \operatorname{Arg}\lambda < 2\pi$, we have that $0 < \operatorname{Arg} v_\lambda < \pi$ and so

$$\operatorname{Arg} c_0 = \frac{\operatorname{Arg}\lambda}{6} < \operatorname{Arg} v_\lambda = \frac{\operatorname{Arg}\lambda}{2} < \frac{\operatorname{Arg}\lambda}{6} + \frac{2\pi}{3} = \operatorname{Arg} c_2$$

for all of these λ-values. Similarly, by symmetry,

$$\operatorname{Arg} c_3 < \operatorname{Arg}(-v_\lambda) < \operatorname{Arg} c_5.$$

It follows that the images of the critical point rays lie outside of the sectors P_0 and P_3 these sectors for each λ. Then, as shown in [2], the set of points whose orbits remain for all iterations in $P_0 \cup P_3$ is a simple closed curve η_λ that passes through both 0 and ∞. F_λ maps the portion of η_λ that lies in P_0 univalently onto all of η_λ (except at ∞, which has both 0 and ∞ as preimages). The curve η_λ contains the external rays of angle 0 and 1/2 in B_λ together with certain of the preimages of these rays that lie in the preimages of B_λ that meet $P_0 \cup P_3$. We include ∞ in the external rays of angle 0 and 1/2, so the union of these two rays is an open arc in η_λ.

Note that F_λ maps η_λ two-to-one over itself. Also, η_λ breaks into two disjoint pieces: the union of the open arcs comprising the external rays and their preimages together with the complement of this set which is a Cantor set that we call Γ_λ. We have that F_λ is conjugate to the full one-sided two shift on Γ_λ. The Cantor set of circles Λ_λ consequently meets Γ_λ in a set that is also a Cantor set; indeed, each circle in Λ_λ meets Γ_λ in exactly two points, one in P_0 and one in P_3. One can identify these points symbolically using the two-shift dynamics on Γ_λ, but we do not need this result here.

Note that there is a preimage of η_λ that lies in $P_1 \cup P_4$ and another preimage in $P_2 \cup P_5$. By symmetry, the circles in Λ_λ also meet the preimages of Γ_λ in these two regions at another pair of points. Therefore we may parametrize any circle in Λ_λ as follows. Let $s \in \Sigma$ and denote by ξ_s^λ the circle in Λ_λ corresponding to the itinerary s. Let $\xi_s^\lambda(0)$ (resp., $\xi_s^\lambda(\pi)$) be the point on ξ_s^λ lying in $\Gamma_\lambda \cap P_0$ (resp., $\Gamma_\lambda \cap P_3$). Similarly, let $\xi_s^\lambda(j\pi/3)$ be the point in ξ_s^λ lying in the preimage of Γ_λ lying in P_j. Then, using the symbolic dynamics induced on the bounded orbits in the P_j, there is a unique way to define $\xi_s^\lambda(\theta)$ so that $F_\lambda \,|\, \xi_s^\lambda$ is conjugate to either $\theta \mapsto 3\theta$ or $\theta \mapsto -3\theta$ mod 1 depending on the first digit of the sequence s. Note that this parametrization therefore depends analytically on λ since $\xi_s^\lambda(j\pi/3)$ all do and is 2π-periodic.

Since $\xi_s^\lambda(\theta)$ is constrained to lie in a particular prepole sector for each λ, and since each of these sectors rotates by $\pi/3$ radians as the argument of λ rotates by 2π, we have the following:

Proposition. *As the argument of λ increases from 0 to 2π, the argument of $\xi_s^\lambda(\theta)$ changes by at most $2\pi/3$ radians.*

When $\operatorname{Arg}\lambda = 0$ or 2π, the situation is a little different. The external rays of angle 0 and 1/2 now lie on the boundaries of $P_0 \cup P_3$, in each case along the real axis. Consequently, the endpoints of these rays also lie in $\mathbb{R}$. But the preimage of this set in $\overline{T}_\lambda$ now lies along the rays $\operatorname{Arg} z = -\pi/3$ and $\operatorname{Arg} z = 2\pi/3$ when $\operatorname{Arg}\lambda = 0$ and along the rays $\operatorname{Arg} z = \pi/3$ and

$\operatorname{Arg} z = -2\pi/3$ when $\operatorname{Arg}\lambda = 2\pi$. All other points in η_λ then lie in the interior of $P_0 \cup P_3$. Since these prepole sectors differ if $\operatorname{Arg}\lambda = 0$ or 2π (they are rotated by $\pi/3$), it follows that the sets η_λ differ if $\operatorname{Arg}\lambda = 0$ or $\operatorname{Arg}\lambda = 2\pi$. Hence the parametrizations of ξ_s^λ differ in these two cases. For this reason, to obtain a parametrization of ξ_s^λ that varies analytically with λ, we must lift these parametrizations to the corresponding covering spaces, exactly as was done in [5].

Let $\tilde{\mathcal{S}}$ denote the universal covering space of $\mathcal{S}$, i.e., the strip $\log(0.014) \leq \operatorname{Re} z \leq -\log(49)$. In the dynamical plane, one checks easily that if $\lambda \in \mathcal{S}$, then the region $|z| \geq 2$ is contained in B_λ. Let $\mathcal{W}$ denote the punctured disk $0 < |z| < 2$ and let $\tilde{\mathcal{W}}$ be the universal covering of $\mathcal{W}$. Also let $\tilde{\mathcal{A}}$ be the universal covering space of $\mathcal{A}$, i.e., the strip $\log(0.238) \leq \operatorname{Re} z \leq \log(0.28)$. So $\tilde{\mathcal{A}} \subset \tilde{\mathcal{W}}$. Let Ψ denote the natural projections from these covering spaces to the respective portions of the parameter and dynamical planes. Given $\lambda \in \mathcal{S}$, we denote by $\tilde{\lambda}$ any point in $\tilde{\mathcal{S}}$ that projects to λ. Similarly, let $\tilde{\xi}_s^\lambda(\theta)$ denote the lift of the circle $\xi_s^\lambda(\theta)$. We always choose this lift so that $\operatorname{Im}\tilde{\xi}_s^\lambda(0)$ lies in the interval $[-\pi/3, \pi/3]$. Finally, let $\tilde{\Lambda}_{\tilde{\lambda}}$ denote the set of points in $\tilde{\mathcal{W}}$ that project to $\Lambda_{\Psi(\tilde{\lambda})}$ in the dynamical plane. So the preimage of a circle in Λ_λ in $\tilde{\mathcal{W}}$ is now a curve extending from $\operatorname{Im} z = -\infty$ to $\operatorname{Im} z = +\infty$

By the earlier Corollary, we have the existence of Γ_λ^K, the union of the certain of the preimages of Λ_λ that lie in $\mathcal{A}$. We may parameterize these curves as follows. Let ξ_s^λ be a circle in Λ_λ and suppose that $\omega_{s,k}^\lambda$ is the preimage of ξ_s^λ in Γ_λ^K that is mapped by F_λ^k to ξ_s^λ. Then, as above, there is a unique point on $\omega_{s,k}^\lambda$ that also lies in $\Gamma_\lambda \cap P_0$. Again we call this point $\omega_{s,k}^\lambda(0)$. We then define $\omega_{s,k}^\lambda(\theta)$ by the rule $F_\lambda^k(\omega_{s,k}^\lambda(\theta)) = \xi_s^\lambda(\theta)$. Note that this curve is now periodic with period $3^k \cdot 2\pi$. We define the parameterization of this curve in the covering space exactly as above. Thus, for each circle in Λ_λ, we now have countably many curves in $\tilde{\mathcal{A}}$ given by $\tilde{\omega}_{s,k}^{\tilde{\lambda}}(\theta)$, each of which is mapped to that circle by $F_\lambda^k \circ \Psi$ for some k. Also, as before, as θ increases from 0 to 2π, the argument of the point $\omega_{s,k}^\lambda(\theta)$ changes by no more than $2\pi/3$ radians. In fact, the argument actually increases by much less than this when $|k|$ is large.

Proposition. *Given k, θ, and s, there exists a unique parameter $\lambda = \lambda_{s,\theta}$ that has the property that $v_\lambda = \omega_{s,k}^\lambda(\theta)$, i.e., the critical orbit lands on the point $\xi_s^\lambda(\theta)$ after k iterations.*

Proof: We have two maps taking $\tilde{\mathcal{S}}$ to $\tilde{\mathcal{W}}$. The first is the map $\tilde{\lambda} \mapsto v_{\tilde{\lambda}}$, where we choose a specific critical value and then lift it continuously to the covering space. Call this map α. Note that α is univalent, so we also have the inverse of α. The second is the map $\tilde{\lambda} \mapsto \tilde{\omega}^{\tilde{\lambda}}_{s,k}(\theta)$. Call this map β. Then consider $G(\tilde{\lambda}) = \alpha^{-1}(\beta(\tilde{\lambda}))$. The map G takes the strip $\tilde{\mathcal{S}}$ to itself. We claim that there is a unique fixed point for this map; this fixed point will be the unique parameter in $\tilde{\mathcal{S}}$ for which $v_{\tilde{\lambda}}$ lands at the chosen point on the lift of the given circle in the dynamical plane, and the projection of $\tilde{\lambda}$ will yield the unique parameter in the statement of the Proposition.

To prove this, recall that, as λ varies in $\mathcal{S}$, v_λ covers an annulus in dynamical plane that contains $\mathcal{A}$ in its interior. Therefore, in the covering spaces, the image of $\mathcal{S}$ under α is a strip that properly contains the closure of the strip that is the image of β. Hence G maps the vertical boundaries of $\mathcal{S}$ to vertical curves that lie strictly inside $\mathcal{S}$ and extend from the top to the bottom of this strip. Next recall that, as $\operatorname{Arg}\lambda$ increases by 2π, the imaginary part of $v_{\tilde{\lambda}}$ increases by π whereas the imaginary part of $\tilde{\omega}^{\tilde{\lambda}}_{s,k}(\theta)$ increases by a smaller amount, namely $2\pi/3$. It follows that, if ℓ is sufficiently large, G takes the rectangle in $\mathcal{S}$ bounded above and below by $\operatorname{Im}\tilde{\lambda} = \pm\ell$ strictly inside itself. By the Schwarz Lemma, we therefore have a unique fixed point for G in $\tilde{\mathcal{S}}$. This is the parameter $\tilde{\lambda}$.

□

Note that the fixed point in the above Proposition depends continuously on θ and s. Hence, this yields the collection of Cantor sets of circles $\mathcal{C}$ in the portion of the parameter plane $\mathcal{S}$ having the property that, for each $\lambda \in \mathcal{C}$, the critical orbit lands in Λ_λ after a certain number of iterations. This then gives:

Theorem. *Suppose $\lambda \in \mathcal{C}$. Then the Julia set of F_λ is a Sierpinski curve.*

Proof: We must show that $J(F_\lambda)$ is compact, connected, locally connected, nowhere dense, and has the property that each pair of complementary domains is bounded by a simple closed curve. Since we have a basin of ∞, we have that $J(F_\lambda) \neq \mathbb{C}$ so it follows that the Julia set is compact and nowhere dense. Since $v_\lambda \notin B_\lambda \cup T_\lambda$, it is known that $J(F_\lambda)$ is connected [7]. Since the critical orbits all eventually land in Λ_λ, we have that the critical orbits are not recurrent. Furthermore, since F_λ is hyperbolic on Λ_λ, there are no parabolic periodic points. By the results in [14], it follows that $J(F_\lambda)$ is locally connected. Finally, since $|v_\lambda| < |c_\lambda|$, we have that B_λ and all of its preimages are bounded by simple closed curves [1]. Since Λ_λ separates ∂B_λ from ∂T_λ, it follows that these two curves are disjoint. Hence all of the preimages of ∂B_λ are disjoint from one another. Therefore $J(F_\lambda)$ is a Sierpinski curve.

□

Since we completely understand the dynamics on the invariant set of simple closed curves for Λ_λ, and this dynamical behavior must be preserved by any conjugacy, we have:

Corollary. *Suppose $\lambda, \mu \in \mathcal{C}$. Then F_λ is topologically conjugate to F_μ only if the critical orbits land on points in the invariant Cantor sets of circles Λ_λ and Λ_μ that have symmetric itineraries, i.e., they land on points on the same circles and that correspond to points in $(s, \theta_\lambda), (s, \theta_\mu) \in \Sigma \times S^1$ where θ_λ and θ_μ have the same itineraries under the maps on S^1 (up to symmetry).*

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