Cantor Webs in the Parameter and Dynamical Planes of Rational Maps *

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Our goal in this paper is to describe a new type of structure that exists in both the parameter plane and dynamical planes of the families of rational maps given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

where $z \in \mathbb{C}$ and $n \geq 3$. Called Cantor webs, these sets are homeomorphic to a model set constructed as follows. Start with an open disk in the plane. Surround this disk with k smaller open disks that are symmetrically arranged around the original disk. Then surround each of these k disks with k smaller disks again symmetrically arranged and continue ad infinitum. These disks are arranged so that they can then be connected by a Cantor set.

This construction generalizes the construction of a Cantor necklace that was described in [4]. A Cantor necklace is a planar set obtained as follows. Start with the Cantor middle-thirds set lying on the x-axis in the plane. Replace each removed open interval with an open disk whose diameter is the length of the removed interval. The union of the Cantor set and these countably many open disks is a Cantor necklace. So a Cantor necklace is a particular example of a Cantor web when $k = 2$.

In the dynamical plane, the behavior of the map F_λ on the Cantor web is as follows. The Cantor set portion of the web is an invariant set on which the map is conjugate to a one-sided shift map on $2n - 2$ symbols, while the open disks consist of points whose orbits eventually land in the immediate basin of attraction of ∞. Each member of this family of maps for which the Julia set is a connected set possesses a homeomorphic copy of this set, and each of these maps has the same dynamics on these sets.

In the parameter plane, the Cantor web is a collection of parameters for which the corresponding map has a Julia set that, with certain exceptions, is a Sierpinski curve. See Figures 1 and 2.

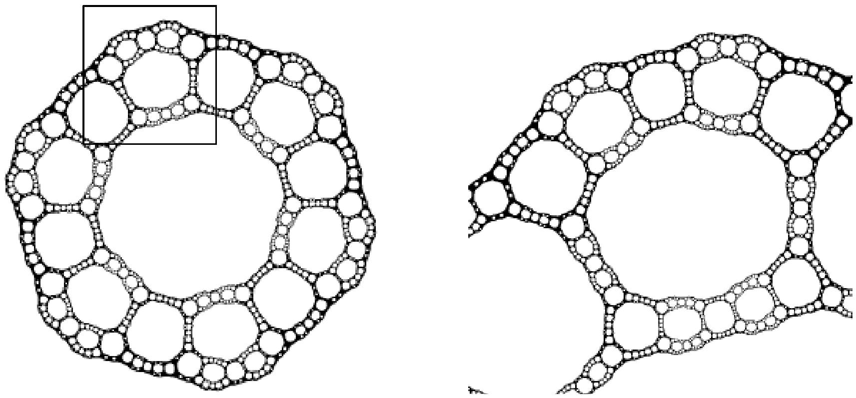

Figure 1: The Julia set of $F_\lambda(z) = z^3 + 0.125/z^3$ and a magnification illustrating a Cantor web.

1 Preliminaries

Let $F_\lambda(z) = z^n + \lambda/z^n$ where $\lambda \in \mathbb{C}$ is a parameter and $n \geq 3$. The reason for choosing $n \geq 3$ will be explained below. When $|z|$ is large, $F_\lambda(z) \approx z^n$, so F_λ has an immediate basin of attraction at ∞ that we denote by B_λ. As is well known [9], there is a Böttcher coordinate ϕ_λ that conjugates F_λ to $z \mapsto z^n$ in a neighborhood of ∞.

Each F_λ also has a pole of order n at the origin; hence there is an open neighborhood of 0 that is mapped into B_λ. Now, either this neighborhood is disjoint from the immediate basin B_λ or else this neighborhood is contained in B_λ. In the former case, we denote the entire preimage of B_λ that contains the origin by T_λ. We call this region the *trap door* since any point $z \notin B_\lambda$ but such that $F_\lambda^k(z)$ does lie in B_λ for some $k > 0$ has the property that there is a unique point on the orbit of z that lies in T_λ.

Besides 0 and ∞, F_λ has $2n$ additional critical points c_λ given by $(c_\lambda)^{2n} = \lambda$. However, F_λ has only two critical values given by $v_\lambda = \pm 2\sqrt{\lambda}$. In fact, there is only one free critical orbit for F_λ up to symmetry. For, if n is even,

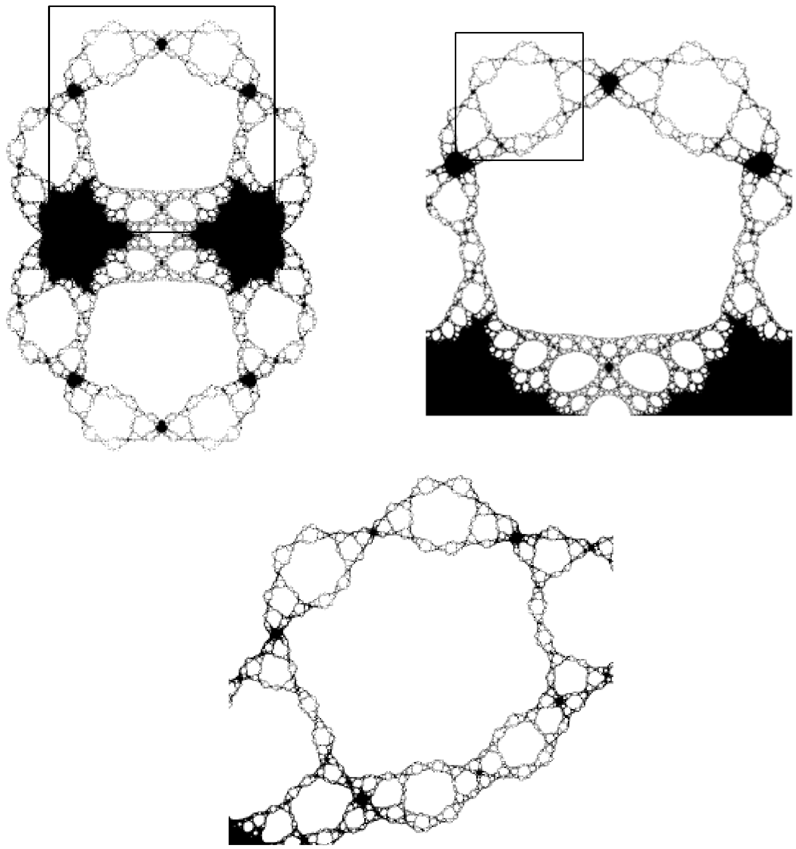

Figure 2: The parameter plane for the family $F_\lambda(z) = z^3 + \lambda/z^3$ and a magnification illustrating a Cantor web together with a further magnification showing a portion of the web.

we have $F_\lambda(2\sqrt{\lambda}) = F_\lambda(-2\sqrt{\lambda})$, so each of the critical orbits land on the same orbit after two iterations. If n is odd, then we have $F_\lambda(-z) = -F_\lambda(z)$, so the orbits of $\pm 2\sqrt{\lambda}$ are always symmetric under $z \mapsto -z$.

We call the straight rays given by tc_λ with $t > 0$ the *critical point rays.* Note that

$$F_\lambda(tc_\lambda) = \lambda^{1/2}\left(t^n + \frac{1}{t^n}\right),$$

so it follows that each critical point ray is mapped two-to-one onto the straight ray that extends from v_λ to ∞. We call this ray the *critical value* *ray.*

Each F_λ also has $2n$ prepoles p_λ given by $(p_\lambda)^{2n} = -\lambda$, so $F_\lambda(p_\lambda) = 0$. The rays tp_λ with $t > 0$ are called *prepole rays.* These rays are mapped one-to-one onto the entire line segment passing through $\pm iv_\lambda$ and extending to ∞ in both directions. Note that these lines are perpendicular to the critical value rays.

For each λ, there is a unique critical point lying in the sector $0 \leq \operatorname{Arg} z < \pi/n$. Call this critical point $c_0 = c_0(\lambda)$. We denote the remaining critical points by c_j and order them in the clockwise direction around the origin. We call the open sector bounded by two adjacent critical point rays a *prepole sector* since each such sector contains a unique prepole. Let $P_j = P_j(\lambda)$ denote the prepole sector bounded by the critical point rays through c_j and c_{j+1}, and let $p_j = p_j(\lambda)$ denote the unique prepole that lies in P_j. An easy computation shows that F_λ maps each P_j univalently onto the complement of the two critical value rays in $\mathbb{C}$. Henceforth, we denote the image of c_0 by v_λ, so we have $F_\lambda(c_{2k}) = v_\lambda$ whereas $F_\lambda(c_{2k+1}) = -v_\lambda$ for each k. (Note that the notation c_j and p_j becomes ambiguous in the special case where $\lambda = 1, 2$; however, we will never specifically deal with these special cases.)

Recall that the *Julia set* $J(F_\lambda)$ for the rational map F_λ has several equivalent characterizations. It is known that the Julia set is the closure of the set of repelling periodic points as well as the boundary of the set of points whose orbits tend to ∞ [9]. The complement of the Julia set is called the *Fatou set.*

There are several symmetries in the dynamical plane. First let $\nu = \exp(\pi i/n)$. Then we have $F_\lambda(\nu z) = -F_\lambda(z)$, so, as above, either the orbits of z and νz coincide after two iterations (when n is even), or else they behave symmetrically under $z \mapsto -z$ (when n is odd). In either event, the dynamical plane and the Julia set both possess $2n$-fold symmetry, as do

B_λ and T_λ. Let $H_\lambda(z)$ be one of the n involutions given by $\lambda^{1/n}/z$. Then $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, so the dynamical plane and Julia set are also symmetric under each H_λ. Note that $H_\lambda(B_\lambda) = T_\lambda$.

The following result is proved in [5].

Theorem *(The Escape Trichotomy). Let $F_\lambda(z) = z^n + \lambda/z^n$ and consider the orbit of v_λ.*

1. *If v_λ lies in B_λ, then $J(F_\lambda)$ is a Cantor set;*

2. *If v_λ lies in T_λ, then $J(F_\lambda)$ is a Cantor set of simple closed curves, each of which surrounds the origin;*

3. *If $F_\lambda^k(v_\lambda)$ lies in T_λ where $k \geq 1$, then $J(F_\lambda)$ is a Sierpinski curve.*

Finally, if v_λ does not lie in either B_λ or T_λ, then $J(F_\lambda)$ is a connected set.

We remark that case 2 of the above result was proved by McMullen [8]. This part of the Theorem does not hold if $n = 1$ or $n = 2$; this is one of the reasons we restrict attention in this paper to the case $n \geq 3$.

A *Sierpinski curve* is any planar set that is homeomorphic to the well-known fractal called the *Sierpinski carpet.* By a result of Whyburn [12], there is a topological characterization of such sets: any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any pair of complementary domains are bounded by simple closed curves that are pairwise disjoint is known to be homeomorphic to the Sierpinski carpet. A Sierpinski curve also has the interesting property that it is a universal plane continuum in the sense that it contains a homeomorphic copy of any compact, connected, one-dimensional planar set.

We turn now to the parameter plane for these families, i.e., the λ-plane. There are some different symmetries in the parameter planes for these maps.

Let $\omega = \exp(2\pi/(n-1))$. Then the parameter plane is easily seen to be symmetric under the maps

1. $\lambda \mapsto \overline{\lambda}$;
2. $\lambda \mapsto \omega\lambda$;
3. $\lambda \mapsto \omega\overline{\lambda}$.

In particular, the parameter plane can be separated into $n-1$ symmetry sectors of the form

$$\frac{2j\pi}{n-1} < \operatorname{Arg}\lambda < \frac{2(j+1)\pi}{n-1}.$$

Because of the Escape Trichotomy, the parameter plane for F_λ (the λ-plane) divides into three distinct regions. Let $\mathcal{L}$ be the set of parameters for which $v_\lambda \in B_\lambda$ so $J(F_\lambda)$ is a Cantor set. We call $\mathcal{L}$ the *Cantor set locus*. As in the case of the Mandelbrot set and quadratic polynomials, there is a well defined Böttcher coordinate Φ defined on $\mathcal{L}$. It is known that $\Phi : \mathcal{L} \to \mathbb{C} - \overline{\mathbb{D}}$ is an analytic homeomorphism and that the preimages of all rational rays in $\mathbb{C} - \overline{\mathbb{D}}$ land on a unique point in the boundary of $\mathcal{L}$ (see [10]).

Let $\mathcal{M}$ denote the set of parameters for which $v_\lambda \in T_\lambda$; $\mathcal{M}$ is called the *McMullen domain*. It is known that $\mathcal{M}$ is an open disk punctured at the origin and bounded by a simple closed curve [1].

Let $\mathcal{C}$ denote the complement of $\mathcal{L} \cup \mathcal{M}$. $\mathcal{C}$ is called the *connectedness locus* since $J(F_\lambda)$ is a connected set if $\lambda \in \mathcal{C}$. It is known that $\mathcal{C}$ contains precisely $(2n)^{k-3}(n-1)$ *Sierpinski holes* with escape time $k \geq 3$ [2], [11]. These are open disks in $\mathcal{C}$ in which each corresponding map has the property that the critical orbit lands in B_λ at iteration k or, equivalently, the orbit of the critical value lands in T_λ at iteration $k-2$. See Figure 3. There is also a Böttcher coordinate on each Sierpinski hole. Let $\mathcal{S}$ be a Sierpinski hole with escape time k so that $F_\lambda^{k-2}(v_\lambda) \in T_\lambda$. Fix a choice of the involution

H_λ. Then the Böttcher coordinate on $\mathcal{S}$ is given by $\Psi : \mathcal{S} \to \mathbb{C} - \overline{\mathbb{D}}$ where $\Psi(\lambda) = \phi_\lambda(H_\lambda(F_\lambda^{k-2}(v_\lambda)))$. See [2], [11].

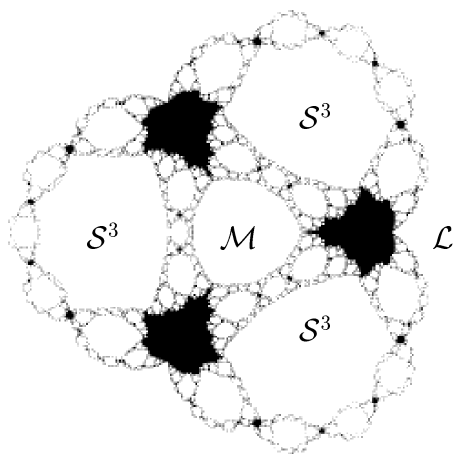

Figure 3: The parameter plane when $n = 4$. The open disks marked $\mathcal{S}^3$ are the Sierpinski holes with escape time 3.

In Figure 3, there are three clearly visible copies of the Mandelbrot set. Indeed, it is known that there are $n - 1$ copies of the Mandelbrot set that straddle the rays given by $\operatorname{Arg} \lambda = s\omega^k$ for $s > 0$ [3]. These sets are called the *principal Mandelbrot sets* in the parameter plane. The cusps of the main cardioids of these sets all lie on the boundary of $\mathcal{L}$ while the tips of the tails of these sets (i.e., the parameters corresponding to $c = -2$ in the usual Mandelbrot set for $z^2 + c$) all lie in the boundary of $\mathcal{M}$. In fact, there are infinitely many other copies of the Mandelbrot set in $\mathcal{C}$ [2].

2 Cantor Webs

In this section we define a set $S_k \subset \mathbb{R}^2$ for each $k \geq 2$ together with a specific dynamical system defined on S_k. A *Cantor k-web* will then be any set that is homeomorphic to S_k. The set S_k will consist of a Cantor set portion on which there is defined a natural one-sided shift map on $2k$ symbols, together with another portion that is a countable union of simply connected open sets. The union of both portions of S_k will be a connected set that is neither open nor closed.

For simplicity, we begin with the case $k = 2$. Let U be the closed unit square in the plane. Consider the four closed subsquares of sidelength $1/3$ and that touch one of the corners of U. Call these subsquares $U_0, \dots, U_3$ with U_0 touching the lower right corner of U and the other subsquares arranged around the square in the clockwise direction. We shall construct an orientation-preserving map F defined on the U_j that expands each subsquare by a factor of 3 and maps it onto U. The map F is defined as follows: on U_0 and U_2, F takes the top and bottom as well as the left and right sides of U_j to the corresponding sides of U. Thus F has fixed points at 1 and i. On U_1 and U_3, F first maps these subsquares to U as in the previous case, but then a half-turn rotation is applied. So F has a 2-cycle at 0 and $1+i$. A computation shows that F is given by $z \mapsto 3z - 2$ on U_0; $z \mapsto -3z + 1 + i$ on U_1; $z \mapsto 3z - 2i$ on U_2; and $z \mapsto -3z + 3 + 3i$ on U_3.

Let Λ denote the set of points whose orbits remain in $\cup U_j$ for all iterations of F. Standard arguments from planar dynamics imply that Λ is a Cantor set and $F \,|\, \Lambda$ is conjugate to the one-sided shift on the four symbols $\{0, \dots, 3\}$. Indeed, in this case Λ is just the product of a pair of middle-third Cantor sets, one on the x-axis and one on the y-axis in U. The set Λ is the Cantor set portion of S_2.

We call the pair of horizontal lines $y = 0$ and $y = 1$ in U the horizontal boundary of U. Note that we can compute explicitly the symbol sequences that correspond to points that lie in the intersection of the horizontal boundary of U (or any of its preimages) and the Cantor set Λ. For example, if a point in Λ lies in the horizontal boundary of U, then in the corresponding sequence a 0 may only be followed by a 0 or 1 while a 1 may only be followed by a 2 or 3. Similarly, 2 may only be followed by 2 or 3, while 3 may only be followed by 0 or 1. That is, the intersection of the horizontal boundary of U with Λ consists of all itineraries generated by the subshift of finite type whose transition matrix is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Points in the preimage of the boundary of U therefore have itineraries that end in a sequence generated by this subshift.

To complete the construction of S_2, we add in certain open horizontal "middle-third" rectangles to Λ. First let R_0 be the open rectangle given by $0 < x < 1$, $1/3 < y < 2/3$. Let R_1 be the preimage of R_0 under F, so R_1 is the union of four horizontal "middle-thirds" rectangles, one in each U_j. Then, for each $j > 1$, let R_j denote the preimage of R_{j-1} under F. So R_j consists of 4^j open horizontal rectangles. Finally, we let S_2 be the union of Λ together with all of the R_j's.

To define the k-web when $k > 2$, we put $2k$ subsquares $U_0, \ldots, U_{2k-1}$ into the unit square as follows. Each of the subsquares has sidelength $1/(k+1)$ and $U_0, \ldots, U_{k-1}$ are arranged equally spaced along the bottom boundary of U with U_0 containing 1 on the x-axis and U_{k-1} containing the origin. $U_k, \ldots, U_{2k-1}$ are similarly arranged along the top boundary of U with U_k containing i and U_{2k-1} containing $1+i$. The map F is now just expansion by a factor of $k+1$ with the boundaries of the U_j mapped by F as follows. The boundaries of the subsquares U_0, U_2, U_4, etc. are mapped to the corresponding boundaries of the unit square S, while the upper and lower as well as the right and left boundaries of the other subsquares are interchanged with those of S. Let Σ be the space of one-sided infinite sequences on the $2k$ symbols $\{0, \ldots, 2k\}$. Then, as above, the set of points whose orbits remain in the U_j for all iterations is a Cantor set Λ and $F \mid \Lambda$ is conjugate to the one-sided shift on Σ. Then the open rectangles R_j are defined exactly as in the case $k = 2$: R_0 is a single open horizontal rectangle given by $0 < x < 1$ and $1/k < y < 1 - 1/k$ and the R_j are the j^{th} preimages of R_0. As before, Λ is a product of a pair of Cantor sets, one on the x-axis and one on the y-axis.

Note that certain points in Λ lie in the boundaries of the R_k and U; we call the set of such points the *unburied portion* of Λ. All other points in Λ lie in the *buried portion* of Λ. Using the symbolic dynamics as above, we compute that the set of points lying in the intersection of the horizontal boundary of U and Λ are again generated by a subshift of finite type whose $2k \times 2k$ transition matrix has rows that consist of n consecutive 0's and 1's in the form $(0 \ldots 0\, 1 \ldots 1)$ or $(1 \ldots 1\, 0 \ldots 0)$. For example, when $k = 3$, the transition matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We denote by Γ the subset of Σ consisting of sequences determined by this subshift.

3 Cantor Webs in the Dynamical Plane

In this section we prove that there is a Cantor $(n-1)$-web Δ_λ in the dynamical plane of $F_\lambda(z) = z^n + \lambda/z^n$ for each λ in the connectedness locus and that Δ_λ moves holomorphically with λ as λ varies in each symmetry sector. The Cantor set portion of the web will lie in $J(F_\lambda)$ while the adjoined open disks will consist of certain preimages of B_λ.

Lemma. *Suppose* $|\lambda| \leq 2$. *If* $|z| \geq 3/2$, *then* $z \in B_\lambda$.

Proof: If $|z| \geq 3/2$, then we have

$$|F_\lambda(z)| \geq |z|^n - \frac{|\lambda|}{|z|^n} \geq |z|^n - \frac{2^{n+1}}{3^n} > |z|^n - 1 > \left(\frac{3}{2}\right)^{n-1} |z| - |z|.$$

Since $((3/2)^{n-1} - 1) > 1$ for $n \geq 3$, $F_\lambda^k(z) \to \infty$ and it follows that the region $|z| \geq 3/2$ lies in B_λ.

□

Let γ denote the circle of radius 2 in the dynamical plane, and let $\mu = \mu_\lambda = H_\lambda(\gamma)$. By the previous Lemma, F_λ maps both γ and μ_λ strictly outside of γ. It follows that γ lies in B_λ and μ lies in T_λ (provided $\lambda \notin \mathcal{L}$; otherwise, there is no trap door and $\mu \subset B_\lambda$).

Let $\mathcal{S}$ denote the portion of the symmetry sector in the parameter plane defined by $|\lambda| < 2$ and $0 < \operatorname{Arg} \lambda < 2\pi/(n-1)$. We first construct Δ_λ for each $\lambda \in \mathcal{S} \cap \mathcal{C}$. Then we extend the construction to $\mathcal{C} \cap \mathbb{R}^+$. By the symmetry in the parameter plane, this produces a Cantor $(n-1)$-web for each $\lambda \in \mathcal{C}$.

For $\lambda \in \mathcal{S}$, consider the $2n$ prepole sectors $P_0, \ldots, P_{2n-1}$. Since $0 < \operatorname{Arg} \lambda < 2\pi/(n-1)$, we have that

$$0 < \operatorname{Arg} v_\lambda = \frac{\operatorname{Arg} \lambda}{2} < \frac{\pi}{n-1}.$$

If z lies in the prepole sector P_{2n-1} that is bounded by the critical point rays through c_0 and c_{2n-1}, then we also have

$$\operatorname{Arg} c_0 = \frac{\operatorname{Arg} \lambda}{2n} < \operatorname{Arg} z < \frac{\operatorname{Arg} \lambda}{2n} + \frac{\pi}{n} = \operatorname{Arg} c_{2n-1}.$$

But

$$\frac{\operatorname{Arg} \lambda}{2n} < \frac{\operatorname{Arg} \lambda}{2} = \operatorname{Arg} v_\lambda < \frac{\operatorname{Arg} \lambda}{2n} + \frac{\pi}{n}$$

since $\lambda \in \mathcal{S}$, so the critical value v_λ always lies in P_{2n-1} for these parameters. Similarly, $-v_\lambda$ lies in P_{n-1} for each $\lambda \in \mathcal{S}$. In particular, the critical value rays also lie in these two sectors. So we exclude from consideration the two sectors P_{n-1} and P_{2n-1} for the moment.

Consider the remaining prepole sectors $P_0, \ldots, P_{n-2}$ and $P_n, \ldots, P_{2n-2}$. Let $I_j = I_j(\lambda)$ denote the region in P_j lying between γ and μ_λ. By construction, each I_j is open and simply connected. Let $\mathcal{I}_\lambda$ denote the union of the I_j where again $j = 0, \ldots, n-2, n, \ldots, 2n-2$. Let Λ_λ be the set of points whose orbits remain in $\mathcal{I}_\lambda$ for all iterations of F_λ. Λ_λ will be the Cantor set portion of the web in the dynamical plane.

Proposition. *The set Λ_λ is homeomorphic to a Cantor set for each $\lambda \in \mathcal{S}$. $F_\lambda \,|\, \Lambda_\lambda$ is conjugate to the shift on the $2n-2$ symbols $0, \ldots, n-2, n, \ldots, 2n-2$. The sets Λ_λ vary analytically with $\lambda \in \mathcal{S}$.*

Proof: For each $\lambda \in \mathcal{S}$, F_λ maps the boundary curves γ and μ_λ strictly outside γ, and hence outside $\mathcal{I}_\lambda$. Moreover, F_λ maps the critical point ray boundaries of each of the I_j in $\mathcal{I}_\lambda$ to the critical value rays, both of which lie in $P_{n-1} \cup P_{2n-2}$ for each λ and hence also outside $\mathcal{I}_\lambda$. Therefore it follows that F_λ maps each I_j univalently onto a region that completely covers each other I_k (except $k = n-1$ or $k = 2n-1$). Standard arguments from complex dynamics then give that Λ_λ is a Cantor set with $F_\lambda \,|\, \Lambda_\lambda$ conjugate to the one-sided shift map on the $2(n-1)$ symbols $0, \ldots, n-2, n, \ldots, 2n-2$. Since the I_j vary analytically with λ, we have that the points in Λ_λ also vary analytically with λ.

$\square$

Now let Σ be the space of one-sided sequences consisting of the symbols $0, \ldots, n-2, n, \ldots, 2n-2$. Since the symbolic dynamics on Λ_λ and $\Lambda \subset S_{n-1}$ are the same (up to the slightly different names of the symbols), the set Λ_λ is the Cantor set portion of the web in the dynamical plane. To complete the construction of Δ_λ, we assume further that $\lambda \in \mathcal{S} \cap \mathcal{C}$. Hence, for each such λ, B_λ and T_λ are disjoint open disks, although the boundaries of B_λ and T_λ may intersect. This happens, for example, for certain parameters on the boundary of $\mathcal{C}$; see [7]. In particular, T_λ lies strictly inside γ.

We next adjoin the open disk T_λ to Λ_λ. Then, there is a unique prepole inside each I_j. Consider the $2n-2$ preimages of T_λ that surround these prepoles. Since the critical values do not lie in T_λ, it follows that each of these preimages is an open disk. So we adjoin these $2n-2$ open disks to $\Lambda_\lambda \cup T_\lambda$. We now continue inductively. Since each I_j is mapped univalently over all the other I_k's, it follows that there are exactly $2n-2$ points in each I_j that are mapped to the $2n-2$ prepoles in $\mathcal{I}_\lambda$. These points are surrounded by open disks that are mapped to T_λ by F_λ^2. So we adjoin these $(2n-2)^2$ second preimages of T_λ to the set. Continuing in this fashion, we produce a set that is the union of Λ_λ and a countable collection of open disks. This is the set Δ_λ.

Proposition. *If $\lambda \in \mathcal{S}$, the set Δ_λ is homeomorphic to the Cantor $(n-1)$-web, and the homeomorphism depends analytically on λ.*

Proof: We only need to show that B_λ, T_λ, and the preimages of T_λ meet the Cantor set Λ_λ in points with the appropriate itineraries under F_λ. First consider the boundary of B_λ, ∂B_λ. Note that each I_j contains a subset of ∂B_λ. There may be several components of the set $I_j \cap \partial B_\lambda$, but we claim that there is a unique component of this set whose closure meets both of the straight line boundaries of I_j. If there were two or more such components, then by symmetry, the same would be true in each of the sets I_j. Since each of these components is then stretched by F_λ over $n-1$ of the I_j's, it follows that there would be more than one point sharing the same itinerary in Λ_λ, and this cannot happen. So let β_j denote this component of $I_j \cap \partial B_\lambda$. By construction of the prepole sectors, we have that $F_\lambda(\beta_0) \supset \beta_0 \cup \cdots \cup \beta_{n-2}$

whereas $F_\lambda(\beta_1) \supset \beta_n \cup \cdots \beta_{2n-2}$. Using symmetry, $F_\lambda(\beta_0)$ (resp. $F_\lambda(\beta_1)$) cannot contain points in β_j where $n \leq j < 2n-2$ (resp. $0 \leq j < n-1$), for otherwise there would be too many preimages of points in ∂B_λ in this boundary. Proceeding clockwise around ∂B_λ, we have

$$\begin{aligned} F_\lambda(\beta_{2j}) &\supset \beta_0 \cup \cdots \cup \beta_{n-1} \\ F_\lambda(\beta_{2j+1}) &\supset \beta_n \cup \cdots \beta_{2n-2} \end{aligned}$$

for each $j < n-1$. For $n \leq j \leq 2n-2$ we have

$$\begin{aligned} F_\lambda(\beta_{2n-2}) &\supset \beta_0 \cup \cdots \cup \beta_{n-1} \\ F_\lambda(\beta_{2n-3}) &\supset \beta_n \cup \cdots \beta_{2n-2} \end{aligned}$$

and this pattern continues alternately as j decreases from $2n-2$ to n. Thus, the set of points in Λ_λ that also lie in ∂B_λ can be coded by the subshift of finite type on $2n-2$ symbols whose transition matrix has rows that contain n 1's and n 0's and, as before, are of the form $(0\ldots 0\,1\ldots 1)$ or $(1\ldots 1\,0\ldots 0)$. But, as shown in the previous section, this is precisely the same set of points that lie in the upper and lower boundaries of the unit square U in the model web S_{n-1}. Taking preimages of these sets of points in ∂T_λ and its preimages. yields the points on the boundaries of the rectangles R_k. This proves the result in case $\lambda \in \mathcal{S}$.

□

We now extend the construction of Δ_λ to the boundaries of the symmetry sectors. To accomplish this, it suffices by symmetry to consider the case where $\lambda \in \mathbb{R}^+ \cap \mathcal{C}$. For such parameters, we still have the situation where each I_j is mapped univalently over all of the other I_k's (excluding, as before, I_{n-1} and I_{2n-1}). However, not all sequences in Σ correspond now to single points: there are three special sequences in Σ that correspond to multiple points in the dynamical plane. Consequently, any preimage of these special sequences also corresponds to multiple points. The first case is that any point on $\mathbb{R}^+$ whose orbit is bounded corresponds to the sequence $s = \overline{0}$ and hence there may be multiple points in I_0 with this property. Also, any point on $\mathbb{R}^-$ whose orbit is bounded has itinerary $\overline{n}$ (if n is odd) or $n\,\overline{0}$ (if n is even). This latter dichotomy results from the fact that $\mathbb{R}^-$ is invariant if n is odd, whereas $\mathbb{R}^-$ is mapped into $\mathbb{R}^+$ if n is even. As a result, any sequence that ends in one of these three sequences corresponds to more than one point in the dynamical plane. However, we may choose a particular point that has such an itinerary in a natural manner and add this point to Δ_λ. Then we can show that this extended Δ_λ with $\lambda \in \mathbb{R}^+ \cap \mathcal{C}$ is actually the limit of any other such set as the parameter approaches λ.

Given $\lambda \in \mathcal{S}$ and a sequence $s \in \Sigma$, let $z_s = z_s(\lambda)$ be the point in Δ_λ that corresponds to s. As we have shown, there is a unique such point in Δ_λ for each $\lambda \in \mathcal{S}$. Moreover, $z_s(\lambda)$ varies analytically with λ.

Lemma. *For $\lambda^* \in \mathbb{R}^+ \cap \mathcal{C}$, let q_{λ^*} be the unique fixed point of F_{λ^*} that lies in $\partial B_{\lambda^*} \cap \mathbb{R}^+$. Let $\{\lambda_j\}$ be a sequence of parameters in $\mathcal{S}$ such that $\lambda_j \to \lambda^*$ as $j \to \infty$. Then $z_{\overline{0}}(\lambda_j) \to q_{\lambda^*}$.*

Proof: Note that there is only one parameter in $\mathbb{R}^+ \cap \mathcal{C}$ for which q_{λ^*} is parabolic, namely, the parameter that corresponds to the cusp of the main cardioid of the Mandelbrot set lying along $\mathbb{R}^+$. So we first assume that we are not in this case, i.e., that q_{λ^*} is a repelling fixed point. Choose $\epsilon > 0$ and let B_ϵ be the disk of radius ϵ centered at q_{λ^*}. We may assume that ϵ is small enough so that F_{λ^*} expands B_ϵ univalently onto a region that properly contains B_ϵ, so q_{λ^*} is the unique fixed point in this disk. Now we have that $F'_{\lambda^*}(q_{\lambda^*}) > 1$, so the portion of this disk below $\mathbb{R}^+$ (that is, below the critical point ray for F_{λ^*}) is mapped over itself.

There exists $\delta > 0$ such that, if $|\lambda - \lambda^*| < \delta$, then F_λ also expands B_ϵ

univalently over a region that strictly contains B_ϵ. Therefore F_λ also has a unique fixed point in B_ϵ. If λ lies in $\mathcal{S}$, then we claim that this fixed point is $z_{\overline{0}}(\lambda)$. Indeed, if we look at the portion of B_ϵ that lies below the critical point ray through c_λ, then this region is mapped univalently over itself by F_λ. This follows since, when $\lambda \in \mathcal{S}$, the critical value ray lies strictly above the critical point ray in the upper half plane and the lower portion of B_ϵ is mapped below the critical value ray. Hence the fixed point in B_ϵ for such a map lies in the corresponding set $I_0(\lambda)$, and so this is the fixed point $z_{\overline{0}}(\lambda)$. Therefore, as $\epsilon \to 0$, we have $z_{\overline{0}}(\lambda) \to q_{\lambda^*}$.

If q_{λ^*} is the parabolic fixed point along $\mathbb{R}^+$, the same result holds by continuity with respect to λ.

$\square$

Similar arguments also hold in the case of the sequences $\overline{n}$ or $n\,\overline{0}$, so, as above, we can choose a unique point in the dynamical plane for F_{λ^*} that is the limit of $z_{\overline{n}}(\lambda)$ or $z_{n\,\overline{0}}(\lambda)$ as $\lambda \to \lambda^*$ in the other exceptional cases. With a slight abuse of notation, we call these points $z_{\overline{n}}(\lambda)$ and $z_{n\,\overline{0}}(\lambda)$ even when $\lambda \in \mathbb{R}^+ \cap \mathcal{C}$. Extending this to preimages of such points, we see that we can extend the definition of Δ_λ continuously to any parameter in $\mathbb{R}^+ \cap \mathcal{C}$. This completes the proof of the Theorem.

4 Cantor Webs in the Parameter Plane

Our goal in this section is to prove that there is a collection of Cantor $(n-1)$-webs in the parameter plane for the family $F_\lambda(z) = z^n + \lambda/z^n$, one in each of the $n-1$ symmetry sectors. The open disks in these webs will be certain Sierpinski holes in the symmetry sector, so any parameter in this portion of the web corresponds to a map with a Sierpinski curve Julia set. The Cantor set portion of these webs will consist of parameters for which $F_\lambda(v_\lambda)$ lies in

Λ_λ, i.e., in the corresponding Cantor set portion of the web in the dynamical plane. In the following section, we shall use the symbolic dynamics associated to the itineraries of the critical orbits to identify the different types of Julia sets that correspond to these types of parameters.

As in the previous section, it suffices (with three exceptions) to restrict attention to $\lambda \in \mathcal{S}$. For such values of the parameter we have shown that each region I_j in the dynamical plane is mapped univalently over the set $\mathcal{I}_\lambda$ which is the union of $I_0, \ldots, I_{n-2}, I_n, \ldots, I_{2n-2}$. Note that F_λ also takes the two omitted regions I_{n-1} and I_{2n-1} univalently over $\mathcal{I}_\lambda$. As earlier, given any itinerary $s = (s_0 s_1 s_2 \ldots) \in \Sigma$, let $z_s(\lambda)$ denote the corresponding point in Λ_λ. Since F_λ takes I_{2n-1} univalently over $\mathcal{I}_\lambda$, there is a unique preimage of each $z_s(\lambda)$ that lies in I_{2n-1}. Call this preimage $w_s(\lambda)$, so $F_\lambda(w_s(\lambda)) = z_s(\lambda)$. (Note that the itinerary of $w_s(\lambda)$ is not s; rather, it is $2n-1$, s, but we will not use this fact.) Hence we have an analytic function $\lambda \mapsto w_s(\lambda)$ that takes $\mathcal{S}$ into the union of all possible sets of the form $I_{2n-1}(\lambda)$ as λ ranges over $\mathcal{S}$. If $z \in I_{2n-1}(\lambda)$ and $\lambda \in \mathcal{S}$, we have

$$0 < \operatorname{Arg} c_0(\lambda) = \frac{\operatorname{Arg}\lambda}{2n} < \operatorname{Arg} z < \operatorname{Arg} c_{2n-1}(\lambda) = \frac{\operatorname{Arg}\lambda}{2n} + \frac{\pi}{n} < \frac{\pi}{n-1}.$$

Let

$$\mathcal{R} = \{z \mid 0 < |z| < 2,\ 0 < \operatorname{Arg} z < \pi/(n-1)\},$$

i.e., $\mathcal{R}$ is exactly one-half of the sector $\mathcal{S}$. So the analytic function $\lambda \mapsto w_s(\lambda)$ takes $\mathcal{S}$ into $\mathcal{R}$.

We have another analytic function taking $\mathcal{S}$ into the dynamical plane. Let $G(\lambda) = v_\lambda$ where $v_\lambda = 2\sqrt{\lambda}$ lies in the upper half plane. So G takes $\mathcal{S}$ univalently over the region

$$\{z \mid 0 < |z| < 2\sqrt{2},\ 0 < \operatorname{Arg} z < \pi/(n-1)\}$$

which contains $\mathcal{R}$. So we can consider the function $L_s(\lambda) = G^{-1}(w_s(\lambda)) =$

$(w_s(\lambda))^2/4$ which maps $\mathcal{S}$ into itself. The following result gives the Cantor set portion of the web in the symmetry region $0 < \operatorname{Arg}\lambda < 2\pi/(n-1)$.

Proposition. *For each sequence $s \in \Sigma$, there is a unique $\lambda = \lambda_s \in \overline{\mathcal{S}}$ such that $L_s(\lambda_s) = \lambda_s$, i.e., a critical value of F_{λ_s} lands on the point $w_s(\lambda_s) \in \Lambda_{\lambda_s}$. Moreover, λ_s varies continuously with s.*

As we shall show, for all but three sequences in Σ, the map L_s takes $\mathcal{S}$ into a compact subset of itself, and so by the Schwarz Lemma, there is a unique fixed point in $\mathcal{S}$ for this map. This fixed point is λ_s. Later we shall deal with the three exceptional sequences for which λ_s lies on the boundary of $\mathcal{S}$.

To prove the result we need the following lemmas.

Lemma. *Suppose*

$$|\lambda|^{\frac{n}{2}-1} \leq \frac{1}{2^{n+2}}.$$

If $|z| \leq |v_\lambda|$, then $F_\lambda(z) \in B_\lambda$.

Proof: Since $|\lambda| < 1$, we have

$$2^n|\lambda|^{\frac{n}{2}} < 2^n|\lambda|^{\frac{n}{2}-1} \leq \frac{1}{4}.$$

If $|z| \leq |v_\lambda| = 2|\lambda|^{1/2}$, then

$$|F_\lambda(z)| \geq \frac{|\lambda|}{|z|^n} - |z|^n \geq \frac{1}{2^n|\lambda|^{\frac{n}{2}-1}} - 2^n|\lambda|^{\frac{n}{2}} \geq 4 - \frac{1}{4}.$$

So $|F_\lambda(z)| > 2$ and therefore $F_\lambda(z) \in B_\lambda$.

□

Lemma. *Suppose $s \in \Sigma$ satisfies*

1. *$s \neq \overline{0}$;*

2. *$s \neq \overline{n-2}$ (if n is even);*

3. $s \neq \overline{2n-2,\, n-2}$ *(if n is odd).*

Then $\{w_s(\lambda) \mid \lambda \in \mathcal{S}\}$ *is contained inside a compact subset of* $\mathcal{R} \cup \{0\}$.

Proof: To prove the result, we need to show that $w_s(\lambda)$ cannot accumulate on the boundary of $\mathcal{R}$ (except at the origin) as λ varies in $\mathcal{S}$.

For each $\lambda \in \mathcal{S}$, the circle of radius 3/2 centered at the origin is contained in B_λ, as we showed in the previous section. Hence we have $|w_s(\lambda)| < 3/2$ and so $w_s(\lambda)$ cannot accumulate on the outer circular boundary of $\mathcal{R}$ given by $|z| = 2$.

For each $\lambda \in \mathcal{S}$, $w_s(\lambda)$ lies in the sector

$$\frac{\operatorname{Arg}\lambda}{2n} < \operatorname{Arg} z < \frac{\operatorname{Arg}\lambda}{2n} + \frac{\pi}{n}.$$

Therefore $w_s(\lambda)$ can accumulate on the straight line boundaries of $\mathcal{R}$ only if $\operatorname{Arg}\lambda \to 0$ or $\operatorname{Arg}\lambda \to 2\pi/(n-1)$, since

$$\frac{2\pi}{(n-1)(2n)} + \frac{\pi}{n} = \frac{\pi}{n-1}.$$

So suppose first that $\operatorname{Arg}\lambda = 0$. Then the positive real axis is invariant under F_λ. Then, as in the previous section, if we were to assign an itinerary to such a point, that itinerary would only contain the digits 0 (and/or $2n-1$ since this point lies on the intersection of the boundaries of I_0 and I_{2n-1}). So let $s \in \Sigma$ be such that $s \neq \overline{0}$. Say $s = s_0 s_1 s_2 \ldots$ where $s_j = 0$ for $j = 0, \ldots, n-1$ but $s_n \neq 0$. Let $I_{s_0 \ldots s_n}$ be the set of points in I_{s_0} whose itinerary begins with $s_0 \ldots s_n$. Then, since $\operatorname{Arg}\lambda = 0$ but $s_n \neq 0, 2n-1$, the closure of the set $I_{s_0 \ldots s_n}(\lambda)$ is bounded away from the real axis. It then follows that $w_s(\lambda)$ is bounded away from this axis as λ varies in $\mathcal{S}$.

As $\operatorname{Arg}\lambda \to 0$, note that the other straight line boundary of $I_{2n-1}(\lambda)$ approaches a portion of the ray $\operatorname{Arg} z = \pi/2n$, which is properly contained in the set $\mathcal{R}$, so there is no problem in this case.

Now suppose that $\operatorname{Arg}\lambda \to 2\pi/(n-1)$. In this case the situation is somewhat different. If $\operatorname{Arg}\lambda = 2\pi/(n-1)$ and $\operatorname{Arg} z = \pi/(n-1)$, then $\operatorname{Arg} F_\lambda(z) = n\pi/(n-1)$. But one checks easily that the ray $\operatorname{Arg} z = n\pi/(n-1)$ is invariant under F_λ if n is even. If n is odd, then this ray is interchanged with its negative by F_λ.

Now the ray $\operatorname{Arg} z = n\pi/(n-1)$ forms part of the boundary of the sets I_{n-2} and I_{n-1} when $\operatorname{Arg}\lambda = 2\pi/(n-1)$, while the negative of this ray is part of the boundary of I_{2n-2} and I_{2n-1}. Therefore points on the ray $\operatorname{Arg} z = n\pi/(n-1)$ have itinerary in Σ given by $\overline{n-2}$ (when n is even) or itinerary $\overline{n-2,\, 2n-2}$ (when n is odd). So when n is odd, points on the negative of this ray have itinerary $\overline{2n-2,\, n-2}$. Hence, arguing exactly as in the case $\operatorname{Arg}\lambda = 0$, only when $s = \overline{n-2}$ (n even) or $s = \overline{2n-2,\, n-2}$ (n odd) does $w_s(\lambda)$ accumulate on the straight line boundary of $\mathcal{R}$ as $\operatorname{Arg}\lambda \to 2\pi/(n-1)$.

$\square$

We now complete the proof of the Proposition. The result follows immediately from the Schwarz Lemma provided that $w_s(\lambda)$ is contained in a compact subset of $\mathcal{R}$ for all $s \in \mathcal{S}$. Thus there are two situations that we must address. The first is the possibility that, for certain sequences s, $w_s(\lambda)$ may accumulate at the origin, and the second is what happens in the case of the three special itineraries listed above.

So suppose first that s is not one of the exceptional sequences $\overline{0}$, $\overline{n-2}$ with n even, or $\overline{2n-2,\, n-2}$ with n odd. Recall that $L_s : \mathcal{S} \to \mathcal{S}$ is given by $L_s(\lambda) = G^{-1}(w_s(\lambda))$ and is analytic on $\mathcal{S}$. By the previous Lemma, $L_s(\mathcal{S})$ is contained in a compact subset of $\mathcal{S} \cup \{0\}$. We claim that L has a fixed point in the interior of $\mathcal{S}$. By the earlier Lemma, if $|\lambda|^{\frac{n}{2}-1} < (1/2)^{n+2}$, then we have $|w_s(\lambda)| > |v_\lambda|$ since all points with $|z| \leq |v_\lambda|$ are mapped into B_λ. Hence $|L_s(\lambda)| > |\lambda|$ for these values of λ. So L_s maps the portion of $\mathcal{S}$ outside the circle of radius $(1/2)^{\frac{2(n+2)}{n-1}}$ into a compact subset of this region.

So by the Schwarz Lemma, there is a fixed point in this region, and hence this must be the unique fixed point for L_s in all of $\mathcal{S}$. Moreover, since L_s varies continuously with s, the parameter λ_s also varies continuously with s. This proves the result except in the case of the three exceptional sequences.

Now suppose that $s = \overline{0}$. Let $\hat{\lambda}$ be the parameter value that lies at the point of intersection of $\mathbb{R}^+$ with the boundary of the McMullen domain. This parameter is the tip of the "tail" of the Mandelbrot set lying along $\mathbb{R}^+$, i.e., the parameter that corresponds to $c = -2$ in the standard Mandelbrot set for $z^2 + c$. Consequently, $F_{\hat{\lambda}}(v_{\hat{\lambda}})$ is the fixed point that lies in $\partial B_{\hat{\lambda}}$, namely the point we called $z_{\overline{0}}(\hat{\lambda})$ in the previous section. So, in terms of the above notation, $v_{\hat{\lambda}}$ lands on $w_{\overline{0}}(\hat{\lambda})$.

By the results in [1], given any point in ∂T_λ with prescribed itinerary s, there is a unique λ for which v_λ lands on the point in ∂T_λ with this itinerary. Furthermore, the boundary of the McMullen domain is a simple closed curve that may be parameterized continuously by the parameters with the corresponding itineraries. Hence it follows that $\hat{\lambda}$ is the unique parameter for which $v_{\hat{\lambda}} = w_{\overline{0}}(\hat{\lambda})$. Moreover, for itineraries s close to $\overline{0}$, λ_s is close to $\hat{\lambda}$. This extends the Cantor set portion of the web in parameter space to $\hat{\lambda}$. The extension to the other two special parameters is similar: both correspond to points at the tip of the tail of the Mandelbrot set lying along $\operatorname{Arg} \lambda = 2\pi/(n-1)$. This completes the proof of the Proposition.

$\square$

Let Λ be the Cantor set portion of the web in the parameter plane that consists of the parameters λ_s for $s \in \Sigma$. Recall that $\Gamma \subset \Sigma$ is the set of all sequences corresponding to points in the dynamical plane that lie in the boundary of B_λ, i.e., the subset corresponding to allowable sequences for the subshift of finite type discussed earlier.

Proposition. *Suppose $s \in \Gamma$. Then λ_s lies in either $\partial\mathcal{L}$ or $\partial\mathcal{M}$.*

Proof: First note that, if s is one of the three special sequences above, then we have already shown that $\lambda_s \in \partial\mathcal{M}$. For other values of s, it suffices to prove that there is a parameter $\tilde{\lambda}$ arbitrarily close to λ_s for which $F_{\tilde{\lambda}}(v_{\tilde{\lambda}})$ lies in $B_{\tilde{\lambda}}$. Toward that end, since periodic sequences are dense in Γ, we may assume that s is a periodic sequence. Then $z_s(\lambda_s)$ is a periodic point lying in ∂B_{λ_s}. Hence there is an external ray $\gamma_t(\lambda_s)$ defined for $t \geq 1$ and having rational angle in B_{λ_s} that lands at $z_s(\lambda_s)$, i.e., that satisfies $\gamma_1(\lambda_s) = z_s(\lambda_s)$.

Now for λ in a neighborhood of λ_s, we have that the function $\lambda \mapsto z_s(\lambda)$ is analytic. As above, there is an external ray $\gamma_t(\lambda)$ that also lands at $z_s(\lambda)$ and, for fixed t, $\gamma_t(\lambda)$ is also analytic in λ. Now λ_s is a root of the function of λ given by $z_s(\lambda) - v_\lambda$. Hence, for t close enough to 1, there is a nearby root of the function $\gamma_t(\lambda) - v_\lambda$. But this root is a λ-value for which v_λ lies on an external ray B_λ, and this therefore yields the nearby parameter $\tilde{\lambda}$ with the required properties.

$\square$

We will denote the Cantor web that lies in the symmetry sector $\mathcal{S}$ by Δ. Thus the set of parameters λ_s with $s \in \Gamma$ gives the portion of Δ that corresponds to points in the model web on the boundary of the unit square. To complete the construction of Δ, we adjoin certain Sierpinski holes to Λ exactly as we added certain preimages of B_λ to Λ_λ to obtain the Cantor web in the dynamical plane. To begin this construction, let $p_\lambda = (-\lambda)^{1/2n}$ be the unique prepole in I_{2n-1}. Consider the function $K(\lambda) = G^{-1}(p_\lambda)$ defined on $\mathcal{S}$. We have

$$K(\lambda) = \frac{(-\lambda)^{1/n}}{4}$$

and a straightforward computation shows that K has a fixed point at

$$\lambda' = (-1)^{1/(n-1)} \left(\frac{1}{4^n}\right)^{1/(n-1)}$$

which lies in $\mathcal{S}$. So we have $v_{\lambda'} = p_{\lambda'}$. Therefore λ' lies at the center of a

Sierpinski hole with escape time 3. So we adjoin this open disk to the Cantor set Λ. This corresponds to adding the rectangle R_0 in the construction of the model Cantor web S_{n-1}. As above, parameters of the form λ_s with $s = s_0 t$ where s_0 is one of the allowed digits in a sequence in Σ and $t \in \Gamma$ lie on the boundary of this Sierpinski hole.

For the other Sierpinski holes, let $t = t_0 \dots t_{n-1}$ be a finite sequence where the digits t_j are as usual drawn from $0, \dots, n-2, n, \dots, 2n-2$. Let $p_t(\lambda)$ be the point in I_{t_0} that satisfies $F_\lambda^j(p_t(\lambda)) \in I_{t_j}$ for $j = 0, \dots, n-1$ and $F_\lambda^n(p_t(\lambda)) = 0$, i.e., the orbit of $p_t(\lambda)$ stays in $\mathcal{I}_\lambda$ until landing on 0 at iteration n. Let $q_t(\lambda)$ be the preimage of $p_t(\lambda)$ that lies in I_{2n-1}.

Proposition. *Suppose $\lambda \in \mathcal{C} \cap \mathcal{S}$. For each allowable finite sequence $t = t_0, \dots, t_{n-1}$, there exists a unique λ_t such that $q_t(\lambda_t) = v_{\lambda_t}$.*

Proof: The proof is essentially the same as in the case of the parameter values in Λ. Let $K_t : \mathcal{S} \to \mathcal{S}$ be given by $G^{-1}(q_t(\lambda))$. There are no special sequences in this case, since the point $q_t(\lambda)$ lies in a compact subset of $\mathcal{R} \cup \{0\}$. This follows from the fact that, when $\operatorname{Arg} \lambda = 0$ or $2\pi/(n-1)$, there are no points on the corresponding boundary lines that are eventually mapped to the origin. Hence, by the Schwarz Lemma, there exists a unique fixed point for K_t, and this parameter is λ_t.

□

This completes the construction of the Cantor web $\Delta \subset \mathcal{S}$. By symmetry, there is a copy of Δ in each of the other symmetry sectors. In fact, we can combine all of these Cantor webs with the McMullen domain to produce a single larger Cantor web in the parameter plane.

5 Julia Sets Corresponding to Parameters in the Cantor Web

In this section we provide a classification of the types of Julia sets that occur for parameters lying in the Cantor web in the parameter plane. Before turning to these sets, we first show that the open set $\mathbb{C}-\overline{B}_\lambda$ is a connected and simply connected set for any parameter in $\mathbb{C}-\mathcal{L}$. Note that this situation is very different from the corresponding situation in the Mandelbrot set: in that case, for Julia sets such as the Douady rabbit or the basilica, the complement of the closure of the basin of ∞ consists of infinitely many disjoint open disks. Then we use this fact to show that, for any parameter in the Cantor web, the boundary of B_λ is always a simple closed curve.

Proposition. *If $\lambda \in \mathcal{C} \cup \mathcal{M}$, then the open set $\mathbb{C} - \overline{B}_\lambda$ is connected and simply connected.*

Proof: Let W_0 denote the open connected component of $\mathbb{C}-\overline{B}_\lambda$ that contains 0. Note that W_0 contains all of T_λ since the boundary of B_λ does not meet T_λ. Hence the closure of W_0 contains ∂T_λ.

Lemma. *W_0 is symmetric under $z \mapsto \nu z$ where $\nu = \exp(i\pi/n)$.*

Proof: Let X denote the set of points z in W_0 for which νz also lies in W_0. Note that X is an open subset of W_0. Note also that $X \supset T_\lambda$ since T_λ possesses $2n$-fold symmetry and lies in W_0. Hence X is nonempty. Now suppose that $X \neq W_0$. Then there must be a point $z_1 \in \partial X \cap W_0$. So $z_1 \in W_0$ but $\nu z_1 \notin W_0$. Therefore νz_1 lies in ∂W_0, which is contained in ∂B_λ. Since $\nu z_1 \in \partial B_\lambda$ and ∂B_λ has $2n$-fold symmetry we have that $z_1 \in \partial B_\lambda$, contradicting our assumption that $z_1 \in W_0$.

□

Lemma. *All $2n$ preimages of any point in W_0 lie in W_0.*

Proof: Since $H_\lambda(B_\lambda) = T_\lambda$ and $T_\lambda \subset W_0$, we have $H_\lambda(\partial B_\lambda) \subset \overline{W}_0$. Therefore $H_\lambda(\partial W_0) \subset \overline{W}_0$ and so H_λ maps $\mathbb{C} - \overline{W}_0$ into W_0.

Now H_λ maps prepoles to prepoles. If one of the prepoles lies in $\mathbb{C} - \overline{W}_0$, then its image under H_λ lies in W_0. This cannot occur since, by the previous lemma, W_0 has $2n$-fold symmetry. Hence each prepole lies in $\overline{W}_0$. In fact, each prepole must lie in W_0 since ∂W_0 is mapped to ∂B_λ and $0 \notin \partial B_\lambda$.

It follows that all $2n$ preimages of 0 lie in W_0. Therefore the entire set $F_\lambda^{-1}(W_0)$ is contained in W_0 for, otherwise, there would be points in $\partial W_0 \subset \partial B_\lambda$ that are mapped into W_0. This cannot happen since ∂B_λ is invariant.

□

We now complete the proof that $\mathbb{C} - \overline{B}_\lambda$ is connected and simply connected. It suffices to show that W_0 is the only component of $\mathbb{C} - \overline{B}_\lambda$. By the above, all preimages of a point in W_0 lie in W_0. Hence all preimages of any point in $\overline{W}_0$ must lie in $\overline{W}_0$. But points in ∂W_0 lie in the Julia set, and it is known that the the union of preimages of such a point under F_λ^k for all k is dense in the Julia set. Hence it follows that the entire Julia set is contained in $\overline{W}_0$. But then $\partial W_0 = \partial B_\lambda$, and the result follows.

□

As a remark, the fact that there is only one component in the complement of $\overline{B}_\lambda$ does not preclude the existence of quadratic-like filled Julia sets with infinitely many pinch points along the boundary. These sets arise when the parameter is drawn from any of the Mandelbrot sets in $\mathcal{C}$. But these sets are properly contained in $\overline{W}_0$.

Corollary. *For each $\lambda \in \Delta$, the Julia set of F_λ is locally connected and the boundary of B_λ is a simple closed curve.*

Proof: If λ lies in any of the Sierpinski holes in Δ, the critical orbits tend to ∞. If λ lies in the Cantor set portion of Δ, then the critical orbits land in the Cantor set portion of Λ_λ after two iterations of F_λ. In either case we have that the critical orbit is non-recurrent. Furthermore, there are no parabolic orbits for F_λ, since the fate of all of the critical orbits is accounted for. It follows that F_λ is semi-hyperbolic. By the results in [13], $J(F_\lambda)$ is locally connected. Therefore $\partial B_\lambda = \partial W_0$ is also locally connected. Since the complement of ∂B_λ in $\overline{\mathbb{C}}$ consists of two open, disjoint, and simply connected regions, namely B_λ and W_0, and their common boundary ∂B_λ is locally connected, it follows that ∂B_λ is a simple closed curve.

□

We finally turn attention to the different types of Julia sets that occur for parameters in Δ. For $\lambda \in \Delta$, we have that $J(F_\lambda)$ is compact, connected, locally connected, and nowhere dense. Also, all of the complementary domains are preimages of B_λ and are therefore bounded by simple closed curves. However, certain of these curves may touch each other, so the Julia set is not always a Sierpinski curve.

By the Escape Trichotomy, if λ resides in one of the Sierpinski holes in Δ, then $J(F_\lambda)$ is a Sierpinski curve. If λ resides in the Cantor set portion of Δ, then there are four possible types of Julia sets. First, if λ lies in the portion of Δ in $\partial\mathcal{L}$, then $J(F_\lambda)$ is a *generalized Sierpinski gasket*. See Figure 4. Here the critical points lie at the intersection of the boundaries of T_λ and B_λ. See [7] for details about these types of sets.

If λ lies in $\partial\mathcal{M}$, then the preimage of ∂T_λ is a chain of $2n$ simple closed curves each of which meets two other such curves at adjacent critical points. Equivalently, ∂T_λ is bounded by a pair of concentric simple closed curves that meet each other at $2n$ points. Then the preimage of this chain is a pair of chains, each with $2n^2$ simple closed curves that meet two other such curves. The preimage of each of these chains is a similar chain, but this time there are $2n^3$ simple closed curves in the chain. And so on. This yields countably

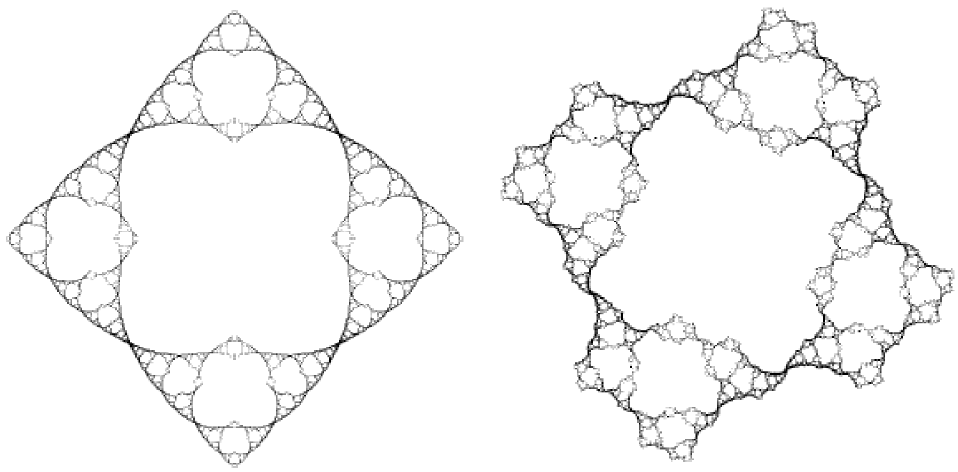

Figure 4: Generalized Sierpinski gasket Julia sets drawn from the family $z^2 + \lambda/z^2$. Similar Julia sets occur in the families with $n > 2$.

many chains in $J(F_\lambda)$. As in the McMullen domain, however, there are uncountably many other components in $J(F_\lambda)$. Each of these components are buried components that are simple closed curves surrounding the origin and without pinches. We call this type of Julia set a *pinched Cantor set of circles*. See Figure 5.

When λ lies on the boundaries of any of the Sierpinski holes in Δ, then $J(F_\lambda)$ is a *hybrid Sierpinski curve*. In these sets, all of the complementary domains are bounded by simple closed curves, but infinitely many of them touch exactly one other such boundary curve, while the rest (infinitely many) do not meet other bounding curves. See Figure 6.

This accounts for all of the non-buried parameters in the Cantor set portion of Δ. Finally, if λ is a buried parameter (i.e., not on the boundaries of $\mathcal{M}, \mathcal{L}$, or any Sierpinski hole), then $J(F_\lambda)$ is again a Sierpinski curve. However, the maps on these Julia sets are, unlike those drawn from Sierpinski holes, structurally unstable. Arbitrarily close to any such parameter are

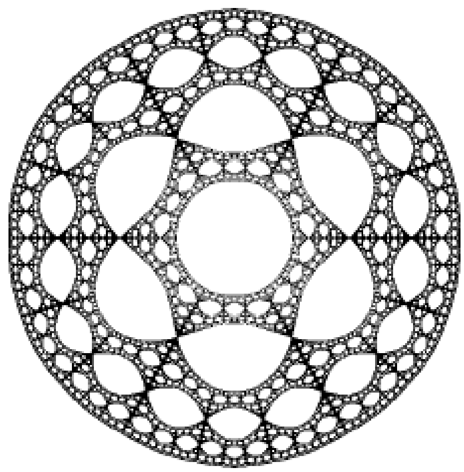

Figure 5: A Julia set that is a pinched Cantor set of circles (drawn from the family $z^3 + \lambda/z^3$).

infinitely many other parameters whose maps are all dynamically distinct from one another. See [4].

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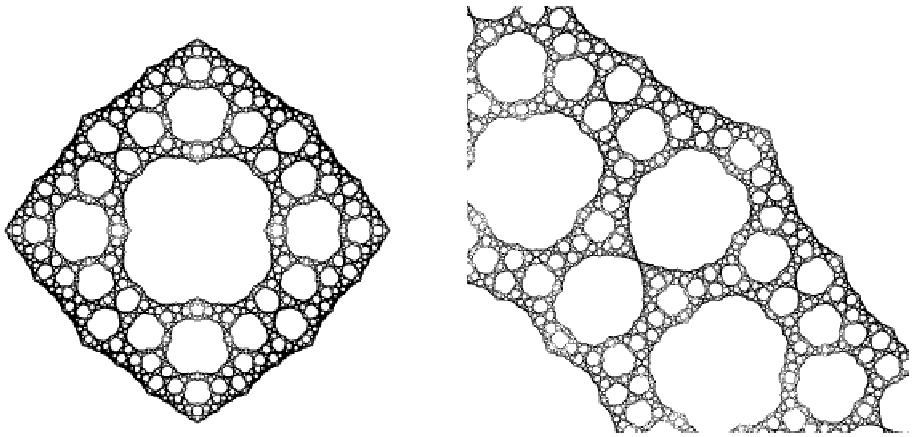

Figure 6: A hybrid Sierpinski curve Julia set (drawn from the family $z^2+\lambda/z^2$ and a magnification. Note that some of the complementary domains appear to be bounded by isolated simple closed curves while others are bounded by simple closed curves that meet another such curve at a single point.

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