Sierpinski Curve Julia Sets and Singular Perturbations of Complex Polynomials

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1 Introduction

In this paper we consider the family of rational maps

$$F_\epsilon(z) = z^2 + \frac{\epsilon}{z^2}$$

where $z \in \mathbb{C}$ and ϵ is a parameter. Our goal is to investigate the Julia set of F_ϵ, which we denote by $J(F_\epsilon)$. By definition, $J(F_\epsilon)$ is the set of points in $\mathbb{C}$ at which the family of iterates of F_ϵ fails to be a normal family in the sense of Montel. Equivalently, $J(F_\epsilon)$ is the closure of the set of repelling periodic points of F_ϵ. It is also the set on which F_ϵ behaves chaotically. The complement of the Julia set is called the Fatou set.

When $\epsilon = 0$ we have the simple map $F_0(z) = z^2$ whose dynamics are well understood. This is a degree two mapping of $\mathbb{C}$ whose Julia set is the unit circle. All orbits in $|z| < 1$ tend to the attracting fixed point at the origin; all orbits in $|z| > 1$ tend to ∞. So the dynamics are quite simple in this case.

When $\epsilon \neq 0$, the map is now a degree four rational map; we say that F_0 has undergone a *singular perturbation* when ϵ becomes nonzero. In this case we witness a dramatic change in the dynamics of F_ϵ. We shall prove:

Theorem. *In any neighborhood of the origin in the complex ϵ-plane, there are infinitely many open sets $\mathcal{O}_n$ such that, if $\epsilon \in \mathcal{O}_n$, the Julia set of F_ϵ is a Sierpinski curve. Hence any two such Julia sets are homeomorphic. However, if ϵ_1 and ϵ_2 lie in distinct $\mathcal{O}_n$'s, then the corresponding maps are not conjugate on their respective Julia sets.*

Recall that a *Sierpinski curve* is, by definition, a compact, connected, locally connected, nowhere dense subset of the plane that has the property that any two boundaries of complementary domains are pairwise disjoint simple closed curves. See Figure 1 for several examples of these types of Julia sets. The *Sierpinski carpet* is perhaps the most well known example of a Sierpinski curve; this set is obtained by dividing the unit square into nine equal-sized subsquares, and then removing the (open) middle square. Next, the open middle subsquare of each of the remaining eight smaller squares is removed leaving 64 smaller closed subsquares. This process is repeated ad infinitum to produce the Sierpinski carpet.

Any two Sierpinski curves homeomorphic. The importance of Sierpinski curves lies in the fact that they are universal objects in the sense that there

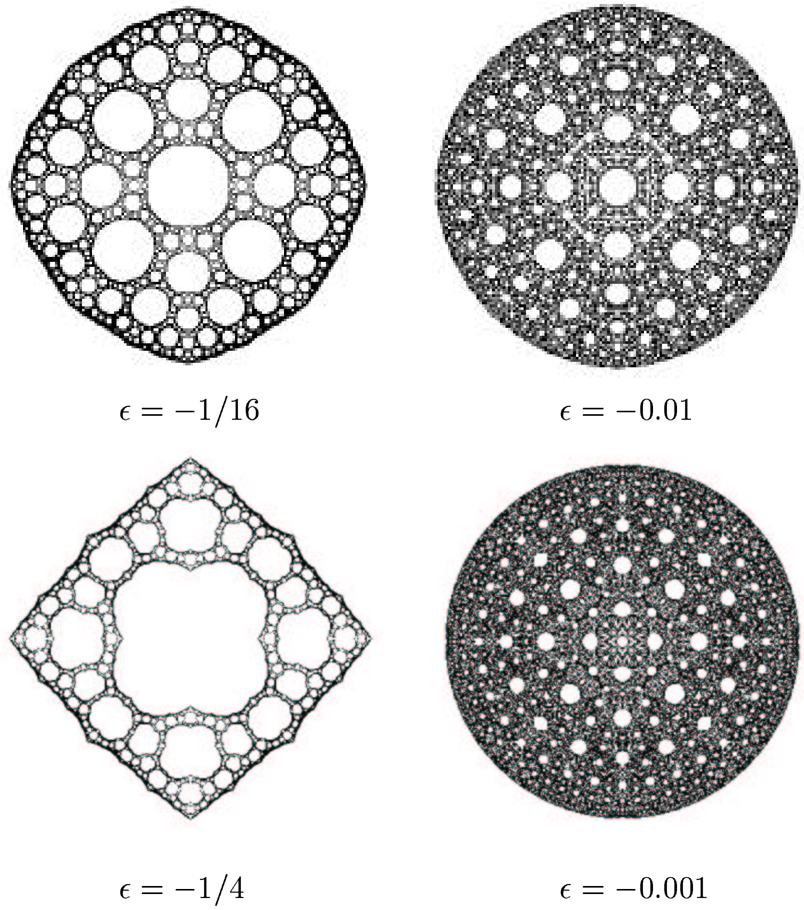

Figure 1: The Julia sets for various values of ϵ.

is a homeomorphic copy of any compact, connected, one-dimensional, planar set contained as a subset of any Sierpinski curve. See [9].

Julia sets that are Sierpinski curves have been observed in other complex dynamical systems. For example, building on work of Wittner [10], Milnor and Tan-Lei [7] have shown that there is a specific degree two rational map having superattracting cycles of periods three and four for which the Julia set is a Sierpinski curve. The examples presented below are somewhat different. In our family we produce infinitely many open intervals I_n on the negative ϵ axis with $n \geq 2$ for which the following properties hold for each $\epsilon \in I_n$:

1. $J(F_\epsilon)$ is a Sierpinski curve;

2. There is a unique attracting cycle for F_ϵ, namely the attracting fixed point at ∞;

3. The complementary domains in the Sierpinski curve Julia set are the components of the basin of attraction of ∞;

4. All four nonzero critical points of F_ϵ enter the immediate basin of attraction of ∞ at iteration n.

The intervals I_n sit inside simply connected open regions $\mathcal{O}_n$ in the complex ϵ-plane. For any complex $\epsilon \in \mathcal{O}_n$, the map F_ϵ has similar properties as those for $\epsilon \in I_n$.

McMullen [5] has considered the family of function $z^n + \epsilon/z^m$ in the case where n and m satisfy $1/n + 1/m < 1$. He finds that, with these restrictions on n and m and ϵ sufficiently small, the Julia set of these maps are given by a Cantor set of simple closed curves surrounding the origin. Hence the singular perturbation that arises in these cases is significantly different from the one that arises in our case. The case where $n = 2$ but $m = 1$ was discussed in [4]. Combining the techniques in that paper with those below shows that a similar collection of Sierpinski curve Julia sets exists in this family when ϵ is small.

The authors wish to thank John Mayer and James Rogers for helpful conversations regarding the topological properties of Sierpinski curves and Curt McMullen, John Milnor, and Kevin Pilgrim regarding their appearance in complex dynamics.

2 Basic properties

In this paper we shall only consider the case where $\epsilon < 0$. However, many of the results are easily extended to the case of certain complex ϵ. The following is a straightforward computation.

Proposition. *For each $\epsilon < 0$:*

1. *F_ϵ has a single pole of order two at 0 and four pre-poles at $(-\epsilon)^{1/4}$;*

2. *The point at ∞ is a superattracting fixed point; we have $F_\epsilon'(\infty) = 0$ and $F_\epsilon''(\infty) \neq 0$.*

3. *The four nonzero critical points of F_ϵ are given by $\epsilon^{1/4}$;*

4. *The two critical values of F_ϵ are given by $\pm v(\epsilon) = \pm 2\sqrt{\epsilon}$;*

5. *The second iterates of the nonzero critical points all land on the same point, namely $F_\epsilon(\pm v(\epsilon)) = 1/4 + 4\epsilon$.*

As we are primarily interested in the singular perturbation that occurs when ϵ becomes nonzero, we henceforth restrict to the case where ϵ belongs to the interval $[-1/16, 0)$. Many of the results below extend to certain values to the left of $-1/16$ as well as to ϵ complex.

The graph of F_ϵ on the real axis shows that there is a pair of repelling fixed points in $\mathbb{R}$. See Figure 2. Let $p = p(\epsilon)$ be the fixed point in $\mathbb{R}^+$. The graph of F_ϵ also shows that the orbit of $x \in \mathbb{R}$ tends directly to ∞ if $|x| > p$.

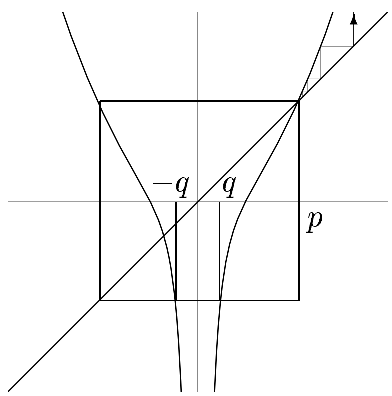

Figure 2: The graph of F_ϵ on the real line for $\epsilon = -1/16$.

Let I_ϵ denote the interval $[-p, p]$. Let $\pm q(\epsilon) = \pm q \in I_\epsilon$ be the points for which $F_\epsilon(\pm q) = -p$ so that $F_\epsilon^2(\pm q) = p$. If $x \in (-q, q)$ then $F_\epsilon(x) < -p$ and $F_\epsilon^2(x) > p$. Hence $F_\epsilon^n(x) \to \infty$ for all $x \in (-q, q)$. We call this interval the *trap door* in $\mathbb{R}$; any orbit in I_ϵ that enters this open interval falls through the trap door and then tends to ∞.

The preimage of $\mathbb{R}$ under F_ϵ consists of the real and imaginary axes; each of these axes is mapped two-to-one over $\mathbb{R}$. The preimage of the imaginary axis consists of two sets: the four rays $\theta = \pm\pi/4, \ \pm 3\pi/4$ and the circle of radius $r_\epsilon = |\epsilon|^{1/4}$ centered at the origin. Note that these rays meet this circle at the four critical points of F_ϵ. Points on the circle given by $r = r_\epsilon e^{i\theta}$ are mapped to points of the form $2\sqrt{|\epsilon|}\ i\ \sin(2\theta)$ on the imaginary axis. Therefore this circle is mapped in four-to-one fashion over the interval $[-v(\epsilon), v(\epsilon)]$ on this axis (except at the endpoints). Each of the four rays is mapped in two-to-one fashion over either $[v(\epsilon), \infty)$ or $(-\infty, -v(\epsilon)]$ on the imaginary axis.

We now investigate the behavior of F_ϵ near ∞.

Proposition. *For each $\epsilon \in [-1/16, 0)$, there is an invariant simple closed curve γ_ϵ encircling the origin on which F_ϵ is conjugate to the map $z \to z^2$. All orbits outside γ_ϵ tend to ∞.*

Proof: Consider the circle $r = (3/4)e^{i\theta}$. For z on this circle, we have

$$|F_\epsilon(z)| = \left|\frac{9}{16}e^{2i\theta} - \frac{16\epsilon}{9}e^{-2i\theta}\right| \leq \frac{9}{16} + \frac{1}{9} < \frac{3}{4}.$$

Let U be the set of $z \in \overline{\mathbb{C}}$ such that $|z| > 3/4$ and let $U' = F_\epsilon^{-1}(U) \cap U$. Then $F : U' \to U$ is a quadratic-like map (see [3]). As a quadratic-like map on U', its filled Julia set is

$$K_{F_\epsilon} = \{z \in U \mid F_\epsilon^n(z) \in U' \text{ for all } n\}.$$

Using the Douady-Hubbard theory, we know that F_ϵ is quasiconformally conjugate to a quadratic polynomial Q on a neighborhood of K_{F_ϵ}. Since $F_\epsilon | K_{F_\epsilon}$ has a superattracting fixed point, $Q(z) = z^2$. The invariant curve γ_ϵ is the image of the Julia set of Q, i.e., the unit circle, under the quasiconformal conjugacy.

$\square$

3 Sierpinski curve Julia sets

In this section we first restrict attention to the special case where $\epsilon = -1/16$. We write $F = F_{-1/16}$. In this case the four critical points of F lie at the points $\omega/2$ where ω is a fourth root of -1. The critical values are $\pm i/2$ and we have $F(\pm i/2) = 0$. Thus the second iterate of each of the critical points lands on the pole at the origin; this is what makes the case $\epsilon = -1/16$ special. There are prepoles at $\pm 1/2$ as well as at $\pm i/2$.

As in the previous section, let I denote the interval $[-p, p]$, where p is the repelling fixed point for F that lies in $\mathbb{R}^+$. Let $\pm q \in I$ be the points for which $F(\pm q) = -p$ so that $F^2(\pm q) = p$. The open interval $(-q, q)$ is the trap door in $\mathbb{R}$. Below we show that the set of points whose orbits remain for all time in I forms a Cantor set; these are the only points in $\mathbb{R}$ whose orbits do not escape to ∞.

As above, the preimage of $\mathbb{R}$ under F consists of the real and imaginary axes while the preimage of the imaginary axis consists of two sets: the four rays $\theta = \pm\pi/4,\ \pm 3\pi/4$ and the circle of radius $1/2$ centered at the origin. Note that all four critical points as well as the four pre-poles lie on this circle. For this reason we call the circle $r = 1/2$ the *critical circle.* Points on the critical circle given by $e^{i\theta}/2$ are mapped to points of the form $(i/2)\sin(2\theta)$ on the imaginary axis. Therefore this circle is mapped in four-to-one fashion over the interval $[-1/2, 1/2]$ on this axis (except at the endpoints, which are the critical values). Each of the four rays is mapped in two-to-one fashion over either $[1/2, \infty)$ or $(-\infty, -1/2]$ on the imaginary axis.

Let γ denote the boundary of the basin of attraction of the superattracting fixed point at ∞. By the Proposition in the previous section, γ is a simple closed curve on which F_ϵ is conjugate to $z \to z^2$. Note that the immediate basin B of ∞ is the exterior of γ and that F is two-to-one on this basin. Since F is conjugate to z^2 on γ, there is a unique fixed point on γ. This must be the fixed point $p \in \mathbb{R}$, since we know that this point lies on the boundary of B.

One of the principal objects contained in the Julia set of F is a Cantor necklace. To define this set, we let Γ denote the Cantor middle thirds set in the unit interval $[0, 1]$. We regard this interval as a subset of the real axis in the plane. For each open interval of length $1/3^n$ removed from the unit interval in the construction of Γ, we replace this interval by a circle of diameter $1/3^n$ centered at the midpoint of the removed interval. Thus this circle meets the Cantor set at the two endpoints of the removed interval. We call the resulting set the *Cantor middle-thirds necklace.* See Figure 3. Any set homeomorphic to the Cantor middle-thirds necklace is called a *Cantor necklace.*

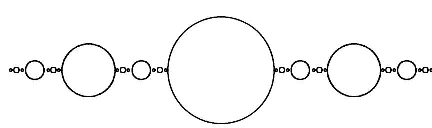

Figure 3: The Cantor middle-thirds necklace.

Let T denote the component of the preimage of B that contains the origin.

We call T the *trap door* in $\mathbb{C}$. The function F maps T in two-to-one fashion (except at the pole at the origin) onto B. The boundary of T which we call τ is mapped in two-to-one fashion onto γ. Note that τ and γ are disjoint; this follows from the fact that the circle of radius 3/4 about the origin is mapped strictly inside itself.

Let V_1 denote the sector given in polar coordinates in the plane by $-\pi/4 \leq \theta \leq \pi/4$. Let V_2 be the sector $3\pi/4 \leq \theta \leq 5\pi/4$. Let $V = V_1 \cup V_2$. Observe that, since the image of each of the rays bounding the V_j is the imaginary axis, these rays meet γ in exactly one point, namely a point whose image under F^2 is $-p$.

Let U be the closed set $V - (T \cup B)$. The set U consists of two closed simply connected regions given by $U_j = U \cap V_j$ for $j = 1, 2$.

Proposition. *Let Λ_U be the set of points whose orbits remain for all iterations in U. Then Λ_U is a Cantor set and $F \mid \Lambda_U$ is conjugate to the one-sided shift on two symbols.*

Proof: Each of the U_j are mapped in essentially one-to-one fashion onto the complement of B in $\mathbb{C}$. Technically, F maps the boundary lines of the V_j in U_j in two-to-one fashion onto the intervals $[\pm 1/2, \pm\zeta]$ on the imaginary axis, where $\pm\zeta$ denotes the point of intersection of this axis with γ. The map is one-to-one at all other points in U_j. Also, the portion of the critical circle $r = 1/2$ in each U_j is mapped onto the interval $(-1/2, 1/2)$ on the imaginary axis. Note that each of these intervals lies in the complement of the U_j. Let $\mathcal{O}$ be the complement of B minus the intervals $[\pm 1/2, \pm\zeta]$ on the imaginary axis. So we have a pair of well-defined inverses G_j of F that map $\mathcal{O}$ into U_j. Standard arguments then show that these inverses are contractions in the Poincaré metric on $\mathcal{O}$. (Technically, we must remove small strips along the imaginary axis and also fatten γ in order to find a simply connected region on which the Poincaré metric resides.) Moreover, for any one-sided sequence $(s_0 s_1 s_2 \ldots)$ of 1's and 2's, the set

$$\bigcap_{j=0}^{\infty} G_{s_0} \circ \ldots \circ G_{s_j}(\mathcal{O})$$

is a unique point and the map that takes the sequence $(s_0 s_1 s_2 \ldots)$ to this point defines a homeomorphism between the space of one-sided sequences of 1's and 2's and Λ_U. Hence Λ_U is a Cantor set and standard arguments show that $F \mid \Lambda_U$ is conjugate to the one-sided shift on two symbols.

□

Now let W denote the union of $\overline{T}$, γ, and U. Clearly, W is a closed subset of $\overline{\mathbb{C}}$. The function F maps W in essentially a two-to-one fashion over $\overline{\mathbb{C}}$. The exceptions are:

1. ∞, which has only one preimage in W;
2. γ together with the open intervals from $\pm i/2$ to γ on the imaginary axis, each point of which has four preimages in W.

We claim that the set of points Λ_W whose orbits remain for all iterations in W is the union of γ and a Cantor necklace connecting the points $\pm p$ and lying along the real axis.

To see this, note that if $z \in \Lambda_W$, then either the orbit of z lands on γ or the orbit of z remains for all time in the U_j. In the latter case, z is in the Cantor set Λ_U lying on the real axis. In the former case, z lies on one of the preimages of τ. We claim that these preimages form the "circles" making up the Cantor necklace.

To see this, consider the closed subset $\overline{T} \cup U$ in W. This set resembles a "bow tie". The preimage of this bow tie is a pair of closed, simply connected regions, one in each of the U_j. Note that each of these preimages is a homeomorphic copy of $\overline{T} \cup U$ that meets both γ and the boundary of T in an arc. That is, each preimage is a smaller bow tie extending across one of the U_j. In particular, $\overline{T}$ has a pair of preimages, one in each of the U_j. The interior of the preimages of $\overline{T}$ is mapped into the trap door by F.

Now we continue: the second preimage of the bow tie consists of four smaller bow ties, each containing a second preimage of the trap door, and each connecting either γ or τ to the preimage of the trap door. Continuing in this fashion, we have:

Proposition. *The set of points whose orbit remains for all time in W is a Cantor necklace extending from $-p$ to p along the real axis together with the simple closed curve γ.*

We now turn to the structure of the Julia set of F. Let $S = \overline{\mathbb{C}} - B$. Since the orbit of each critical point eventually enters B, it follows that all of the stable domains in the complement of the Julia set have this property. Hence $J(F)$ is the set of points whose orbits remain for all time in S. That is, $J(F)$ is the complement of the basin of attraction of ∞. Now the points whose orbits leave S must lie in one of the preimages of the trap door T.

Each preimage of T is a finite union of disjoint, open, simply connected sets. Thus $J(F)$ is just S with countably many open disks removed. Hence $J(F)$ is connected. Since all critical points tend to ∞, it is also known that $J(F)$ is locally connected. Also, $J(F)$ is nowhere dense, for if $J(F)$ contains an open subset, then it must be all of $\mathbb{C}$, which it is not. See [1] for these standard facts about the Julia set.

It remains to show that the boundaries of all of the complementary domains in $J(F)$ are pairwise disjoint. Note that this is indeed the case along the real axis, where the boundaries of the complementary domains are just the endpoints of the Cantor set.

Now each of these complementary domains in S is a particular preimage of the trap door. The preimage of γ (not equal to γ) is τ, which we now denote by $F^{-1}(\gamma)$. We have that γ and $F^{-1}(\gamma)$ are disjoint, since we know that F maps the circle of radius $3/4$ strictly inside itself. Hence $F^{-1}(\gamma)$ lies inside this circle and so is disjoint from γ. The function F maps the annular region in S lying between γ and $F^{-1}(\gamma)$ onto S, with both of the boundary curves mapped onto γ. Call this annular region A_1. Hence the preimages of $\overline{T}$ lie in the interior of A_1 and so their boundary curves are disjoint from γ and $F^{-1}(\gamma)$. Since none of the critical points lie in these preimages, it follows that these boundary curves are pairwise disjoint and each is mapped homeomorphically onto τ. Call these boundary curves $F^{-2}(\gamma)$.

Now remove from A_1 each of the four open regions bounded by $F^{-2}(\gamma)$. The remaining set A_2 is a disk with five holes (counting T). The boundary curves of A_2 are mapped to the boundary curves of A_1 and so the preimages of $F^{-2}(\gamma)$ lie in the interior of A_2. There are only twelve such preimages, since four of the preimages contain critical points and these are mapped two-to-one onto their images. Nonetheless, each is contained in the interior of A_2 and so their bounding curves are disjoint from the previous boundaries. They are also pairwise disjoint, since there are no critical points along these boundaries.

Continuing in this fashion, we see that all of the preimages of γ are disjoint from each other. We have thus shown that $J(F)$ is a Sierpinski curve.

For the more general case, we consider ϵ in the interval $(-1/16, 0)$. We assume further that there exists $n \geq 2$ such that the n^{th} iterate of F_ϵ maps each of the critical values into the trap door T_ϵ, that is, $F_\epsilon^n(\epsilon^{1/4}) \in (-q(\epsilon), q(\epsilon)) \subset T_\epsilon$. The proof that there is an invariant Cantor set on the real line goes through without change. The only modification necessary to prove the existence of a Cantor necklace along $\mathbb{R}$ is to note that the image of the rays bounding the V_j are now intervals in the imaginary axis extending from $\pm 2\sqrt{|\epsilon|}$ to the points $\pm\zeta_\epsilon$ on γ_ϵ. Also, the image of the critical circle $r = |\epsilon|^{1/4}$ in each U_j is now the interval $[-2\sqrt{|\epsilon|}, 2\sqrt{|\epsilon|}\,]$ along the imaginary axis. This interval is then mapped by F_ϵ into $\mathbb{R}$ and its image is strictly contained in the interval $(-p(\epsilon), p(\epsilon))$ for each ϵ. In particular, the critical values do not lie in B_ϵ. The proof now goes through as above. We have proved:

Theorem. *If $\epsilon \in [-1/16, 0)$ and $F_\epsilon^n(\epsilon^{1/4})$ lies in the trap door for some n, then $J(F_\epsilon)$ is a Sierpinski curve.*

If the orbit of the critical point meets the boundary of the trap door, then certain preimages of the trap door have boundaries that meet at a single point. This point is one of the critical points (or their preimages). Hence $J(F_\epsilon)$ is not a Sierpinski curve in this case.

4 Conjugacy questions

We continue to deal with the case where ϵ is negative with $-1/16 \leq \epsilon < 0$. Let T_ϵ and B_ϵ be the trap door and the basin of ∞ respectively for F_ϵ. Let c_ϵ be any of the four critical points of F_ϵ. We have

$$F_\epsilon^2(c_\epsilon) = 4\epsilon + \frac{1}{4}.$$

Thus, after two iterations, each of the critical points land on the same point on the real axis.

Proposition. *There is an increasing sequence $\epsilon_2, \epsilon_3, \ldots$ with $\epsilon_j \to 0$ and $F_{\epsilon_j}^j(c_{\epsilon_j}) = 0$.*

Proof: Since $F_\epsilon^2(c_\epsilon) = 4\epsilon + 1/4$, it increases monotonically toward $1/4$ as $\epsilon \to 0$. Now the orbit of $1/4$ remains in $\mathbb{R}^+$ under F_0 and decreases monotonically to 0. Hence, given N, for ϵ sufficiently small, $F_\epsilon^j(c_\epsilon)$ lies in $\mathbb{R}^+$ for $2 \leq j \leq N$ and moreover this finite sequence is decreasing.

Now suppose $\beta < \alpha < 0$. We have $F_\beta(x) < F_\alpha(x)$ for all $x \in \mathbb{R}^+$. Also, $F_\beta^2(c_\beta) < F_\alpha^2(c_\alpha) < 1/4$. Hence $F_\beta^j(c_\beta) < F_\alpha^j(c_\alpha)$ for all j for which $F_\beta^j(c_\beta) \in \mathbb{R}^+$. The result then follows by continuity of F_ϵ with respect to ϵ. □

Note that $\epsilon_2 = -1/16$. If $\epsilon_2 < \epsilon < 0$, then the Proposition in Section 1 shows that the boundary γ_ϵ is a simple closed curve and $F_\epsilon \mid \gamma_\epsilon$ is conjugate to $z \to z^2$.

Using the previous Proposition, we may find open intervals I_j about ϵ_j for $j = 2, 3, \ldots$ having the property that, if $\epsilon \in I_j$, then $F_\epsilon^j(c_\epsilon) \in T_\epsilon$, and so $F_\epsilon^{j+1}(c_\epsilon) \in B_\epsilon$. Therefore, $F_\epsilon^n(c_\epsilon) \to \infty$ as $n \to \infty$, and so $J(F_\epsilon)$ is a Sierpinski curve.

Now let $C(c_\epsilon)$ denote the component of the Fatou set of F_ϵ containing c_ϵ. Note that F_ϵ is two-to-one on each of the four components containing these critical points, and we have $F_\epsilon^j(C(c_\epsilon)) = T_\epsilon$. Now suppose that $F_\epsilon|J(F_\epsilon)$ is conjugate to $F_\alpha|J(F_\alpha)$ for some $\alpha \in \cup I_j$. This conjugacy must take the boundaries of B_ϵ and T_ϵ to the corresponding boundaries of B_α and T_α. Similarly the boundaries of the four regions $C(c_\epsilon)$ must be mapped to the corresponding regions by the conjugacy, since these are the only complementary domains (besides B_ϵ and T_ϵ) on which F_ϵ is two-to-one. If, however, $\epsilon \in I_j$ and $\alpha \in I_k$ with $j \neq k$, then these maps cannot be conjugate, since a conjugacy maps each of the j^{th} preimages of the T_ϵ to one of the j^{th} preimages of T_α. Such a conjugacy would also have to map boundaries of domains on which F_ϵ and F_α were two-to-one to each other. Since $j \neq k$, this is impossible. We therefore have:

Theorem. *Let $\epsilon \in I_j$ and $\alpha \in I_k$ with $j \neq k$. Then F_ϵ is not conjugate to F_α on their corresponding Julia sets.*

References

[1] Blanchard, P. Complex Analytic Dynamics on the Riemann Sphere, *Bull. Amer. Math. Soc.* Vol. II, No. 1, 1984, 85-141.

[2] Douady, A. and Hubbard, J. Itération des Polynômes quadratiques complexes, *C.R. Acad. Sci. Paris*, t.29, Serie I-1982, pp. 123-126.

[3] Douady, A. and Hubbard, J. On the Dynamics of Polynomial-like Mappings, *Ann. Scient., Éc. Norm. Sup.* 4^e *séries*, t.18, (1985), 287-343.

[4] Devaney, R., Josic, K. and Shapiro, Y. Singular Perturbations of Quadratic Maps. To appear in *Intl. J. Bifurcation and Chaos.*

[5] McMullen, C. Automorphisms of Rational Maps. *Holomorphic Functions and Moduli*. Vol. 1. Math. Sci. Res. Inst. Publ. **10**. Springer, New York, 1988.

[6] Milnor, J. Dynamics in One Complex Variable. Vieweg, 1999.

[7] Milnor, J. and Tan Lei. A "Sierpinski Carpet" as Julia Set. Appendix F in Geometry and Dynamics of Quadratic Rational Maps. *Experiment. Math.* **2** (1993), 37-83.

[8] Sullivan, D., Quasiconformal Maps and Dynamical Systems I, Solutions of the Fatou-Julia Problem on Wandering Domains. *Ann. Math.* **122** (1985), 401-418.

[9] Whyburn, G. T. Topological Characterization of the Sierpinski Curve. *Fund. Math.* **45** (1958), 320-324.

[10] Wittner, B. On the Bifurcation Loci of Rational Maps of Degree Two. Thesis, Cornell University (1988).