Mandelpinski Spokes in the Parameter Planes of Rational Maps *

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Abstract

In this paper we describe a new structure that arises in the parameter plane of the family of maps $z^n + \lambda/z^d$ where $n \geq 2$ is even but $d \geq 3$ is odd. We call these structures Mandelbrot-Sierpinski spokes (or, for short, “Mandelbiski spokes”). It is known that there are infinitely many baby Mandelbrot sets in these parameter planes that are part of what is called the Mandelbiski maze for these maps. We show here that there are infinitely many “spokes” emanating from each of these Mandelbrot sets. Each spoke consists of infinitely many alternating Mandelbrot sets and Sierpinski holes that lie along a certain arc that tends away from the given Mandelbrot set in a certain direction.

In this paper we will concentrate on the family of maps $F_\lambda(z) = z^2 + \lambda/z^3$, though everything we discuss goes over to the more general case of $z^n + \lambda/z^d$ where $n \geq 2$ is even $d \geq 3$ is odd. It is known [4] that there is a very elaborate structure called a Mandelbiski maze that branches away from the negative real axis in the parameter planes for these maps. Roughly speaking, this maze consists of infinitely many baby Mandelbrot sets and Sierpinski holes that alternate along each edge of a specific planar graph that has infinitely many vertices. A Sierpinski hole is a disk in which all parameters correspond to maps whose Julia sets are Sierpinski curves, i.e., they are homeomorphic to the well known Sierpinski carpet.

In this paper we will look in detail at a neighborhood of each of these Mandelbrot sets in the maze. We shall show that there are infinitely many “spokes” emanating from this set. Along these spokes there are infinitely many alternating copies of Mandelbrot sets and Sierpinski holes. Roughly speaking, the spokes along which these sets lie are the analogues of the external rays of angle $j/2^k$ in the parameter plane for the usual Mandelbrot set, though, of course, in this case, these rays are not in the region where the critical values lie in the basin of $\infty$.

In Figures 1 and 2, we display the parameter plane for $z^2 + \lambda/z^3$, i.e., the $\lambda$-plane. Along the negative real axis, there are infinitely many red
disks: these are the Sierpinski holes. Between any two Sierpinski holes, there is then a (very small) Mandelbrot set, as shown in the first magnification in this figure. Each of the four spokes displayed in this magnification pass through infinitely many more Mandelbrot sets and Sierpinski holes. The next two magnifications in Figure 2 show more of the spokes emanating from this Mandelbrot set. The Sierpinski holes are again visible, but the intermediate Mandelbrot sets are too small to be seen at this level.

![Mandelbrot set magnifications](image)

**Figure 1:** The parameter plane for $z^2 + \lambda/z^3$. The magnification shows a small Mandelbrot set with four spokes $(0, 1/4, 1/2, 3/4)$ emanating. Figure 2 shows further magnifications around this Mandelbrot set.

## 1 Preliminaries

This paper describes what we call the Mandelpinski spokes that live in the parameter plane of the family of rational maps given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$
Figure 2: Two further magnifications of the parameter plane. In the first figure, the spokes with angle $j/8$ where $j = 1, 3, 5, \text{and } 7$ are displayed, and, in the second, those with angle $j/16$ where $j = 1, 3, 5, \text{and } 7$ are displayed. Again, the intermediate Mandelbrot sets are too small to be seen.

where $n \geq 2$ is even and $d \geq 3$ is odd. However, for simplicity, we shall concentrate only on the case

$$F_{\lambda}(z) = z^2 + \frac{\lambda}{z^d}.$$  

The extensions from this case to the more general case are straightforward; see [4] for more details.

When $|z|$ is large, we have that $|F_{\lambda}(z)| > |z|$ and so the point at $\infty$ is an attracting fixed point in the Riemann sphere. We denote the immediate basin of attraction of $\infty$ by $B_\lambda$. There is also a pole at the origin for each of these maps, and so there is a neighborhood of the origin that is mapped into $B_\lambda$. If the preimage of $B_\lambda$ surrounding the origin is disjoint from $B_\lambda$, we call this region the trap door and denote it by $T_\lambda$.

The Julia set of $F_{\lambda}$, $J(F_{\lambda})$, has several equivalent definitions. $J(F_{\lambda})$ is
the set of all points at which the family of iterates of \(F_\lambda\) fails to be a normal family in the sense of Montel. Equivalently, \(J(F_\lambda)\) is the closure of the set of repelling periodic points of \(F_\lambda\), and it is also the boundary of the set of all points whose orbits tend to \(\infty\) under iteration of \(F_\lambda\), not just those in the boundary of \(B_\lambda\). See [11].

There are five critical points for the map \(F_\lambda\) that are given by \((3\lambda/2)^{1/5}\). We denote the critical point that lies in \(\mathbb{R}^-\) when \(\lambda \in \mathbb{R}^-\) by \(c_0 = c_0^\lambda\) (and then \(c_0^\lambda\) varies analytically with \(\lambda\)). We denote the other critical points by \(c_j = c_j^\lambda\) for \(-2 \leq j \leq 2\) where the \(c_j\) are now arranged in the clockwise order as \(j\) increases. As \(\lambda\) moves half way around the origin from \(\mathbb{R}^-\), \(c_0\) rotates exactly one-tenth of a turn in the corresponding direction. Thus, when Arg \(\lambda\) decreases from \(\pi\) to 0, \(c_2\) lies in \(\mathbb{R}^+\) and when Arg \(\lambda\) increases from \(\pi\) to \(2\pi\), \(c_{-2}\) now lies in \(\mathbb{R}^+\). The critical values of \(F_\lambda\) are then given by \(v^\lambda = \kappa \lambda^{2/5}\) where \(\kappa\) is the constant given by \(5/(2^{2/5}3^{3/5})\). One computes easily that \(\kappa \approx 1.96\). We denote by \(v_j^\lambda\) the critical value that is the image of \(c_j^\lambda\).

There are also five prepoles for \(F_\lambda\) given by \((-\lambda)^{1/5}\). We denote the prepole that lies in \(\mathbb{R}^+\) when \(\lambda \in \mathbb{R}^-\) by \(p_2 = p_2^\lambda\). The other prepoles are denoted by \(p_j = p_j^\lambda\) where again \(-2 \leq j \leq 2\) and the \(p_j\) are arranged in the clockwise order as \(j\) increases. Note that, when \(\lambda \in \mathbb{R}^-\), the critical point \(c_0\) lies between the two rays starting at the origin and passing through \(p_0\) and \(p_{-1}\).

The straight ray extending from the origin to \(\infty\) and passing through the critical point \(c_j^\lambda\) is called a critical point ray. This ray is mapped two-to-one onto the portion of the straight ray from the origin to \(\infty\) that starts at the critical value \(v_j^\lambda\) and extends to \(\infty\). A similar straight line extending from 0 to \(\infty\) and passing through a prepole \(p_j^\lambda\) is a prepole ray, and this ray is mapped one-to-one onto the entire straight line passing through both the origin and the point \((-\lambda)^{2/5}\).
Let $\omega$ be a fifth root of unity. Then we have $F_\lambda(\omega z) = \omega^2 F_\lambda(z)$, and so it follows that the dynamical plane is symmetric under the rotation $z \mapsto \omega z$. In particular, all of the critical orbits have “similar” fates. If one critical orbit tends to $\infty$, then all must do so. If one critical orbit tends to an attracting cycle of some period, then all other critical orbits also tend to an attracting cycle, though these cycles may be different and also may have different periods. Nonetheless, the points on these attracting cycles are all symmetrically located with respect to the rotation by $\omega$. As a consequence, each of $B_\lambda$, $T_\lambda$, and $J(F_\lambda)$ are symmetric under rotations by $\omega$.

There is an Escape Trichotomy \cite{7} for this family of maps. One scenario in this trichotomy occurs when one and hence, by symmetry, all of the critical values lie in $B_\lambda$. In this case it is known that $J(F_\lambda)$ is a Cantor set. The corresponding set of $\lambda$-values in the parameter plane is called the Cantor set locus. The second scenario is that the critical values all lie in $T_\lambda$ (which we assume is disjoint from $B_\lambda$). In this case the Julia set is a Cantor set of simple closed curves surrounding the origin. This can only happen when $n, d \geq 2$ but not both equal to 2 \cite{10}. We call the region $E^1$ in parameter plane where this occurs the “McMullen domain”; it is known that $E^1$ is an open disk surrounding the origin \cite{2}. A third scenario is that the orbit of a critical point enters $T_\lambda$ at iteration 2 or higher. Then, by the above symmetry, all such critical orbits do the same. In this case, it is known that the Julia set is a Sierpinski curve \cite{6}, i.e., a set that is homeomorphic to the well known Sierpinski carpet fractal. The regions in the parameter plane for which this happens are the open disks that we call Sierpinski holes \cite{13}. If the critical orbits do not escape to $\infty$, then it is known \cite{8} that the Julia set is a connected set. Thus we call the set of parameters for which the critical orbits either do not escape or else enter the trap door at iteration 2 or higher the connectedness locus. This is the region between the Cantor set locus and
the McMullen domain. In [1] it has been shown that there is a “principal” Mandelbrot set in the parameter plane that lies along the positive real axis and extends from the Cantor set locus down to the McMullen domain. See Figure 3 for a display of these regions in the parameter plane. For more details about the dynamical properties of these maps and structure of the parameter plane, see [3].

![Diagram](image)

Figure 3: The parameter plane for the family $z^2 + \lambda/z^3$. The external region $C$ is the Cantor set locus. All of the disks visible in these pictures are Sierpinski holes, except for the McMullen domain $M$, which is the tiny disk pointed to in the magnification. The principal Mandelbrot set lies along the positive real axis between the Cantor set locus and the McMullen domain.

2 The Initial Mandelpinski Arc

In this section, we construct a Mandelpinski arc. This will be an arc in the parameter plane that passes alternately along the spines of infinitely many baby Mandelbrot sets and through the centers of the same number of
Sierpinski holes. By the spine of the Mandelbrot set we mean the analogue of the portion of the real axis lying in the usual Mandelbrot set associated with the quadratic family $z^2 + c$. As a remark, the construction in this section replicates the one in [4], but we include these ideas here since they are essential for what comes later.

In this first case, there will be infinitely many Mandelbrot sets $\mathcal{M}^k$ with $k \geq 2$ along this arc. Here $k$ is the period of the attracting cycle for parameters drawn from the main cardioid of $\mathcal{M}^k$, i.e., the base period of $\mathcal{M}^k$. There will also be infinitely many Sierpinski holes $\mathcal{E}^k$ with $k \geq 1$ where $k$ is the escape time in $\mathcal{E}^k$, i.e., the number of iterations it takes for the orbit of the critical points to enter $T_\lambda$. In this special case, the arc will be the portion of the negative real axis in the parameter plane extending from the McMullen domain $\mathcal{E}^1$ down to the endpoint on the boundary of the connectedness locus. Then the Mandelbrot sets and Sierpinski holes will be arranged along this arc as follows:

$$\ldots \mathcal{M}^4 < \mathcal{E}^3 < \mathcal{M}^3 < \mathcal{E}^2 < \mathcal{M}^2 < \mathcal{E}^1.$$ 

In each case there will be an interval of nonzero length between any adjacent Mandelbrot set and Sierpinski hole lying along this arc. The Mandelpinski spokes we construct later will emanate from each of the $\mathcal{M}^k$.

To construct the objects lying along this arc, we will restrict attention at first to the $\lambda$-values lying in the annular region $\mathcal{O}$ in parameter plane given by $10^{-10} \leq |\lambda| \leq 2$. Also, let $\mathcal{A}$ be the annulus in the dynamical plane given by $\kappa 10^{-4} \leq |z| \leq \kappa 2^{2/5}$ where $\kappa \approx 1.96$ is defined as above.

**Proposition.**

1. For any $\lambda \in \mathcal{O}$, all points on the outer circular boundary of $\mathcal{A}$ lie in $B_\lambda$, while all points on the inner circular boundary of $\mathcal{A}$ lie in $T_\lambda$. Moreover, $F_\lambda$ maps each of these boundaries strictly outside the boundary of $\mathcal{A}$. 

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2. If $\lambda$ lies on the inner circular boundary of $\mathcal{O}$, then each critical value lies on the inner circular boundary of $\mathcal{A}$ and so $\lambda$ lies in the McMullen domain.

3. If $\lambda$ lies on the outer circular boundary of $\mathcal{O}$, then each critical value lies on the outer circular boundary of $\mathcal{A}$ and so $\lambda$ lies in the Cantor set locus in the parameter plane.

**Proof:** First, if $|z| = \tau \kappa^{2/5}$ for any $\tau \geq 1$, we have for each $\lambda \in \mathcal{O}$:

$$|F_\lambda(z)| \geq |\tau^2 \kappa^{2/5} | - \left| \frac{\lambda}{\tau^3 \kappa 2^{6/5}} \right|$$

$$\geq \tau^2 1.95^{2/5} - \frac{2}{\tau^3 \kappa 2^{6/5}}$$

$$\geq 6\tau^2 - 1/(7\tau^3)$$

$$> \tau \kappa^{2/5} = |z|.$$ 

So all points outside of the circle $|z| = \kappa^{2/5}$ lie in $B_\lambda$ when $\lambda \in \mathcal{O}$.

Similarly, if $|z| = \kappa 10^{-4}$, then we have

$$|F_\lambda(z)| \geq \frac{|\lambda|}{\kappa^{3} 10^{-12}} - \kappa^2 10^{-8} \geq \frac{10^{-10}}{\kappa^3 10^{-12}} - \kappa^2 10^{-8} \geq 100/\kappa^2 - \epsilon$$

where $\epsilon \approx 4 \cdot 10^{-8}$. So this inner boundary is mapped into $B_\lambda$ and outside of $\mathcal{A}$, and so are all smaller circles around the origin. Hence this circle lies in $T_\lambda$ (when $\lambda$ lies in the connectedness locus).

Now if $\lambda$ lies on the inner circular boundary of $\mathcal{O}$, then $|\lambda| = 10^{-10}$ so that $|v_j^\lambda| = \kappa 10^{-4}$ for each $j$. Hence, for these $\lambda$-values, $v_j^\lambda$ lies on the inner circular boundary of $\mathcal{A}$, which lies in $T_\lambda$, and $\lambda$ therefore lies in the McMullen domain. If $\lambda$ lies on the outer circular boundary of $\mathcal{O}$, then $|\lambda| = 2$ so that $|v_j^\lambda| = \kappa 2^{2/5}$ (the outer boundary of $\mathcal{A}$) and thus this boundary circle lies in the Cantor set locus in the parameter plane.

$\square$
We now restrict attention to a “smaller” subset of $\mathcal{O}$. Let $\mathcal{O}'$ be the subset of $\mathcal{O}$ containing parameters $\lambda$ for which $0 \leq \text{Arg} \lambda \leq 2\pi$. Despite the overlap of this region along the real axis, we will think of $\mathcal{O}'$ as being a closed disk (not an annulus) in the parameter plane with $\text{Arg} \lambda = 0$ and $\text{Arg} \lambda = 2\pi$ considered as different portions of the boundary. We do this because, as $\text{Arg} \lambda$ increases from 0 to $2\pi$, the critical point $c_0$ that we will be following rotates one-fifth of a turn in the dynamical plane. So this point will migrate to the position of a different critical point as $\text{Arg} \lambda$ rotates one full turn.

For any parameter in $\mathcal{O}'$, let $L^\lambda$ be the closed “portion of the wedge” in the annulus $\mathcal{A}$ in the dynamical plane that is bounded by the two prepole rays through $p_0$ and $p_{-1}$. When $\lambda \in \mathbb{R}^-$, $L^\lambda$ is thus bounded by the rays extending from 0 and passing through $\exp(2\pi i (2/5))$ and $\exp(2\pi i (3/5))$. So the critical point $c_0$ lies in the interior of $L^\lambda$. Next, let $R^\lambda$ be the portion of the wedge in $\mathcal{A}$ that is bounded by the critical point rays passing through $c_2$ and $c_{-2}$. When $\lambda \in \mathbb{R}^-$, this wedge is bounded by the critical point rays extending from 0 and passing through $\exp(\pm 2\pi i /10)$. Note that $R^\lambda$ is the symmetric image of $L^\lambda$ under $z \mapsto -z$. See Figure 4.

**Proposition.** For each $\lambda \in \mathcal{O}'$:

1. $F_\lambda$ maps the interior of $R^\lambda$ in one-to-one fashion onto a region that contains the interior of $R^\lambda \cup L^\lambda$ together with a portion of $T_\lambda$ that contains 0;

2. $F_\lambda$ maps the interior of $L^\lambda$ two-to-one over a region that contains the interior of $R^\lambda$;

3. As $\lambda$ winds once around the boundary of $\mathcal{O}'$, the critical value $F_\lambda(c_0^\lambda) = v_0^\lambda$ winds once around the boundary of $R^\lambda$, (i.e., the winding index of
Figure 4: The wedges $L^\lambda$ and $R^\lambda$ for $\lambda = -0.09$.

the vector connecting this critical value to the prepole $p_2^\lambda$ lying in the interior of $R^\lambda$ is one).

**Proof:** For the first case, recall that the straightline boundaries of $R^\lambda$ are mapped two-to-one onto the critical value rays passing through $v_2^\lambda$ and $v_{-2}^\lambda$. When $0 < \text{Arg} \, \lambda < 2\pi$, one checks easily that these rays are disjoint from both $R^\lambda$ and $L^\lambda$. The reason for this is that the arguments of the rays containing the critical values increase/decrease twice as fast as the arguments of the critical point and prepole rays as $\lambda$ varies. However, when $\text{Arg} \, \lambda = 0$, the critical value ray $v_2^\lambda$ now reaches the boundary of $R^\lambda$ on the real line, and when $\text{Arg} \, \lambda = 2\pi$, the same thing is true for the critical value ray $v_{-2}^\lambda$. By the previous Proposition, the outer boundary curve of $R^\lambda$ is mapped to an arc that lies in $B_\lambda$ and also lies outside the outer circular boundaries of $R^\lambda$ and $L^\lambda$. This image arc connects the two critical value rays in $B_\lambda$, and lies to the right of these rays in the basin. The inner boundary is mapped to a similar
arc connecting these rays but now lying to the left of $L^\lambda$. Consequently, the image of $R^\lambda$ contains the interiors of both $R^\lambda$ and $L^\lambda$ and a portion of $T_\lambda$, since the critical values never land at the origin.

For the second case, we have that the straightline boundaries of $L^\lambda$ contain the prepoles $p_0^\lambda$ and $p_{-1}^\lambda$, which are both mapped to straight lines passing through the origin. In the case of $p_0^\lambda$, this straight line passes through $\exp(4\pi i/5)$ when $\lambda \in \mathbb{R}^-$. Then as $\text{Arg} \lambda$ increases or decreases by at most $\pi$, the argument of this image line rotates by at most one-fifth of a turn in the corresponding direction. Hence this line lies strictly outside $R^\lambda$ (except when $\text{Arg} \lambda = 2\pi$, in which case this line is now the real axis, which meets the boundary of $R^\lambda$). Similar arguments work for the image of the other prepole ray. For the circular boundaries of $L^\lambda$, by the previous Proposition, they are both mapped to curves in $B_\lambda$ that lie outside of the outer boundary of $A$, but now these curves are arcs that connect the images of the prepole rays passing to the right of these lines. Hence $F_\lambda$ maps $L^\lambda$ over the interior of $R^\lambda$ in two-to-one fashion.

For the third case, when $\text{Arg} \lambda = 0$, the image of $c_0^\lambda$ lies on the ray passing through $\exp(-2\pi i/5)$, and when $\text{Arg} \lambda = 2\pi$, this critical value lies on the complex conjugate ray. So, for these parameters, the critical value lies on a line that includes the straight line boundary of $R^\lambda$. For the circular boundaries of $O'$, the previous Proposition shows that the critical value now rotates around in a region outside the corresponding circular boundary of $R^\lambda$. Hence the critical value $F_\lambda(c_0^\lambda)$ winds with index one around $R^\lambda$ as $\lambda$ winds around the boundary of $O'$.

\[ \square \]

Before constructing this Mandelpinski arc, we recall the concept of a polynomial-like map. Let $G_\mu$ be a family of holomorphic maps that depends analytically on the parameter $\mu$ lying in some open disk $D$. Suppose each $G_\mu :$
$U_\mu \to V_\mu$ where both $U_\mu$ and $V_\mu$ are open disks that also depend analytically on $\mu$. $G_\mu$ is then said to be polynomial like of degree 2 if, for each $\mu$:

- $G_\mu$ maps $U_\mu$ two-to-one onto $V_\mu$ and so there is a unique critical point in $U_\mu$;

- $V_\mu$ contains $U_\mu$;

- As $\mu$ winds once around the boundary of $\mathcal{D}$, the critical value winds once around $U_\mu$ in the region $V_\mu - U_\mu$.

As shown in [9], for such a family of polynomial-like maps, there is a homeomorphic copy of the Mandelbrot set in the disk $\mathcal{D}$. Moreover, for $\mu$-values in this Mandelbrot set, $G_\mu | U_\mu$ is conjugate to the corresponding quadratic map given by this homeomorphism.

We can now prove

**Theorem.** Along the negative real axis in the parameter plane, there exist infinitely many alternating Mandelbrot sets $\mathcal{M}^k$ with $k \geq 2$ and Sierpinski holes $\mathcal{E}^k$ with $k \geq 1$. Here $k$ denotes the base period of $\mathcal{M}^k$ and the escape time of $\mathcal{E}^k$.

**Proof:** We first consider the escape time case. By construction, for each $\lambda \in \mathcal{O}'$, there is a unique prepole $p_2^\lambda$ in the interior of $R^\lambda$. Since $F_\lambda$ maps $R^\lambda$ one-to-one over itself, there is a unique preimage of this prepole, $z_3^\lambda$, in $R^\lambda$, so $F_\lambda^2(z_3^\lambda) = 0$. Continuing, for each $\lambda \in \mathcal{O}'$, there is a unique point $z_k^\lambda$ in $R^\lambda$ for which we have $F_\lambda(z_k^\lambda) = z_{k-1}^\lambda$ and so $F_\lambda^{k-1}(z_k^\lambda) = 0$. Now the points $z_k^\lambda$ vary analytically with $\lambda$ and are strictly contained in the interior of $R^\lambda$. So we may consider the function $H^k(\lambda)$ defined on $\mathcal{O}'$ by $H^k(\lambda) = v_0^\lambda - z_k^\lambda$ where $v_0^\lambda = F_\lambda(c_0^\lambda)$. When $\lambda$ rotates once around the boundary of $\mathcal{O}'$, $v_0^\lambda$ rotates once around the boundary of $R^\lambda$ while $z_k^\lambda$ remains in the interior of $R^\lambda$. Hence $H^k(\lambda)$ has winding number one along the boundary of $\mathcal{O}'$ and so
there must be a unique zero in \( \mathcal{O}' \) for each \( H^k \). This is then the parameter that lies at the center of the escape time region \( \mathcal{E}^k \). It is well known [13] that \( \mathcal{E}^k \) is then an open disk in the parameter plane. Note that, as \( \lambda \) decreases along \( \mathbb{R}^- \), both \( v_0^\lambda \) and \( z_k^\lambda \) increase along \( \mathbb{R}^+ \). It then follows that the portion of \( \mathcal{E}^{k+1} \) in \( \mathbb{R}^- \) lies to the left of \( \mathcal{E}^k \) in the parameter plane.

To prove the existence of the Mandelbrot sets \( \mathcal{M}^k \), recall that the orbit of the point \( z_k^\lambda \) under \( F_\lambda \) remains in \( R^\lambda \) before entering \( T_\lambda \) and landing at 0 at iteration \( k - 1 \) (here \( z_2^\lambda = p_2^\lambda \)). For each \( k \geq 2 \), let \( E^k_\lambda \) be the open set surrounding \( z_k^\lambda \) in \( R^\lambda \) that is mapped onto \( T_\lambda \) by \( F^{k-1}_\lambda \). Let \( D^k_\lambda \) be the set in \( R^\lambda \) consisting of points whose first \( k - 2 \) iterations lie in \( R^\lambda \) but whose \((k - 1)\)st iterate lies in the interior of \( L^\lambda \). Since \( F_\lambda \) is univalent on \( R^\lambda \), each \( D^k_\lambda \) is an open disk. Furthermore, the boundary of \( D^k_\lambda \) meets a portion of the boundaries of both \( E^{k-1}_\lambda \) and \( E^k_\lambda \) (where \( E^1_\lambda = T_\lambda \)). Since \( F^{k-1}_\lambda \) maps \( D^k_\lambda \) one-to-one over the interior of \( L^\lambda \) and then \( F_\lambda \) maps \( L^\lambda \) two-to-one over a region that contains \( R^\lambda \), we have that \( F^k_\lambda \) maps \( D^k_\lambda \) two-to-one over a region that completely contains \( R^\lambda \). Moreover, the critical value for \( F^k_\lambda \) is just \( v_0^\lambda \), which, by the preceding Proposition, winds once around the exterior of \( R^\lambda \) as \( \lambda \) winds once around the boundary of \( \mathcal{O}' \). Hence \( F^k_\lambda \) is a polynomial-like map of degree two on \( D^k_\lambda \) and this proves the existence of a baby Mandelbrot set \( \mathcal{M}^k \) lying in \( \mathcal{O}' \) for each \( k \geq 2 \). When \( \lambda \) is real and negative, we have that the centers of the escape regions \( \mathcal{E}^k \) lie along \( \mathbb{R}^- \) and, since the real line is invariant under \( F_\lambda \) when \( \lambda \in \mathbb{R}^- \), both \( \alpha_0^\lambda \) and \( v_0^\lambda \) also lie on the real axis. Then, by the \( \lambda \mapsto \overline{\lambda} \) symmetry in the parameter plane, the spines of these Mandelbrot sets also lie in \( \mathbb{R}^- \).

Next, since the \( E^k_\lambda \) and \( D^k_\lambda \) are arranged along the postive real axis in the following fashion:

\[
T_\lambda = E^1_\lambda < D^2_\lambda < E^2_\lambda < D^3_\lambda < E^3_\lambda < \ldots
\]
and, as shown above, the $\mathcal{E}^k$ decrease along $\mathbb{R}^-$ as $k$ increases. Thus we have that the $\mathcal{E}^k$ and $\mathcal{M}^k$ are arranged along the negative real axis in the parameter plane in the opposite manner:

$$\ldots \mathcal{E}^3 < \mathcal{M}^3 < \mathcal{E}^2 < \mathcal{M}^2 < \mathcal{E}^1.$$  

See Figure 5.

Finally, when $\lambda \in \mathbb{R}^-$, there is a non-empty interval lying between each adjacent $\mathcal{M}^k$ and $\mathcal{E}^j$ (where $j = k$ or $k-1$). This interval contains parameters for which $F^k_\lambda(c_0^\lambda)$ lies in $L^\lambda$, but then $F^{k+1}_\lambda(c_0^\lambda)$ is back in $R^\lambda$ and close to $\partial B_\lambda$. As a consequence, it takes more than $k$ additional iterations for this critical orbit to reach $T_\lambda$ or return to $L^\lambda$.

\[\square\]

![Figure 5: The Mandelbrot arc along the negative real axis. The $\mathcal{M}^k$ are so small that they are not visible in this picture. However, the magnification shows $\mathcal{M}^3$.](image)

In the remainder of this paper, we shall concentrate on a specific Mandelbrot set $\mathcal{M}^k$ and describe the infinite collection of Mandelbiski spokes
emanating from this set. With an eye toward how we shall proceed with this construction, note that, at this stage, we have a single infinite Mandelpinski arc extending to the left of $\mathcal{M}^k$ which contains the sets $\mathcal{M}^j$ with $j > k$ and $\mathcal{E}^j$ with $j \geq k$. And there is a finite Mandelpinski arc lying on the other side of $\mathcal{M}^k$ which now contains finitely many sets $\mathcal{M}^j$ where $2 \leq j < k$ and $\mathcal{E}^j$ where $1 \leq j < k$. These will be the initial portions of two of the Mandelpinski spokes emanating from $\mathcal{M}^k$.

3 The First Mandelpinski Spoke

In this next phase of the construction, we shall show that, on each side of the Mandelbrot set $\mathcal{M}^k$ in the first spoke, there are a pair of infinite spokes, each extending over to one of the adjacent Sierpinski holes $\mathcal{E}^k$ and $\mathcal{E}^{k-1}$. We think of this as extending the two previously constructed arcs emanating from $\mathcal{M}^k$. In addition, we shall show that there are a pair of new finite spokes extending above and below each $\mathcal{M}^k$. As above, a finite spoke means that there are only finitely many Mandelbrot sets and Sierpinski holes that alternate along this spoke. These will be the initial portions of the first four spokes emanating from $\mathcal{M}^k$.

To begin this phase of the construction, let us assume that the critical value $v_0^\lambda$ now lies in a particular open disk $D^k_\lambda$ for some fixed $k \geq 2$. Let $\mathcal{O}_k \subset \mathcal{O}'$ denote the set of parameters for which this happens. Now the boundary of $D^k_\lambda$ is mapped by $F^{k-1}_\lambda$ one-to-one onto the boundary of $L^\lambda$, and the boundary of $L^\lambda$ varies analytically with $\lambda$. So we can construct a natural parametrization of this boundary which also varies analytically with $\lambda$. Then we can pull back this parameterization to the boundary of each $D^k_\lambda$. Again, as we saw earlier, as $\lambda$ rotates around the boundary of the original disk $\mathcal{O}'$ in the parameter plane, $v_0^\lambda$ rotates once around the boundary of $R^\lambda$. Hence,
arguing just as in the previous section, there must be a unique parameter \( \lambda \) for which \( v^{\lambda}_0 \) lands on any given point in the parametrization of the boundary of \( D^k_{\lambda} \). Hence we have that \( \mathcal{O}_k \) is a disk contained inside \( \mathcal{O}' \) and, as \( \lambda \) rotates once around the boundary of \( \mathcal{O}_k \), the critical value has winding number one around the boundary of the disk \( D^k_{\lambda} \).

Now consider the set of preimages in \( L^\lambda \) of all of the \( D^j_{\lambda} \) and \( E^j_{\lambda} \) under \( F_{\lambda} \). Since we have assumed that \( v^{\lambda}_0 \) lies in \( D^k_{\lambda} \), it follows that there is a unique preimage of \( D^k_{\lambda} \) in \( L^\lambda \) which is a disk that contains \( c^\lambda_0 \) and is mapped two-to-one onto \( D^k_{\lambda} \). Call this special disk \( L^\lambda_k \). For each other \( D^j_{\lambda} \) (with \( j \neq k \)), there are now two preimage disks lying in \( L^\lambda \). Note that, when \( \lambda \in \mathbb{R}^- \) and \( j > k \), there are a pair of preimages of \( D^j_{\lambda} \) lying along \( \mathbb{R}^- \), one to the right of \( L^\lambda_k \) and one to the left. These preimages tend away from \( D^k_{\lambda} \) in either direction as \( j \) increases. When \( 2 \leq j < k \), there are again two preimages of \( D^j_{\lambda} \), but when \( \lambda \in \mathbb{R}^- \), these preimages no longer lie on the negative axis; rather they branch out more or less perpendicularly above and below \( L^\lambda_k \) on this axis.  

As for the preimages of \( E^j_{\lambda} \) in \( L^\lambda \), we have the same situation: there are infinitely many pairs of preimages of each \( E^j_{\lambda} \) lying along \( \mathbb{R}^- \) on either side of the preimage of \( D^k_{\lambda} \) when \( j \geq k \) and \( \lambda \in \mathbb{R}^- \), and finitely many pairs extending above and below this preimage when \( 1 \leq j < k \). Thus we have a pair of infinite chains of alternating preimages of the disks \( D^k_{\lambda} \) and \( E^k_{\lambda} \) extending away from \( L^\lambda_k \) and another pair of chains consisting of finitely many such preimages extending in a “perpendicular” direction away from \( L^\lambda_k \).

Since \( F_{\lambda}^{k-1} \) maps \( D^k_{\lambda} \) one-to-one over \( L^\lambda \), we thus have a similar collection of preimages that lie inside the disk \( D^k_{\lambda} \). We denote by \( D^{kj}_\lambda \) each of the two disks in \( D^k_{\lambda} \) that are mapped onto \( D^j_{\lambda} \) by \( F^k_{\lambda} \) when \( j \neq k \). And we let \( D^{kk}_\lambda \) denote the single preimage of \( D^k_{\lambda} \) under \( F^k_{\lambda} \) that is contained in \( D^k_{\lambda} \), i.e., the preimage of \( L^\lambda_k \) under \( F_{\lambda} \). So points in \( D^{kj}_\lambda \) have orbits that remain in \( R^\lambda \) for
the first $k - 2$ iterations, then map to $L^\lambda$ under the next iteration, and then map into $D^j_\lambda$ under the next iteration. Then $F^{j-1}_\lambda$ maps this set onto $L^\lambda$. So $F^{k+j-1}_\lambda$ maps each $D^{kj}_\lambda$ one-to-one onto all of $L^\lambda$ (assuming $k \neq j$). Then the next iteration takes this set two-to-one onto all of $R^\lambda$. Now the critical value for $F^{k+j}_\lambda$ is again $v^\lambda_0$, and, as we showed above, as $\lambda$ rotates around the boundary of $\mathcal{O}_k$, $v^\lambda_0$ circles around the boundary of $D^k_\lambda$. Hence $F^{j+k}_\lambda$ is polynomial-like of degree two on each of the two disks $D^{kj}_\lambda$ (where we again emphasize that we are assuming $j \geq 2$ and $j \neq k$). So this produces a pair of Mandelbrot sets $\mathcal{M}^{kj}$ with base period $k + j$ in $\mathcal{O}_k$. As in the previous construction, the Mandelbrot sets $\mathcal{M}^{kj}$ with $j > k$ all have spines lying along $\mathbb{R}^-$, one on each side of $\mathcal{M}^k$. The other Mandelbrot sets with $j < k$ now lie off the real axis, one above $\mathcal{M}^k$ and the other below $\mathcal{M}^k$.

Similar arguments as in the preceding section also produce a pair of Sierpinski holes $\mathcal{E}^{kj}$ on each side of $\mathcal{M}^k$ along the real axis where now $j \geq k$. And there are a pair of Sierpinski holes $\mathcal{E}^{kj}$, one above and one below $\mathcal{M}^k$, where now $1 \leq j < k$. As earlier, these Mandelbrot sets and Sierpinski holes alternate along each of these spokes. For parameters in the Sierpinski hole $\mathcal{E}^{k1}$, the critical orbit $F^{i}_\lambda(c^\lambda_0)$ lies in $R^\lambda$ for iterations $1 \leq i \leq k - 1$. Then $F^k_\lambda(c^\lambda_0)$ returns to $L^\lambda$, and then $F^{k+1}_\lambda(c^\lambda_0)$ enters $T_\lambda$.

Note that the Mandelbrot sets $\mathcal{M}^{kj}$ are not subsets of the larger Mandelbrot set $\mathcal{M}^k$. This follows since the orbit of the critical point returns to $L^\lambda$ only at iterations $k$ and $k + j$ with $j \neq k$ when $\lambda \in \mathcal{M}^{kj}$, whereas these returns occur at iterations $k$ and $2k$ when $\lambda \in \mathcal{M}^k$. This also follows from the fact that there is a Sierpinski hole separating each of these baby Mandelbrot sets from $\mathcal{M}^k$ along the new spoke. In Figure 6 we display a portion of these smaller spokes around $\mathcal{M}^4$. To summarize the results at this phase of the construction, we have shown:

**Theorem.** *In the original Mandelbński arc, between each $\mathcal{E}^{k-1}$ and $\mathcal{E}^k$,*
there exist a pair of infinite spokes, each containing Mandelbrot sets $\mathcal{M}^{kj}$ where $j > k$ and Sierpinski holes $\mathcal{E}^{kj}$ where $j \geq k$ in the same alternating arrangement as earlier. One spoke extends from $\mathcal{M}^k$ to $\mathcal{E}^{k-1}$, the other from $\mathcal{M}^k$ to $\mathcal{E}^k$. There are also a pair of finite spokes extending away from $\mathcal{M}^k$ in different directions. These finite spokes contain the Mandelbrot sets $\mathcal{M}^{kj}$ where $2 \leq j < k$ and the Sierpinski holes $\mathcal{E}^{kj}$ where now $1 \leq j < k$. The Mandelbrot sets $\mathcal{M}^{kj}$ have base period $k+j$ and the Sierpinski holes $\mathcal{E}^{kj}$ have escape time $k+j$.

![Image of Mandelbrot and Sierpinski sets](image)

Figure 6: The finite spoke above and below $\mathcal{M}^4$ as well as a magnification showing the infinite spokes along the real axis.

### 4 Final Phase

We now continue the construction of the Mandelpinski spokes by induction. For simplicity, we will only consider the next phase of the construction; all subsequent phases follow in exactly the same way. This time we will adjoin
four infinite spokes that lie closer to $\mathcal{M}^k$ to those already in place, and then we will add four new finite spokes in between each of these infinite spokes.

To be precise, in the previous phase, we assumed that the critical value resided in a particular disk $D^k_\lambda$, and so there was a special disk $D^{kk}_\lambda \subset D^k_\lambda$ that was mapped two-to-one onto $D^k_\lambda$ by $F^k_\lambda$. At this stage we make the further assumption that $v^\lambda_0$ lies in $D^{kk}_\lambda$. Let $\mathcal{O}_{kk} \subset \mathcal{O}_k$ be the set of parameters for which this occurs. Note that $\mathcal{M}^k$ lies in $\mathcal{O}_{kk}$. We have that $F^{kk}_\lambda$ maps the boundary of $D^{kk}_\lambda$ two-to-one onto the boundary of $D^k_\lambda$. Thus we may pull back the parametrization of $\partial D^k_\lambda$ constructed earlier to produce a natural parametrization of $\partial D^{kk}_\lambda$ which varies analytically with $\lambda$. Thus there is a unique $\lambda$ for which $v^\lambda_0$ lands on a given point in the boundary of $D^{kk}_\lambda$, and so, as $\lambda$ winds once around the boundary of $\mathcal{O}_{kk}$, $v^\lambda_0$ winds once around $\partial D^{kk}_\lambda$.

By the prior construction, we have a pair of infinite chains each of which consists of the disks $D^{kj}_\lambda$ with $j > k$ and $E^{kj}_\lambda$ with $j \geq k$ lying in the annular region $D^k_\lambda - D^{kk}_\lambda$ as well as a pair of finite chains consisting of the disks $D^{kj}_\lambda$ and $E^{kj}_\lambda$ with $j < k$ lying in the same annulus. Since $F^k_\lambda$ maps $D^{kk}_\lambda$ two-to-one onto the entire disk $D^k_\lambda$, we therefore have four new infinite chains inside $D^{kk}_\lambda$ that are the preimages of the two infinite chains in the annular region. These chains consist of disks that we denote by either $D^{kkj}_\lambda$ with $j > k$ or $E^{kkj}_\lambda$ with $j \geq k$. Each of these chains then connects to one of the two infinite or finite chains in the outer annular region. This follows since these outer chains were all mapped onto the left or right portion of the original chain by $F^k_\lambda$. We also have four finite chains in $D^{kk}_\lambda$ consisting of disks $D^{kkj}_\lambda$ and $E^{kkj}_\lambda$ with $j < k$ that are preimages of the finite chains in the annular region. These chains do not connect to the previously constructed chains in the annular region.

Then the same arguments as above produce the corresponding spokes in the parameter plane. Each of the two finite and infinite spokes constructed earlier now have an added infinite spoke that lies in the region between $\mathcal{M}^k$.
and that spoke. The Mandelbrot sets and Sierpinski holes in this new portion of the spoke are given by $\mathcal{M}^{kj}$ where $j > k$ and $\mathcal{E}^{kj}$ where $j \geq k$ and the four new finite spokes consist of similar sets with now $j < k$. These are all associated with rays of angle $\ell/8$ with $\ell$ even for the infinite spokes and $\ell$ odd for the finite spokes.

At this stage we now have eight Mandelpinski spokes emanating from $\mathcal{M}^k$, four finite spokes and four infinite spokes. Continuing inductively, at the next stage, we then add eight infinite spokes between each of these spokes and $\mathcal{M}^k$ as well as eight new finite spokes, one between each of these newly added infinite spokes. In the limit, we get an infinite collection of Mandelpinski spokes emanating from $\mathcal{M}^k$.

References


