Cantor Necklaces and Structurally Unstable Sierpinski Curve Julia Sets for Rational Maps

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In this paper we consider families of rational maps of degree $2n$ on the Riemann sphere $F_\lambda : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

where $\lambda \in \mathbb{C} - \{0\}$ and $n \geq 2$. The case where $n = 1$ is very different and hence is excluded from this study. We denote the family of all such rational maps of degree $2n$ by Λ_n. A function in Λ_n is called a *singular perturbation* of z^n.

One of our goals in this paper is to describe a type of structure that we call a *Cantor necklace* that occurs in both the dynamical and the parameter plance for F_λ. Such a set is homeomorphic to a set constructed as follows. Start with the Cantor middle thirds set embedded on the x-axis in the plane. Then replace each removed open interval with an open circular disk whose diameter is the same as the length of the removed interval. A Cantor necklace is a set that is homeomorphic to the resulting union of the Cantor set and the adjoined open disks.

Of primary interest in the dynamical plane for F_λ is the *Julia set* which we denote by $J(F_\lambda)$. The second goal of this paper is to use the Cantor necklaces in the parameter plane to prove the existence of several new types of Sierpinski curve Julia sets (defined below) that arise in these families of rational maps. Unlike most examples of this type of Julia set, the maps on these Julia sets are structurally unstable. That is, small changes in the parameter λ give rise to Julia sets on which the dynamical behavior is quite different. In addition, we also describe a new type of related Julia set which we call a hybrid Sierpinski curve.

Despite the possibly high degree of the rational map F_λ, there are essentially only three critical orbits for this function. One critical point occurs at ∞, which is therefore a superattracting fixed point for F_λ, and thus there always exists an immediate basin of attraction at ∞ which we call B_λ. The origin is a pole for each of these maps, and since $n > 1$, this point is a second critical point which is mapped onto the fixed point at ∞. In particular, there is an open set T_λ containing 0 that is mapped onto B_λ by F_λ. (We remark that either $B_\lambda = T_\lambda$ or B_λ and T_λ are disjoint.) Finally, there are $2n$ other critical points, but the orbits of each of these points are symmetric about the origin, so there is essentially only one "free" critical orbit for F_λ.

The dynamics on and the topology of $J(F_\lambda)$ are quite rich when this free critical orbit tends to ∞. The following theorem is proved in [2] and is called the *escape trichotomy.*

Theorem. *Suppose the free critical orbit tends to* ∞*.*

1. *If the critical values lie in* B_λ*, then* $J(F_\lambda)$ *is a Cantor set;*
2. *If the critical values lie in* T_λ *(and* $B_\lambda \neq T_\lambda$*), then* $J(F_\lambda)$ *is a Cantor set of simple closed curves;*
3. *If the critical values lie in some other preimage of* B_λ*, then* $J(F_\lambda)$ *is a Sierpinski curve.*

A *Sierpinski curve* is a planar set that is characterized by the following five properties: it is a compact, connected, locally connected and nowhere dense set whose complementary domains are bounded by simple closed curves that are pairwise disjoint. It is known [14] that any two Sierpinski curves are homeomorphic. In fact, they are homeomorphic to the well-known Sierpinski carpet fractal. From the point of view of topology, a Sierpinski curve is a universal set in the sense that this set contains a homeomorphic copy of any planar, compact, connected, one-dimensional set.

The case where the Julia sets of these maps is a Cantor set of simple closed curves was first observed by McMullen [7]. For some examples of each of these three types of Julia sets drawn from the family when $n = 4$, see Figure 1.

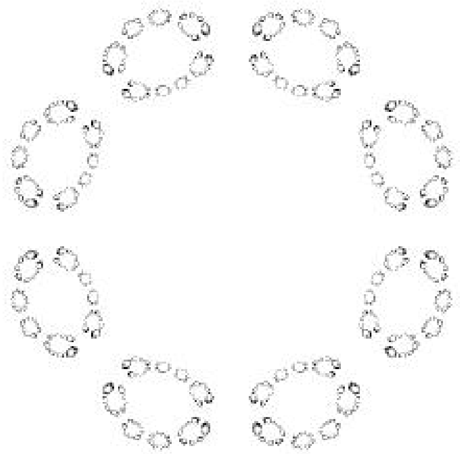
Even though Sierpinski curve Julia sets are always homeomorphic, from a dynamical systems point of view, these sets can be quite different from one another. In [2] it is shown that, for each $n \geq 2$, there are infinitely many disjoint open sets $\mathcal{O}_j$ in Λ_n such that, if $\lambda_j \in \mathcal{O}_j$, then $J(F_{\lambda_j})$ is a Sierpinski curve. However, F_{λ_j} is not topologically conjugate to F_{λ_k} if $j \neq k$. So the Julia sets of these maps are the same but the dynamics on these sets are different. In [1] it is shown that, in the special case where $n = 2$, there are infinitely many such $\mathcal{O}_j$ in every neighborhood of 0 in the parameter plane. See Figure 2 for several examples of Sierpinski curve Julia sets drawn from the family with $n = 2$.

There are other ways besides having all critical orbits tend to ∞ that the Julia sets of F_λ can be Sierpinski curves. For example, in [5], it is shown that each of these families also posseses infinitely many Sierpinski curve Julia sets whose complementary domains consist of a finite number of basins of attraction of different attracting cycles including the basin at ∞, together with all of their preimages. Again, each of these Julia sets comes with different dynamics.

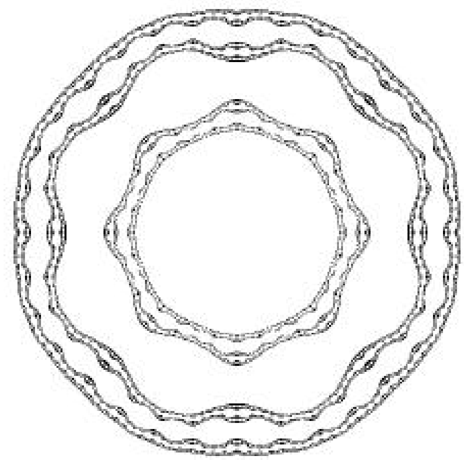
One of the main properties of all of the Julia sets mentioned above is that the corresponding maps are structurally stable within Λ_n. That is, there is an open neighborhood $\mathcal{U}$ about such a $\lambda \in \Lambda_n$ having the property that if $\mu \in \mathcal{U}$, then F_λ and F_μ are topologically conjugate on their Julia sets. This can be seen by noting that both F_λ and F_μ have the same symbolic dynamics on their Julia sets as described in [4].

In this paper, we present a very different collection of Sierpinski curve Julia sets. Our main result here is:

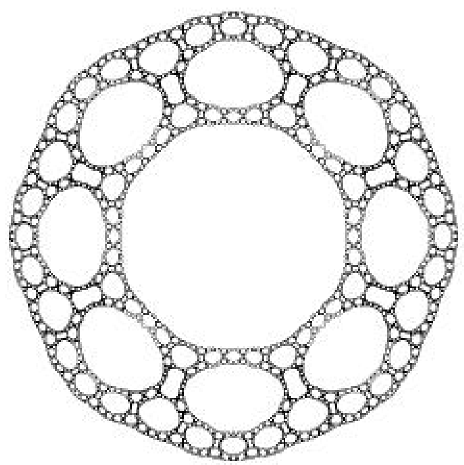
Theorem. *There exist infinitely many parameter values λ in each Λ_n such*



$\lambda = 0.23$

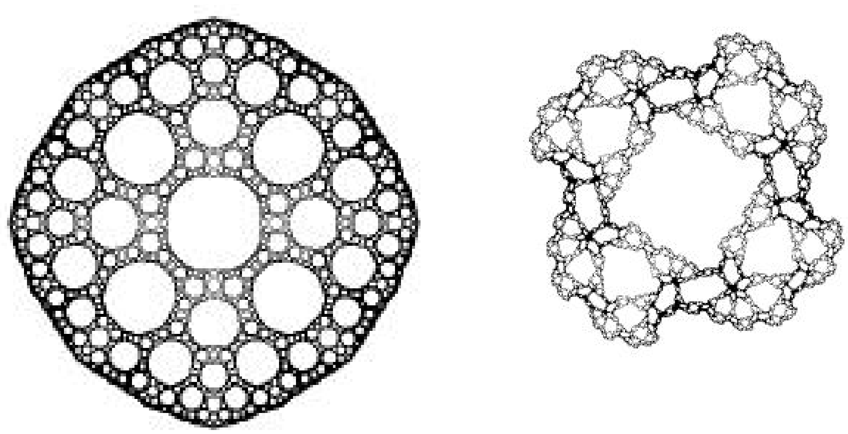


$\lambda = 0.03$



$\lambda = -0.1$

Figure 1: Some Julia sets for $z^4 + \lambda/z^4$: if $\lambda = 0.23$, $J(F_\lambda)$ is a Cantor set; if $\lambda = 0.03$, $J(F_\lambda)$ is a Cantor set of circles; and if $\lambda = -0.1$, $J(F_\lambda)$ is a Sierpinski curve.



$\lambda = -1/16$ $\lambda = 0.132 + 0.097i$

Figure 2: Two Sierpinski curve Julia sets for $z^2 + \lambda/z^2$.

that $J(F_\lambda)$ is a Sierpinski curve but F_λ is not structurally stable on this Julia set. In particular, in every neighborhood $\mathcal{U}$ of each such λ:

1. *There exist infinitely many parameter values $\mu_j \in \mathcal{U}$ such that $J(F_{\mu_j})$ is a structurally stable Sierpinski curve Julia set, but F_{μ_j} is not conjugate to F_{μ_k} for $j \neq k$;*

2. *There exist infinitely many parameter values $\nu_j \in \mathcal{U}$ such that $J(F_{\nu_j})$ is a Sierpinski curve Julia set which is not structurally stable and again, F_{ν_j} is not conjugate to F_{ν_k} when $j \neq k$ nor is F_{ν_j} conjugate to any of the F_{μ_ℓ};*

3. *Finally, there exist infinitely many parameter values $\tau_j \in \mathcal{U}$ such that $J(F_{\tau_j})$ is a hybrid Sierpinski curve, and again, F_{τ_j} is not conjugate to F_{τ_k} for $j \neq k$, nor to any of the F_{μ_ℓ} or F_{ν_ℓ} above.*

A *hybrid* Sierpinski curve is a set that has the same five properties as a Sierpinski curve except that infinitely many of the complementary domains have a boundary point in common with exactly one other complementary domain, while infinitely many other complementary domains have boundaries that do not meet any other such boundary. See Figure 3 for an example of such a hybrid Sierpinski curve Julia set. We conjecture that, as in the case of an ordinary Sierpinski curve, any two hybrid Sierpinski curves are homeomorphic.

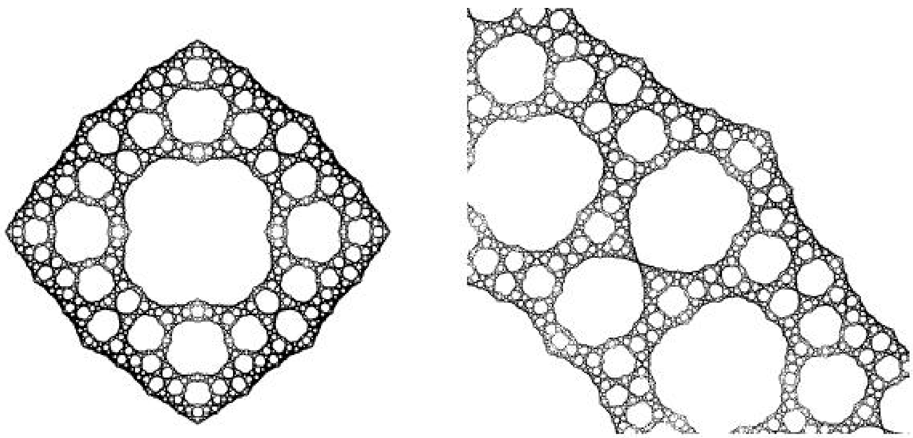


Figure 3: A hybrid Sierpinski curve Julia set and a magnification. Note that some of the complementary domains appear to be bounded by isolated simple closed curves while others are bounded by simple closed curves that meet another such curve at a single point.

1 Basic Properties

For simplicity, for most of this paper we shall concentrate on the specific family of maps given by

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2},$$

i.e., the case where $n = 2$. In the final section, we briefly describe the straightforward modifications necessary to extend these results to the case

$n > 2$.

We begin by reviewing some basic properties of functions in this family. See [1] or [3] for proofs of these facts.

Note first that 0 is the only pole for each function in this family. The points $(-\lambda)^{1/4}$ are prepoles for F_λ since they are mapped by F_λ directly to 0. The critical points for F_λ occur at the four points $\lambda^{1/4}$. Since $F_\lambda(\lambda^{1/4}) = \pm 2\lambda^{1/2} = \pm v_\lambda$, there are only two critical values for F_λ. Also, $F_\lambda^2(\lambda^{1/4}) = 1/4 + 4\lambda$, so each of the four critical points lies on the same forward orbit after two iterations. Thus the orbit of $1/4 + 4\lambda$ is the tail of the free critical orbit.

The circle given by $|z| = |\lambda^{1/4}|$ is known as the *critical circle* for this family and is denoted by C_λ. A computation shows that the critical circle is mapped onto a straight line segment connecting the two critical values and passing through the origin. The map takes C_λ in 4 to 1 fashion onto this segment except at the critical values, each of which has only two preimages.

Recall that the point at ∞ is a superattracting fixed point for F_λ and that B_λ is its immediate basin of attraction. Let ∂B_λ denote the boundary of B_λ. The basin B_λ is a (forward) invariant set for F_λ in the sense that, if $z \in B_\lambda$, then $F_\lambda^n(z) \in B_\lambda$ for all $n \geq 0$. The same is true for ∂B_λ.

Recall that $J = J(F_\lambda)$ is the *Julia set* of F_λ. By definition, $J(F_\lambda)$ is the set of points at which the family of iterates of F_λ fails to be a normal family in the sense of Montel. Equivalently, $J(F_\lambda)$ is the closure of the set of repelling periodic points of F_λ and it is also the boundary of the set of points whose orbits tend to ∞. See [9] for proofs of these equivalences.

As in the escape trichotomy, we may have different types of Julia sets when the free critical orbit escapes to ∞. However, in the special case where $n = 2$, the Cantor set of circles case does not occur. As shown in [3], the critical values cannot lie in T_λ and so we really have a dichotomy in this case:

Theorem. *For the family of rational maps given by*

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2},$$

1. *If the critical values lie in B_λ, then $J(F_\lambda)$ is a Cantor set;*

2. *Otherwise, $J(F_\lambda)$ is a compact, connected set and B_λ is open and simply connected. In particular, if the critical orbit escapes to ∞ but the critical values do not lie in B_λ, then $J(F_\lambda)$ is a Sierpinski curve.*

In case 2 of this theorem, since the critical values do not lie in B_λ, it is known that the preimage of B_λ surrounding the origin, T_λ, is disjoint from B_λ. The map is 2 to 1 on both B_λ and T_λ. Since F_λ has degree 4, these two sets contain all of the preimages of points in B_λ. We call T_λ the *trap door*, since any orbit that eventually enters B_λ must pass through T_λ.

Each of the maps F_λ possess certain symmetries. For example, we have that $F_\lambda(-z) = F_\lambda(z)$ and $F_\lambda(iz) = -F_\lambda(z)$ so that $F_\lambda^2(iz) = F_\lambda^2(z)$ for all $z \in \mathbb{C}$. As a consequence, each of the sets B_λ, T_λ, and $J(F_\lambda)$ are invariant under $z \mapsto iz$. We therefore say that these sets possess fourfold symmetry.

There is a second symmetry present for this family. Consider the map $H(z) = \sqrt{\lambda}/z$. Note that there are two such maps depending upon which square root of λ we choose. H is an involution and we have $F_\lambda(H(z)) = F_\lambda(z)$. As a consequence, H also preserves J. The involution H also preserves the circle of radius $|\lambda|^{1/4}$, the critical circle, and interchanges the interior and exterior of this circle. Hence J is symmetric about the critical circle with respect to the action of H.

Finally, in analogy with the well-studied quadratic polynomial family $z \mapsto z^2 + c$, since F_λ has degree two on B_λ (in case 2 of the above Theorem), it is known that F_λ is conjugate to $z \mapsto z^2$ on B_λ when the critical values do not lie in B_λ. That is, there is an analytic homeomorphism $\phi_\lambda : B_\lambda \to \overline{\mathbb{C}} - \overline{\mathbb{D}}$

that satisfies $\phi_\lambda \circ F_\lambda(z) = (\phi_\lambda(z))^2$ for all $z \in B_\lambda$. Here $\mathbb{D}$ is the open unit disk in $\mathbb{C}$. As is well known, the map $z \mapsto z^2$ preserves the straight rays $\operatorname{Arg} z =$ constant, so the inverse images of these straight rays under ϕ_λ are preserved by F_λ. These curves are known as *external rays.* In particular, there is an external ray corresponding to the ray $\operatorname{Arg} z = 0$. It is known (see [11]) that this ray limits on a unique point p_λ in ∂B_λ and that p_λ is the unique fixed point of F_λ that lies in ∂B_λ. Similarly, the external ray corresponding to $\operatorname{Arg} z = \pi$ limits on the point $-p_\lambda \in \partial B_\lambda$.

2 Cantor Necklaces in Dynamical Plane

One of the principal objects contained in the dynamical plane of F_λ is a Cantor necklace. To define this set, we let Γ denote the Cantor middle thirds set in the unit interval $[0, 1]$. We regard this interval as a subset of the real axis in the plane. For each open interval of length $1/3^n$ removed from the unit interval in the construction of Γ, we replace this interval by an open disk of diameter $1/3^n$ centered at the midpoint of the removed interval. Thus the boundary of this open disk meets the Cantor set at the two endpoints of the removed interval. We call the resulting set the *Cantor middle-thirds necklace.* See Figure 4. Any set homeomorphic to the Cantor middle-thirds necklace is called a *Cantor necklace.* We do not include the boundary of the open disks in the Cantor necklace for the following technical reason: it is sometimes difficult in practice to verify that these bounding curves are simple closed curves. On the other hand, in our case, the open regions will be preimages of the basin at ∞, and we know that these preimages are open and simply connected.

Our aim in this section is to describe various Cantor necklaces in the dynamical plane for F_λ. In the next section we prove the existence of similar sets in the parameter plane.

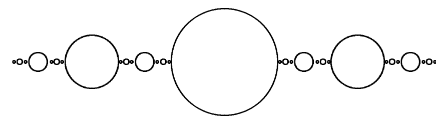


Figure 4: The Cantor middle-thirds necklace.

Write $\lambda = |\lambda|e^{i\eta}$. For the rest of this paper, we will consider only the case where $0 < \eta < 2\pi$. All of the results below also hold when $\eta = 0$, but the arguments are slightly different and we will not consider this case in the sequel anyway.

When $|\lambda| < 1$, we have the following escape criterion.

Proposition. (The Escape Criterion). *Suppose $|\lambda| < 1$ and $|z| \geq 2$. Then $z \in B_\lambda$, so $J(F_\lambda)$ is contained in the open disk $|z| < 2$.*

Proof: If $|z| \geq 2$, then we have

$$|F_\lambda(z)| \geq |z|^2 - \frac{|\lambda|}{|z|^2} \geq 2|z| - \frac{1}{4} > \frac{3}{2}|z|.$$

Inductively, we find

$$|F_\lambda^n(z)| \geq \left(\frac{3}{2}\right)^n |z|.$$

Therefore the orbit of any such z tends to ∞ so all points on or outside the circle of radius 2 lie in B_λ.

$\square$

From the results in [2], it is known that $J(F_\lambda)$ is a Cantor set if $|\lambda| \geq 1$, so we also exclude this case for the remainder of this paper. Therefore we assume throughout that $\lambda = |\lambda|e^{i\eta}$ with $0 < \eta < 2\pi$ and $|\lambda| < 1$.

Recall that the critical points of F_λ are given by $\lambda^{1/4}$. Therefore one of the critical points of F_λ lies on the straight line through the origin given by $t\exp(i\eta/4)$ with $t > 0$. The image of this line lies in the straight line with argument $\theta = \eta/2$, and F_λ maps the line $t\exp(i\eta/4)$ with $t > 0$ in two-to-one fashion over the portion of this straight line that lies beyond the critical value $2\sqrt{\lambda}$ whose argument is $\eta/2$. Note that the image of the line is disjoint from the line itself since we have assumed that $0 < \eta < 2\pi$. There is a second critical point of F_λ lying on the line with argument $\theta = \eta/4 - \pi/2$, and this line is mapped in two-to-one fashion to the opposite line $\theta = -\eta/2$ exactly as in the previous case.

By the escape criterion, we know that any point on or outside $r = 2$ is mapped closer to ∞. Let β_λ denote the image of this circle, so that $\beta_\lambda \subset B_\lambda$. Using the involution H, there is a second circle, namely $r = |\lambda|^{1/2}/2$, that is also mapped two-to-one onto β_λ.

Consider the open region R_λ bounded by the rays $\theta = \eta/4$ and $\theta = \eta/4 - \pi/2$ and the two circular preimages of β_λ. The set R_λ is a quarter of an annulus. Let $L_\lambda = -R_\lambda$. We call R_λ (resp., L_λ) the right (resp., left) fundamental sector. These fundamental sectors are a pair of disjoint, open, simply connected regions in $\mathbb{C}$. Note that, for each λ, R_λ lies in the right half plane $\operatorname{Re} z > 0$, while L_λ lies in the left half plane. See Figure 5.

Proposition. *F_λ maps each of the fundamental sectors in one-to-one fashion onto the open set $\mathcal{O}$ bounded by β_λ minus the portions of the two straight lines $\theta = \pm\eta/2$ extending from the critical values $\pm v_\lambda$ to β_λ. So the image of each of these fundamental sectors contains the closures of both R_λ and L_λ in its interior.*

Proof: The images of the straight rays bounding R_λ and L_λ are contained in the rays $\theta = \pm\eta/2$, both of which lie outside these sectors. The image of

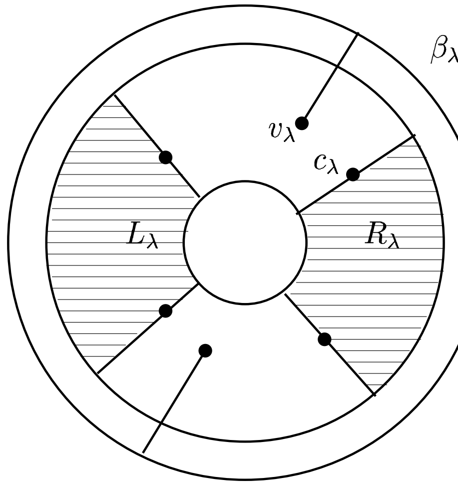


Figure 5: R_λ and L_λ and their image under F_λ, which is the interior of β_λ minus the two segments connecting this circle to the critical values.

the outer circular boundary of each fundamental sector is exactly one-half of β_λ, while the inner boundary of each sector is mapped to the opposite half of β_λ. Hence each fundamental sector is mapped onto the open disk bounded by β_λ minus the two portions of the rays $\theta = \pm\eta/2$ lying beyond the critical values. This set is $\mathcal{O}$. By fourfold symmetry, this map must be one-to-one on each fundamental sector.

□

Since F_λ maps the union of the fundamental sectors strictly outside itself, many points in $R_\lambda \cup L_\lambda$ have orbits that leave this set at some iteration. Let Γ_λ be the set of points whose orbits remain for all iterations in $R_\lambda \cup L_\lambda$. Then we have:

Proposition. *The set* Γ_λ *is a Cantor set and* $F_\lambda \,|\, \Gamma_\lambda$ *is conjugate to the one-sided shift on two symbols.*

Proof: By the previous result, each of the fundamental sectors is mapped in one-to-one fashion onto the open region $\mathcal{O}$ that properly contains $R_\lambda \cup L_\lambda$ in

$\mathbb{C}$. So we have a pair of well-defined inverses G_0 (resp., G_1) of F_λ that map $\mathcal{O}$ into R_λ (resp., L_λ). Standard arguments then show that these inverses are contractions in the Poincaré metric on $\mathcal{O}$. Moreover, for any one-sided sequence $(s_0 s_1 s_2 \ldots)$ of 0's and 1's, the set

$$\bigcap_{j=0}^{\infty} G_{s_0} \circ \ldots \circ G_{s_j}(\mathcal{O})$$

is a unique point and the map that takes the sequence $(s_0 s_1 s_2 \ldots)$ to this point defines a homeomorphism between the space of one-sided sequences of 0's and 1's endowed with the usual topology and Γ_λ. Hence Γ_λ is a Cantor set and we have that $F_\lambda \,|\, \Gamma_\lambda$ is conjugate to the one-sided shift on two symbols. □

We remark that when $\lambda \in \mathbb{R}^-$, the Cantor set Γ_λ lies on the real axis. Indeed, a glance at the graph of the real function F_λ shows that F_λ maps the interval $[-p_\lambda, p_\lambda]$ in two-to-one fashion over itself, where p_λ is the fixed point for F_λ on the positive real axis and on the boundary of B_λ. See Figure 6. See [1].

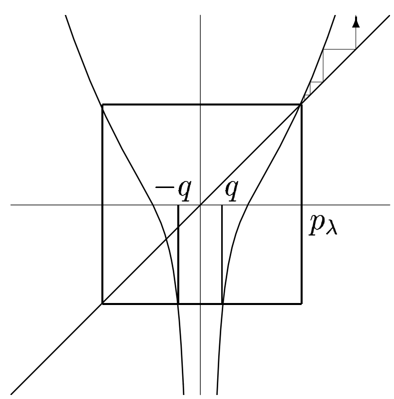


Figure 6: The graph of F_λ on the real line for $\lambda < 0$. The points $\pm q$ bound the trap door on the real axis.

Now suppose in addition that the critical values do not lie in B_λ. So, by the dichotomy of the previous section, $J(F_\lambda)$ is a connected set and B_λ is a simply connected open set. Our goal is to construct a Cantor necklace in the dynamical plane. The Cantor set portion of the necklace will be the set Γ_λ constructed above, whereas the open disks will be certain of the preimages of the basin of ∞ lying in R_λ and L_λ.

To construct the necklace, recall that, as discussed in Section 1, there are a pair of external rays in B_λ that limit on $\pm p_\lambda \in \Gamma_\lambda$. Let $\pm q_\lambda$ be the preimages of $-p_\lambda$ in L_λ and R_λ. Then there is a unique curve in T_λ passing through 0 and connecting q_λ to $-q_\lambda$ that is mapped onto these two external rays together with ∞. Since T_λ is open and simply connected, we may define a homeomorphism that takes T_λ onto a disk centered at 1/2 on the real line and having radius 1/6. This is the central disk in the Cantor middle-thirds necklace. Moreover, we may arrange that this homeomorphism extends to the points $\pm q_\lambda$ in ∂T_λ so that the point $-q_\lambda$ is sent to 1/3 and the point q_λ is sent to 2/3.

Now consider the preimages of the trap door. Since the critical values do not lie in the trap door, there are four preimages of this set, but only two of them lie in the fundamental sectors by fourfold symmetry. These give a pair of simply connected open sets that contain the preimages of $\pm q_\lambda$ in their boundaries; one of these sets lies in R_λ, the other lies in L_λ. These may be mapped homeomorphically to the open disks in the Cantor middle-thirds necklace whose diameter is 1/9 with the property that an extension of this homeomorphism takes the preimages of $\pm q_\lambda$ to the corresponding endpoints of the Cantor middle thirds set. We then continue in this fashion by taking appropriate preimages of the trap door under compositions of G_0 and G_1 and then extending the map to all points in Γ_λ in the natural manner. This then gives a homeomorphism between the points in Γ_λ together with certain preimages of the trap door and the Cantor middle thirds necklace. We have proved:

Theorem. *Suppose that the critical values of F_λ do not lie in B_λ and that $0 < \operatorname{Arg}\lambda < 2\pi$. Then the set Γ_λ together with the preimages of B_λ under compositions of the maps G_0 and G_1 forms a Cantor necklace.*

3 Cantor Necklaces in Parameter Plane

In this section we show that there is an analogue of the Cantor necklace in the parameter plane and that the Cantor set portion of this necklace lies along the negative real axis.

Let Σ_2 denote the space of one-sided sequences of 0's and 1's endowed with the usual topology. Let Σ_2' denote the subset of Σ_2 consisting of all sequences whose first digit is 1. Under the conjugacy between $F_\lambda \mid \Gamma_\lambda$ and the shift map, the set Σ_2' corresponds to points in Γ_λ that lie in L_λ. Let $w_\lambda(s)$ denote the point in Γ_λ whose itinerary is s. We will prove that, for each $s \in \Sigma_2'$, there is a unique $\lambda = \lambda_s$ such that $F^2_{\lambda_s}(c_{\lambda_s}) = w_{\lambda_s}(s)$. That is, there is a unique parameter for which the second image of the critical points lies on the specified point $w_\lambda(s) \in \Gamma_\lambda$.

Let D be the half-disk $\{\lambda \mid \operatorname{Re}\lambda < 0, |\lambda| < 1\}$. For each $s \in \Sigma_2'$, let $I_s : D \to \mathbb{C}$ be given by $I_s(\lambda) = w_\lambda(s)$. That is, I_s assigns to each $\lambda \in D$ the unique point in $\Gamma_\lambda \cap L_\lambda$ whose itinerary is s. It is well known that I_s depends analytically on λ and continuously on s. Note that I_s takes values in the half disk given by $\operatorname{Re} z \leq 0$ and $|z| \leq 2$, since each L_λ is completely contained in this region.

We also have an analytic homeomorphism Φ defined on D by $\Phi(\lambda) = F^2_\lambda(c_\lambda) = 4\lambda + 1/4$. Note that Φ maps D to the half-disk determined by $\operatorname{Re} z < 1/4$ and $|z - 1/4| < 4$. This half-disk contains the closure of L_λ for each $\lambda \in D$. In particular, $I_s(\lambda) \in \Phi(D)$ for all $\lambda \in D$ and all $s \in \Sigma_2'$.

Therefore we may define a new map $\Psi_s : D \to D$ by $\Psi_s(\lambda) = \Phi^{-1}(I_s(\lambda))$. Since $I_s(\lambda)$ is contained inside a compact subset of $\Phi(D)$, it follows that Ψ_s maps D into a compact subset of itself. Moreover, Ψ_s is an analytic function of λ. By the Schwarz Lemma, it follows that Ψ_s has a unique fixed point in D. We call this fixed point λ_s. So λ_s is a parameter value for which $\Phi(\lambda_s) = I_s(\lambda_s) = w_{\lambda_s}(s)$. That is, for the parameter λ_s, the second iterate of the critical point lands exactly on the point in Γ_λ whose itinerary is $s \in \Sigma_2'$. Moreover, λ_s is the unique parameter value for which this occurs. Also, since Ψ_s depends continuously on s, it follows that λ_s is also a continuous function of s for $s \in \Sigma_2'$. Therefore the set of parameters in D of the form λ_s as s runs over Σ_2' forms a Cantor set in D which we denote by Γ. We emphasize that Γ is a Cantor set in the parameter plane, whereas Γ_λ is a Cantor set in the dynamical plane for each λ.

If $\lambda \in \mathbb{R}^-$, then the graph of F_λ (Figure 6) shows that the entire dynamical Cantor set Γ_λ lies in $\mathbb{R}$ and, in particular, the portion of Γ_λ that lies in L_λ lies in $\mathbb{R}^-$. As $\lambda \to 0$ for $\lambda \in \mathbb{R}^-$, $F_\lambda^2(c_\lambda) \to 1/4$, so $F_\lambda^2(c_\lambda)$ lies to the right of the portion of Γ_λ in $\mathbb{R}^-$. However, for λ near -1, $F_\lambda^2(c_\lambda) = 4\lambda + 1/4$ approaches -3.75 and so lies in $\mathbb{R}^-$ to the left of -2. Hence $F_\lambda^2(c_\lambda)$ lies to the left of the portion of Γ_λ in $\mathbb{R}^-$. Since the portion of Γ_λ in L_λ is trapped in the interval $[-2, 0]$, and since any given point $w_\lambda(s)$ moves continuously as a function of λ, it follows that there is at least one λ-value in $\mathbb{R}^-$ for which $F_\lambda^2(\lambda) = w_\lambda(s)$. By the uniqueness property of this parameter, it follows that this value must in fact be the (unique) λ_s-value found above. In particular, it follows that the parameter plane Cantor set Γ lies in $\mathbb{R}^-$ and has the same order along the real line as each Γ_λ. That is, if s_1 and s_2 lie in Σ_2', then both $w_\lambda(s_1)$ and $w_\lambda(s_2)$ lie in $\mathbb{R}^-$, and if $w_\lambda(s_1) < w_\lambda(s_2)$, then $\lambda_{s_1} < \lambda_{s_2}$ in Γ. Also, if s is a sequence in Σ_2' that ends in all 0's (so that $w_\lambda(s)$ is a point whose orbit eventually lands on p_λ), then λ_s is a parameter for which the critical orbit lies on the boundaries of preimages of B_λ. We call such values of λ endpoints for the Cantor set Γ. Now it is known by work of Roesch [12] that the set of parameters for which the critical orbit lies in a given preimage of B_λ forms a simply connected open subset of parameter plane. Hence if we append any such open set to the corresponding endpoints of Γ just as we did in the dynamical plane construction, we see that this new set has the structure of a Cantor necklace. We have shown:

Theorem. *There is a Cantor necklace $\mathcal{C}$ in the parameter plane whose Cantor set portion lies along the negative real axis and has the following properties:*

1. *Any point in the Cantor set portion Γ of $\mathcal{C}$ is a parameter value λ_s for which $F^2_{\lambda_s}(c_{\lambda_s})$ has itinerary s in Γ_{λ_s};*

2. *Any point in the open regions of $\mathcal{C}$ are λ-values for which the critical orbit tends to ∞, so $J(F_\lambda)$ is a Sierpinski curve and F_λ is structurally stable;*

3. *Any endpoint λ_s of Γ in $\mathcal{C}$ is a parameter value for which the critical orbit eventually lands at the fixed point p_{λ_s} on the boundary of B_{λ_s}.*

We display in black in Figure 7 the set of λ values for which the orbit of the critical point remains bounded under F_λ. The white holes in this picture are the "Sierpinski holes" where the critical orbit eventually falls through the trap door and so the Julia set is a Sierpinski curve. In Figure 8, we display several magnifications of the "tail" of the parameter plane that shows the Cantor necklace $\mathcal{C}$ running down the negative real axis.

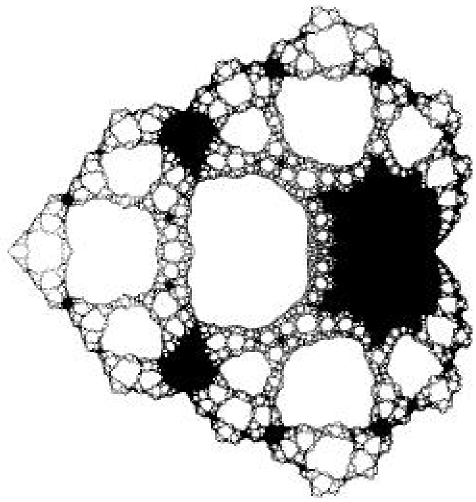


Figure 7: The parameter plane for the family $z^2 + \lambda/z^2$. White regions correspond to λ-values for which the critical orbit escapes to ∞.

4 Sierpinski Curve Julia Sets

In this section we prove the theorem concerning structurally unstable Sierpinski curve Julia sets. We assume throughout that $\lambda = \lambda_s$ is one of the negative real parameter values described in the previous section for which the critical orbit (eventually) lands on a repelling periodic point $w_\lambda(s)$ that is not equal to p_λ. In particular, the sequence s is periodic or eventually periodic but does not end in all 0's. Hence the (eventually) periodic point $w_\lambda(s)$ does not lie at an endpoint of the corresponding Cantor necklace in dynamical plane since these are points that eventually land on p_λ. Such a point is therefore said to be *buried* since it is inaccessible from any given preimage of the trap door lying along the real axis.

We first prove that the Julia set corresponding to such a λ is a Sierpinski

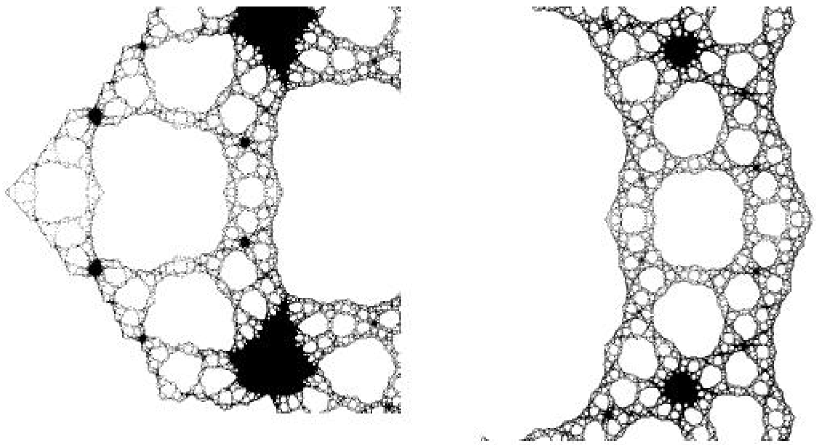


Figure 8: Two magnifications of the parameter plane for the family $z^2 + \lambda/z^2$ along the negative real axis. In the first image, $-0.4 \leq \operatorname{Re}\lambda \leq -0.06$ and, in the second, $-0.2 \leq \operatorname{Re}\lambda \leq -0.15$

curve. From the Theorem in Section 1, $J(F_\lambda)$ is compact and connected. Since $J(F_\lambda)$ is not the entire plane, standard facts from complex dynamics show that $J(F_\lambda)$ is nowhere dense. Also, since the orbits of the critical points are all preperiodic, it follows that $F_\lambda \mid J(F_\lambda)$ is subhyperbolic. Hence $J(F_\lambda)$ is a locally connected set. (See [9] for proofs of these facts.) Thus it suffices to prove that all of the complementary domains of the Julia set are bounded by simple closed curves and that no two of these curves meet one another.

We need a few lemmas. Recall that B_λ is the immediate basin of ∞ and C_λ is the critical circle.

Lemma. *The intersection of B_λ and C_λ is empty, as is the intersection of B_λ and the interval $(-p_\lambda, p_\lambda) \subset \mathbb{R}$.*

Proof: Suppose first that $z_0 \in B_\lambda \cap (-p_\lambda, p_\lambda)$. Since $\lambda \in \mathbb{R}^-$, the real line is preserved by F_λ and B_λ is symmetric under the reflection $z \mapsto \overline{z}$. Since the critical values do not lie in B_λ, B_λ is connected. Therefore $\pm p_\lambda$ lie in different components of $\mathbb{C} - B_\lambda$. Thus $J(F_\lambda)$ is disconnected. This gives a contradiction, so B_λ cannot meet $(-p_\lambda, p_\lambda)$.

If $z_0 \in B_\lambda \cap C_\lambda$, then $F_\lambda^2(z_0) \in B_\lambda \cap (-p_\lambda, p_\lambda)$ which is impossible by the previous observation. This proves the lemma.

□

Lemma. *The boundary of B_λ does not intersect the interval $(-p_\lambda, p_\lambda)$ nor does it meet C_λ.*

Proof: Suppose that $x_0 \in \partial B_\lambda \cap (-p_\lambda, p_\lambda)$. Let $\pm\nu_\lambda$ be the two preimages of 0 on the real axis. The points $\pm\nu_\lambda$ also lie on C_λ. The point x_0 cannot lie in the interval $(-\nu_\lambda, \nu_\lambda)$ since this interval lies inside C_λ, and so B_λ would then meet C_λ. Hence $x_0 \in (-p_\lambda, -\nu_\lambda]$ or $x_0 \in [\nu_\lambda, p_\lambda)$. Assume first that $x_0 \in [\nu_\lambda, p_\lambda)$. Then, since F_λ is decreasing on $\mathbb{R}^+$ and maps $[0, \nu_\lambda)$ to $[-\infty, 0)$, it follows that there exists a first integer k such that $F_\lambda^k(x_0) \in [0, \nu_\lambda)$. Since ∂B_λ is invariant, it follows that $F_\lambda^k(x_0) \in \partial B_\lambda$, so this again says that B_λ meets C_λ. This yields a contradiction. In the case where $x_0 \in (-p_\lambda, -\nu_\lambda]$, note that $F_\lambda(x_0) \in [0, p_\lambda)$, so the previous argument applies to show that $F_\lambda(x_0)$ does not lie in ∂B_λ. Consequently, x_0 does not lie there as well. The fact that ∂B_λ misses C_λ follows from forward invariance of ∂B_λ and the fact that C_λ maps into $(-p_\lambda, p_\lambda)$ after two iterations.

□

Consider the open set $\mathbb{C} - \overline{B}_\lambda$. The origin lies in this set, so let W_0 denote the component of $\mathbb{C} - \overline{B}_\lambda$ containing 0. We claim that W_0 is in fact the only component of $\mathbb{C} - \overline{B}_\lambda$.

Lemma. *If $z \in W_0$, then all four preimages of z lie in W_0.*

Proof: By the previous lemma, $(-p_\lambda, p_\lambda)$ does not meet ∂B_λ, and so this entire interval is contained in W_0. The preimage of this interval lying on the imaginary axis likewise does not meet ∂B_λ, and so this set lies in W_0 since it intersects $(-p_\lambda, p_\lambda)$ at 0. Consequently, all four prepoles lie in W_0, so that all four preimages of 0 lie in W_0.

Now let $\mathcal{U}$ be the set of points in W_0 for which all four preimages also lie in W_0. The set $\mathcal{U}$ is open and nonempty since $0 \in \mathcal{U}$. We claim that $\mathcal{U} = W_0$. If this is not the case, let $z_0 \in \partial\mathcal{U} \cap W_0$. Then z_0 lies in W_0 but at least one of the preimages of z_0 does not. This preimage must then lie on the boundary of W_0, but $\partial W_0 \subset \partial B_\lambda$. Since ∂B_λ is forward invariant, we must have $z_0 \in \partial B_\lambda$ as well. But this contradicts our assumption that $z_0 \in W_0$.

□

It follows immediately from this lemma that, if $z_0 \in \partial W_0$, then all four preimages of z_0 also lie in ∂W_0. This implies that $\partial W_0 = \partial B_\lambda$. To see this, note that if there were a point $z_1 \in \partial B_\lambda - \partial W_0$, then $z_1 \in J(F_\lambda)$. Let V be a neighborhood of z_1 that is disjoint from $\overline{W}_0$. By Montel's Theorem, the forward images of V must hit all (except possibly two) points in $\overline{\mathbb{C}}$, and so some points in V must map into W_0. But, by the previous Lemma, this cannot happen. Therefore, $\partial B_\lambda = \partial W_0$. Equivalently, we have:

Proposition. *The open and connected set W_0 is equal to $\mathbb{C} - \overline{B}_\lambda$.*

We now prove that ∂B_λ is a simple closed curve. As mentioned earlier, since ∞ is a superattracting fixed point of order two, there is a conjugacy

$$\phi_\lambda : B_\lambda \to \overline{\mathbb{C}} - \overline{\mathbb{D}}$$

between $F_\lambda \mid B_\lambda$ and $z \mapsto z^2$ in the complement of the unit disk in $\mathbb{C}$. Recall that the preimage of a straight ray given by $te^{i\theta}$ for fixed θ and $t > 1$ under ϕ_λ is called an external ray of angle θ for F_λ and is denoted by γ_θ. Since $J(F_\lambda)$ is locally connected, it is known that each external ray limits on a single point in ∂B_λ (see [9]). That is,

$$\lim_{t\to 1} \phi_\lambda^{-1}(te^{i\theta})$$

is a single point in ∂B_λ.

To prove that B_λ is a simple closed curve, it then suffices to show that no two external rays limit on the same point in ∂B_λ. Suppose that this happens. Say γ_θ and γ_ψ both limit on $z_0 \in \partial B_\lambda$. Then one of two things must occur. Either all external rays corresponding to angles in one of the arcs between θ and ψ must also limit on z_0. But this implies that in fact all external rays limit on z_0, which cannot happen. The only other possibility is that these two rays together with the point z_0 divide $\partial B_\lambda - \{z_0\}$ into two disjoint, nonempty sets, each of which contains pieces of $J(F_\lambda)$. But the proof of the previous proposition shows that this cannot occur. We conclude that no two rays limit on the same point and so ∂B_λ is a simple closed curve.

Finally, we need to show that all of the preimages of ∂B_λ are simple closed curves that are pairwise disjoint. By assumption, the forward orbits of the critical points do not land on p_λ and hence do not lie in any of the preimages of $\overline{B_\lambda}$. Therefore all of the preimages of ∂B_λ are mapped in one-to-one fashion onto ∂B_λ, and so each of these preimages is a simple closed curve. Also, by one of the lemmas above, ∂B_λ lies outside of C_λ. Hence ∂T_λ lies inside C_λ and so ∂B_λ and ∂T_λ are disjoint simple closed curves. If two different preimages of ∂B_λ intersect, then iterating these preimages forward shows that some preimage of ∂B_λ must intersect ∂B_λ. Since ∂B_λ is forward invariant, we may iterate forward again to show that in fact ∂T_λ and ∂B_λ must intersect. But, by the above, this does not happen. Therefore no two of these simple closed curves meet each other. We have proved:

Theorem. *Let $s \in \Sigma_2'$ be a (pre)-periodic sequence that does not end in all zeroes. Let λ_s be the unique parameter value for which $F_{\lambda_s}^2(c_{\lambda_s}) = w_{\lambda_s}(s)$, i.e., the second image of the critical point lands on the point in Γ_{λ_s} that has itinerary s. Then the Julia set of F_{λ_s} is a Sierpinski curve.*

5 Structural Instability

We now turn attention to values of the parameter that lie in a neighborhood of λ_s, where $\lambda_s \in \mathbb{R}^-$ is one of the parameter values described in the previous two sections for which the critical orbit lands on a repelling periodic point not equal to p_λ. Our goal is to show that there are infinitely many dynamically different maps in any neighborhood of λ_s.

Proposition. *Let $\mathcal{U}$ be a neighborhood of λ_s. There exist infinitely many parameter values $\mu_j \in \mathcal{U}$ such that $J(F_{\mu_j})$ is a structurally stable Sierpinski curve Julia set, but F_{μ_j} is not conjugate to F_{μ_k} for $j \neq k$.*

Proof: In any neighborhood of λ_s in the parameter plane, there are infinitely many intervals in $\mathbb{R}^-$ that contain parameters for which the critical orbit eventually escapes into B_λ. These correspond to Sierpinski holes in the parameter plane and it is known [3] that any two maps drawn from the same hole are conjugate on their Julia sets. Moreover, since λ_s is a buried point in the Cantor set in parameter plane, we may choose an infinite subset of these intervals for which the number of iterations that it takes for the critical orbit to enter B_λ is different in each hole. As shown in [4], this implies that functions drawn from distinct holes are not conjugate on their Julia sets. This proves the existence of infinitely many values of the parameter μ_j for which $J(F_{\mu_j})$ is a structurally stable Sierpinski curve, but F_{μ_j} is not conjugate to F_{μ_k} for $j \neq k$.

$\square$

Proposition. *Let $\mathcal{U}$ be a neighborhood of λ_s. There exist infinitely many parameter values $\nu_j \in \mathcal{U}$ such that $J(F_{\nu_j})$ is a Sierpinski curve Julia set which is not structurally stable and again, F_{ν_j} is not conjugate to F_{ν_k} when $j \neq k$ nor is F_{ν_j} conjugate to any of the F_{μ_ℓ}.*

Proof: Since λ_s is a buried point in the Cantor set in the parameter plane, we may find in any neighborhood of λ_s infinitely many buried preperiodic points (of different periods) in Γ_λ. These then yield additional parameters for which the Julia set is a structurally unstable Sierpinski curve. As above, no two of these maps are conjugate if the periods of these periodic points are different.

$\square$

Proposition. *Let $\mathcal{U}$ be a neighborhood of λ_s. There exist infinitely many parameter values $\tau_j \in \mathcal{U}$ such that $J(F_{\tau_j})$ is a hybrid Sierpinski curve, and again, F_{τ_j} is not conjugate to F_{τ_k} for $j \neq k$, nor to any of the F_{μ_ℓ} or F_{ν_ℓ} above.*

Proof: There are infinitely many endpoints of Γ_{λ_s} in any neighborhood of w_{λ_s}. Hence there are infinitely many parameters λ_{τ_j} for which the critical orbits land on the fixed point p_λ that lies on the boundary of the basin at ∞. We claim that the Julia set associated to such a parameter is a hybrid Sierpinski curve. To see this, first note that the proof in the previous section that ∂B_λ is a simple closed curve goes through without change. Suppose that the n^{th} iterate of the critical points land on p_λ where $n \geq 3$. Then the i^{th} preimage of ∂B_λ is a collection of disjoint simple closed curves for each $i < n$. By four-fold symmetry, among these simple closed curves, only the boundary of the trap door surrounds the origin. However, the particular n^{th} preimages that contain a critical point now consist of a pair of simple closed curves that meet at the critical point. We cannot have more than two such curves meeting, for by fourfold symmetry, a chain of more than two such curves would necessarily contain all of the critical points. But then the image of this set of curves would be a simple closed curve that contains both critical values and hence, by symmetry, surrounds the origin. But this cannot happen, as we observed above. Hence there are exactly 4 such figure eight curves in the n^{th} preimage of ∂B_λ, one corresponding to each critical point of F_λ. The remaining preimages are necessarily simple closed curves. Now continue pulling back either these figure eight or simple closed curve preimages of ∂B_λ. We obtain infinitely many distinct preimages that are figure eights as well as infinitely many that are simple closed curves. This shows that the Julia set is a hybrid Sierpinski curve.

6 Concluding Remarks

In this paper we have concentrated on the family

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2},$$

but all of the results go over more or less intact for the higher degree families

$$G_\lambda(z) = z^n + \frac{\lambda}{z^n}$$

where $n \geq 3$. When n is even, there is an analogous Cantor necklace along the negative real axis in parameter plane, and all of the above results go through without change. When n is odd, one must choose a different straight line in parameter space dictated by symmetries present in the system. On this line one finds small copies of the Mandelbrot sets interspersed with Cantor necklaces, but the results about structurally unstable Sierpinski curve Julia sets remain the same. In Figure 9 we display the parameter planes for the cases $n = 3$ and $n = 4$.

For example, when $n = 3$, we consider parameters along the imaginary axis. Let ω be a primitive eighth root of unity. Then, for $\lambda \in \mathbb{R}$, the function $F_{i\lambda}$ interchanges the two lines $t\omega$ and $t\omega^3$. A computation shows that

$$F_{i\lambda}(t\omega) = \omega^3 \left(t^3 - \frac{\lambda}{t^3}\right)$$

and

$$F_{i\lambda}(t\omega^3) = \omega \left(t^3 + \frac{\lambda}{t^3}\right).$$

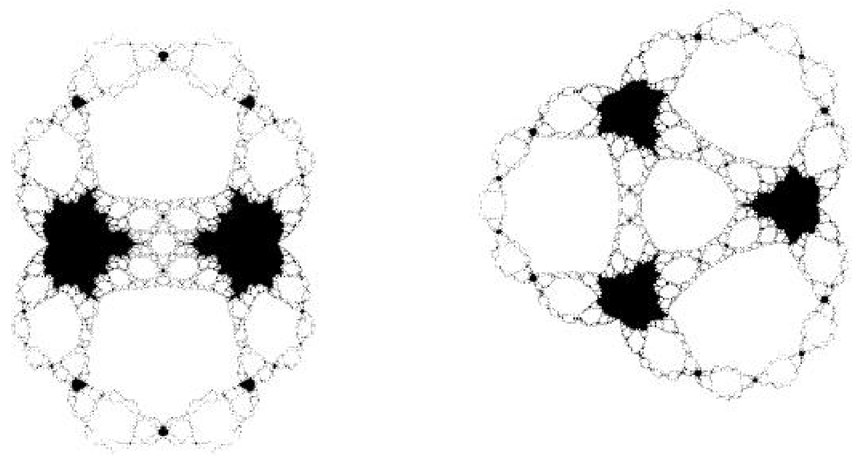


Figure 9: The parameter planes in the cases $n = 3$ and $n = 4$.

Let
$$g_\lambda^{\pm}(t) = t^3 \pm \frac{\lambda}{t^3}.$$
Then the second iterate of $F_{i\lambda}$ on the line $t\omega^3$ is conjugate to the function $g_\lambda^-(g_\lambda^+(t))$ on the real line. The graph of this function on the real line shows that critical points become entangled in a Cantor set as the parameter λ is varied. See Figure 10. Then the same arguments as above produce λ-values surrounded by infinitely many Sierpinski holes centered on the imaginary axis, although the complete structure here is more intricate than a Cantor necklace. We leave the straightforward details to the reader.

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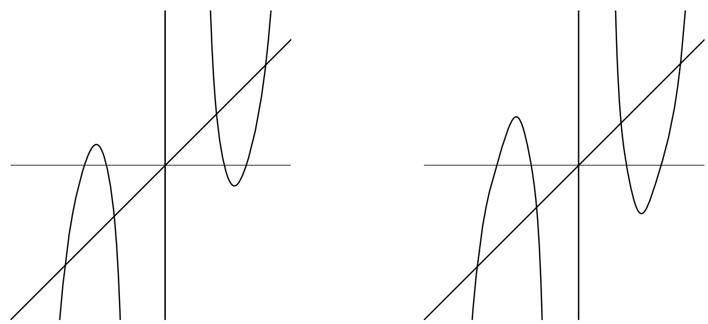

Figure 10: The graphs of $g_\lambda^-(g_\lambda^+(t))$ for $\lambda = 0.05$ and $\lambda = 0.09$. Note that each of these functions has an invariant Cantor set on either side of the origin and that the critical points on opposite sides of 0 map into this Cantor set regime as λ varies.

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