TOPOLOGICAL BIFURCATIONS

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ABSTRACT. In complex dynamics, the important object of study is the Julia set of a given holomorphic function. This set contains all of the points where the map is chaotic. As parameters change, the Julia set sometimes undergoes rather remarkable changes in topology. We call these changes topological bifurcations. In this paper we describe a number of different topological bifurcations, all of which occur in the family

$$F_{\alpha,\beta}(z) = \frac{1}{\alpha + \beta e^{-z}}.$$

We present bifurcations in which the Julia sets of this family is transformed abruptly

(1) From a Cantor bouquet to a simple closed curve in the Riemann sphere;
(2) From a Cantor bouquet to a Cantor set;
(3) From a Cantor bouquet to the whole Riemann sphere, including the appearance of infinitely many indecomposable continua.

1. INTRODUCTION

Our goal in this paper is to describe the topology of the Julia sets of the family of meromorphic functions given by $F_{\alpha,\beta} : \mathbb{C} \to \mathbb{C}$ where

$$F_{\alpha,\beta}(z) = \frac{1}{\alpha + \beta e^{-z}}.$$

with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. As we shall show, this set undergoes some remarkable bifurcations as the parameters α and β are varied. In this section we give some preliminary information about the dynamics and topology of this type of meromorphic function.

One important feature of this family of complex analytic maps is that it has constant Schwarzian derivative. The *Schwarzian derivative* of a function F is defined by

$$SF(z) = \frac{F'''(z)}{F'(z)} - \frac{3}{2}\left(\frac{F''(z)}{F'(z)}\right)^2.$$

A computation shows that, in our case, $SF_{\alpha,\beta}(z) \equiv -1/2$. Schwarzian derivatives play a role in determining the "curvature" of complex functions. They also are important in the study of real one-dimensional maps, where the assumption that the functions have negative Schwarzian derivative has implications for the number of attracting cycles present in such a map.

Functions with constant Schwarzian derivative have the following special properties. First, there are no critical points for these types of functions. Secondly, these functions have a pair of *asymptotic values* v_1 and v_2. In such a family, this means

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that there are a pair of disjoint half planes H_j extending to ∞ which are wrapped infinitely often by the function around a disk punctured at v_j. More precisely, there is a neighborhood U_j of each v_j for which the preimage is an open, simply connected set extending to ∞, containing a half plane, and on which $F\colon F^{-1}(U_j) \to U_j - \{v_j\}$ is a universal covering map. More generally, a function whose Schwarzian derivative is a polynomial of degree n has no critical points and $n+2$ asymptotic values. In this case, a neighborhood of ∞ is broken up into $n+2$ sectors of angle $2\pi/(n+2)$ on each of which the map acts like a universal cover of a punctured disk. For more details, see [12], [13].

For $F_{\alpha,\beta}$, the asymptotic values are 0 and $1/\alpha$. $F_{\alpha,\beta}$ takes the far left half plane and wraps it infinitely often around a disk punctured at the origin. Similarly $F_{\alpha,\beta}$ maps the right half plane infinitely often around a disk punctured at $1/\alpha$. When $\alpha = 0$, the asymptotic value is at ∞ and our family degenerates to the entire function $E_\beta(z) = (1/\beta)e^z$.

When $\alpha \neq 0$, $F_{\alpha,\beta}$ has infinitely many poles. These are given by points of the form $\log(-\beta/\alpha)$ (recall that we always assume that $\beta \neq 0$).

For a complex analytic function F, the most important object is the *Julia set* of F. The Julia set $J(F)$ is defined by any of the following equivalent conditions:

(1) $J(F)$ is the set of points z at which the family of iterates of F fails to be a normal family in the sense of Montel in any neighborhood of z.
(2) $J(F)$ is the closure of the set of repelling periodic points of F.
(3) $J(F)$ is the closure of the set of prepoles of F. By definition, a prepole is a point whose forward orbit lands on a pole and hence is mapped to ∞ by some iterate of F.
(4) $J(F)$ is the chaotic regime for F (using just about any definition of chaos in the literature.)

The topology of the Julia set and the dynamics of F on $J(F)$ is largely determined by the fate of the orbits of the asymptotic values. The following results are well-known. See [3] and [13].

Theorem 1.1. (1) *If F has an attracting cycle, then this cycle must attract at least one of the asymptotic values. In this case, $J(F)$ is a nowhere dense subset of the plane.*
(2) *If both asymptotic values are prepoles, then $J(F) = \mathbb{C}$.*
(3) *If F is an entire function and the finite asymptotic value tends to ∞ under iteration, then $J(F) = \mathbb{C}$.*

2. An Exploding Cantor Bouquet

In this section we assume that $\alpha = 0$ so that our family becomes the family of entire functions given by $F_{0,\beta}(z) = E_\beta(z) = (1/\beta)e^z$. The family E_β undergoes a remarkable bifurcation as β passes through the parameter value e. The following is known; see [14].

Theorem 2.1. *If $\beta \geq e$, then $J(E_\beta)$ is a Cantor bouquet. If $0 < \beta < e$, then $J(E_\beta) = \mathbb{C}$.*

Roughly speaking, a *Cantor bouquet* consists of uncountably many disjoint curves with endpoints. Each of these curves tends to ∞ in a certain direction. To be more precise, using a definition introduced by Aarts and Oversteegen (see [1]), a Cantor bouquet is a subset of $\mathbb{C}$ which is homeomorphic to a *straight brush*. To define a straight brush, let $\mathcal{B}$ be a subset of $[0,\infty)\times(\mathbb{R}-\mathbb{Q})$ having the following properties:

(1) Hairiness. For each $(t,s)\in\mathcal{B}$, there is a $t_s\in[0,\infty)$ such that $\{t\,|\,(t,s)\in\mathcal{B}\}=[t_s,\infty)$. The point $e_s=(t_s,s)$ is called the endpoint of the hair h_s defined by $h_s=[t_s,\infty)\times\{s\}$.

(2) Density. The set $\{\alpha\,|\,(y,\alpha)\in\mathcal{B}$ for some $y\in[0,\infty)\}$ is dense in $\mathbb{R}-\mathbb{Q}$. Moreover, each endpoint of a hair is the limit from above and from below of other endpoints of hairs.

(3) Closed. $\mathcal{B}$ is closed in $\mathbb{R}^2$.

It is relatively easy to see that these three properties imply the following property of $\mathcal{B}$:

4. Endpoint density. If $(t,s)\in\mathcal{B}$, then there are sequences $\{x_i\}\to s$ and $\{y_i\}\to s$ with $x_i,y_i\in\mathbb{R}-\mathbb{Q}$ and $x_i<s$, $y_i>s$ for all i and
$$(t_{x_i},x_i)\to(t,s)\quad\text{and}\quad(t_{y_i},y_i)\to(t,s).$$
That is, each point in $\mathcal{B}$ is an accumulation point of endpoints from above and below.

The proof of the following may be found in [1] or [14].

Theorem 2.2. *If the function λe^z has an attracting fixed point, then the Julia set is a Cantor bouquet.*

In particular, in our case, $J(E_\beta)$ is a Cantor bouquet if $\beta\ge e$. Indeed, when $\beta>e$, the graph of E_β shows that E_β has an attracting fixed point on the real line. See Figure 1. When $\beta=e$, E_β has a neutral fixed point at $z=1$. When $0<\beta<e$, the orbit of the asymptotic value at 0 now tends to ∞. Hence $J(E_\beta)=\mathbb{C}$ for these values of β; we say that the Julia set of E_β explodes as β decreases through e.

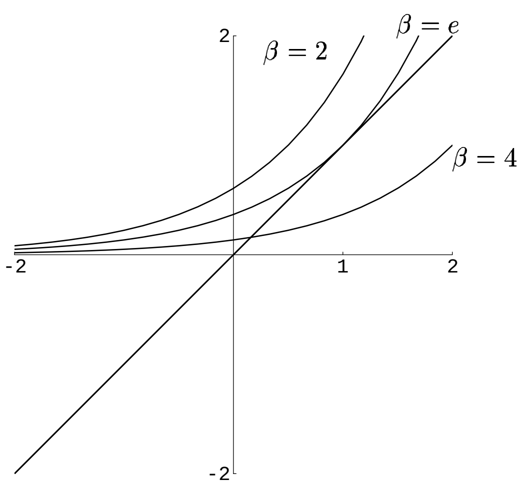

FIGURE 1. The graphs of E_β for $\beta=2,e$, and 4.

Before discussing this explosion, we give a brief indication of why $J(E_\beta)$ is a Cantor bouquet when $\beta=e$. Consider the half plane $H=\{z\,|\,\mathrm{Re}\,z\le 1\}$. E_e maps H onto the disk of radius 1 centered at the origin (minus the origin). E_e has a neutral fixed point at 1 and E_e contracts H inside itself (except at the boundary point $x = 1$), and so all points in H have orbits that tend to the neutral fixed point. Hence $J(E_e)$ lies in the right half plane $\mathrm{Re}\, z \geq 1$.

To compute the Julia set, we determine its complement, which is known to be the basin of attraction $\mathcal{B}$ of the fixed point at 1. The lines $y = (2k+1)\pi i$ are all mapped into the left half plane by E_e; hence these lines also lie in the basin of $x = 1$. We may remove open neighborhoods of these lines which also lie in $\mathcal{B}$. Note that these neighborhoods extend to ∞ to the right. We are left with infinitely many "fingers" C_k as displayed in Figure 2. Each of these fingers is mapped in one-to-one

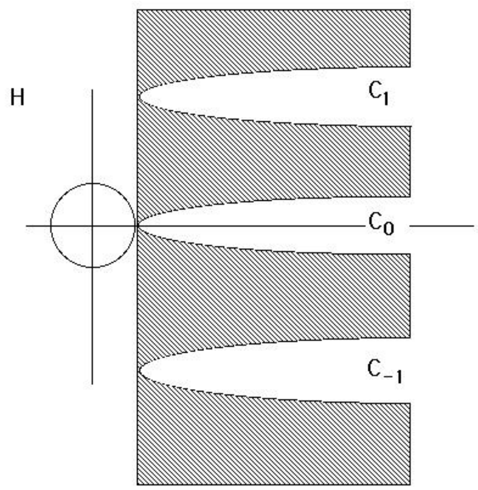

FIGURE 2. The preimage of H consists of H and the shaded region.

fashion onto $\mathrm{Re}\, z \geq 1$. Hence we may also remove all points in each C_k that map into the removed neighborhoods of the lines $y = (2k+1)\pi i$. Continuing in this fashion, we remove at each stage infinitely many narrower "strips" extending to ∞ from each of the C_k. In the limit, we end up with infinitely many curves lying in $J(E_e)$, each of which has an endpoint. These curves are also called "hairs."

For example, the straight line $[1, \infty)$ lies in $J(E_e)$. 1 is the neutral fixed point, while if $x > 1$, we have $E_e^n(x) \to \infty$. This is typical; it is known that, on each curve in $J(E_e)$, all non-endpoints have orbits that tend to ∞. Hence the bounded orbits lie in the set of endpoints. But the repelling periodic points are dense in $J(E_e)$ and these have bounded orbits; hence the endpoints must also be dense in $J(E_e)$, as required in the definition of a straight brush.

This leads us to some of the amazing topological and geometric properties of Cantor bouquets. It is known that:

(1) In the Riemann sphere, consider the set S consisting of all endpoints of $J(E_e)$ together with the point at ∞. Mayer [18] has shown that S is a connected set. However, if we remove just one point from S, namely the point at ∞, then Mayer also shows that $S - \{\infty\}$ is not just disconnected but totally disconnected.
(2) $J(E_e)$ has Hausdorff dimension 2 but Lebesgue measure 0 [19].

(3) The large subset of $J(E_e)$ consisting of the non-endpoints in $J(E_e)$ has Hausdorff dimension 1, but the relatively small subset of endpoints S has much larger Hausdorff dimension, namely 2. See [16], [17].

(4) The basin of attraction $\mathcal{B}$ of the neutral fixed point is simply connected and hence may be uniformized by the Riemann Mapping Theorem. The boundary of $\mathcal{B}$, namely $J(E_e)$, is nowhere locally connected (except at ∞). Nonetheless, all radial limits of the uniformizing map do exist and equal either ∞ or an endpoint of the bouquet.

We remark that the Julia set $J(E_\beta)$ is a Cantor bouquet for all $\beta \geq e$, not just when $\beta = e$. The proof is essentially the same.

To return to the bifurcation at $\beta = e$, we observe that no new periodic points are born at this bifurcation point. Locally, two fixed points coalesce on the real axis as β decreases through e, but then they reappear in the plane with $\operatorname{Im} z \neq 0$. In the plane, all other periodic points move continuously with β as a consequence of the Implicit Function Theorem. When $\beta \geq e$, they all lie in $\operatorname{Re} z \geq 1$. As soon as $\beta < e$, however, they are dense in $\mathbb{C}$. While each repelling periodic point moves continuously, the set of all such points moves discontinuously and abruptly fills the whole plane.

3. How the Julia Set Explodes

When $0 < \beta < e$, $J(E_\beta)$ is the entire complex plane. It is known that many of the hairs in $J(E_\beta)$ persist as β decreases through e. Other hairs, however, explode and become much more complicated from a topological point of view.

For example, consider the set of points Γ whose entire orbit lies in the intersection of the Julia set with the strip S given by the set $0 \leq \operatorname{Im} z \leq \pi$. When $\beta \geq e$, this set is the hair on the real axis $[1, \infty)$; all other points have orbits that either leave S or else tend to the fixed point in $\mathbb{R}$. But when $\beta < e$, suddenly all points on the real axis escape to ∞. So $\mathbb{R} \subset J(E_\beta)$. Much more happens, however. The line $y = \pi$ is mapped to the real axis, so these points also lie in Γ. This line has a preimage in S consisting of a curve whose real part extends to $+\infty$ in both directions. This curve bounds the region L_1 in S consisting of points whose first image lies above $\operatorname{Im} z = \pi$. The preimage of this curve is a second curve bounding the set of points L_2 whose image lies in L_1. And there is a third curve bounding L_3, the region which is mapped above S by E_β^3. Continuing, we find infinitely many regions L_n consisting of points whose orbit leaves S at the n^{th} iteration. The boundaries of these regions are curves that are eventually mapped to $\mathbb{R}$ and hence consist of points whose orbits tend to ∞. Hence we see that there are infinitely many curves lying in Γ. See Figure 3.

There are many more points in Γ, however. Indeed, Γ is homeomorphic to an indecomposable continuum with countably many points removed. To see this, we compactify the collection of curves by first scaling them horizontally so that the set lies in, say, $-1 \leq \operatorname{Re} z \leq 1$. Then we add the endpoints of each of the above curves in $\operatorname{Re} z = \pm 1$. Finally, we identify adjacent points on these curves in $\operatorname{Re} z = \pm 1$. More precisely, we first identify the points at $-\infty$ in $\mathbb{R}$ and in $y = \pi$. Then we identify the point at ∞ on $y = \pi$ with the upper endpoint of the boundary of L_1. Then we identify the lower boundary of L_1 with the upper boundary of L_2. Continuing, we make countably many such identifications. See Figure 4. We end up with a continuous curve in the plane that accumulates everywhere on itself without

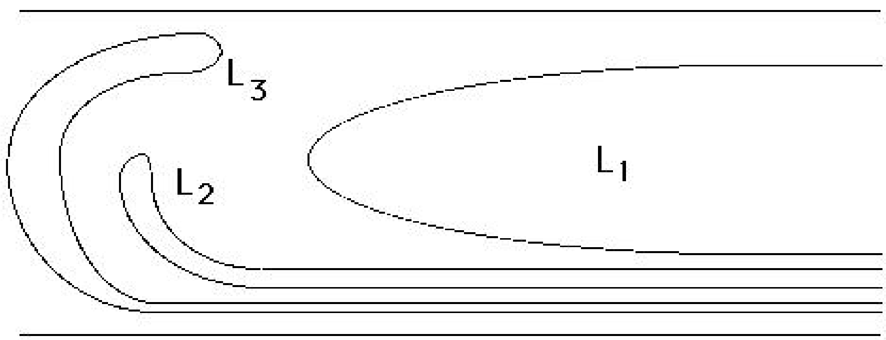

FIGURE 3. Construction of the L_n.

separating the plane. The reason that this curve does not separate the plane is as follows. If there were an open set U contained in S that was not in one of the L_j's, then the entire orbit of U would lie in S. But by Montel's Theorem, the union of the images of U is $\mathbb{C} - \{0\}$. This gives a contradiction.

By a theorem of Curry [6], the closure of this set is an indecomposable continuum. Hence Γ is a homeomorphic copy of this set with the backward orbit of 0 removed. Thus, when $\beta \geq e$, Γ is a simple hair. But when $\beta < e$, Γ explodes to be an indecomposable continuum. It is known that there are many other hairs that undergo such transformations. See [11].

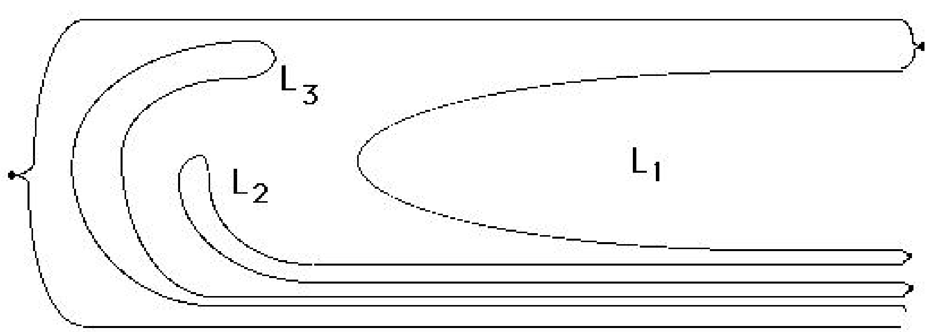

FIGURE 4. Embedding Γ in the plane.

4. ACCUMULATING ON THE CANTOR BOUQUET

Now fix $\beta > e$ and let α vary. When $\alpha = 0$ we know that $J(F_{0,\beta})$ is a Cantor bouquet. When $\alpha \neq 0$, this changes dramatically. We have:

Theorem 4.1. *Fix $\beta > e$. There is an interval I_β containing $\alpha = 0$ such that*

(1) *If $\alpha \in I_\beta$ and $\alpha < 0$, then $J(F_{\alpha,\beta})$ is a Cantor set in $\overline{\mathbb{C}}$.*

(2) *If* $\alpha \in I_\beta$ *and* $\alpha > 0$, $J(F_{\alpha,\beta})$ *is a simple closed curve passing through the point at* ∞ *in* $\overline{\mathbb{C}}$.

Here is a sketch of the proof when $\alpha < 0$. Consider a neighborhood U of the attracting fixed point p in $\mathbb{R}$. We may assume that U contains the interval $[0, p]$ since this interval lies in the basin of p. We construct U so that $F_{\alpha,\beta}(U) \subset U$. The graph of $F_{\alpha,\beta}$ (see Figure 5) shows that $F_{\alpha,\beta}(\mathbb{R}^-) \subset U$ so that the basin of p contains infinitely many open sets extending from $p + 2k\pi i$ to ∞ in the left half

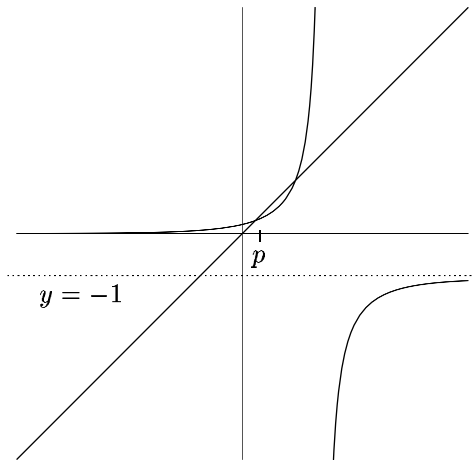

FIGURE 5. The graphs of $F_{\alpha,\beta}$ for $\alpha = -1$ and $\beta = 6$.

plane. Now the far left half plane is mapped to a small disk about 0, so this half plane lies in the basin of p as well. Also, the horizontal lines $x + (2k+1)\pi i$ are mapped to $\mathbb{R}^-$ so these lines together with open strips about them are also in the basin. Finally, the far right half plane is mapped to a disk about the asymptotic value $1/\alpha$ which lies in $\mathbb{R}^-$. So this half plane also lies in the basin. Let $\mathcal{B}$ be the union of these pieces of the basin. So $F_{\alpha,\beta}(\mathcal{B}) \subset \mathcal{B}$. Thus the complement of $\mathcal{B}$ consists of infinitely many closed, simply connected regions V_j, one for each $j \in \mathbb{Z}$. Each V_j meets the line $y = 2j\pi i$ in the right half plane.

Since $F_{\alpha,\beta}(\mathcal{B}) \subset \mathcal{B}$, we have that $F_{\alpha,\beta}$ maps each V_j in one-to-one fashion onto $\overline{\mathbb{C}} - \mathcal{B}$, so $F_{\alpha,\beta}(V_j) \supset V_k$ for each k. Also, each V_j contains a pole, so $F(V_j) \supset \{\infty\}$. Thus we may use symbolic dynamics to associate to each $z \in J(F_{\alpha,\beta})$ an itinerary of one of the following two forms:

$$S(z) = s_0 s_1 s_2 \ldots \quad \text{or} \quad S(z) = s_0 s_1 s_2 \ldots s_{n-1}\infty.$$

Here each $s_j \in \mathbb{Z}$ and $s_j = k$ if and only if $F^j_{\alpha,\beta}(z) \in V_k$. We associate the finite sequence $s_0 s_1 s_2 \ldots s_{n-1}\infty$ to z if $F^n_{\alpha,\beta}(z) = \infty$. That is, $F^{n-1}_{\alpha,\beta}(z)$ is a pole.

Let Σ denote the set of all possible such itineraries. We topologize Σ by taking the usual neighborhood basis about an infinite itinerary. If $s = s_0 s_1 \ldots s_{n-1}\infty$, a neighborhood basis of s consists of all (finite or infinite) sequences $s_0 s_1 \ldots s_{n-1}\tau t_{n+1}\ldots$ where $|\tau| \geq K$ for some $K \in \mathbb{Z}^+$. In this topology, Σ is homeomorphic to a Cantor set. One shows easily (using the Poincaré metric on V_j) that $F_{\alpha,\beta} \mid J(F_{\alpha,\beta})$ is conjugate to the shift map on Σ, and, in particular, $J(F_{\alpha,\beta})$ is a Cantor set in $\overline{\mathbb{C}}$. For more details, see [4], [12].

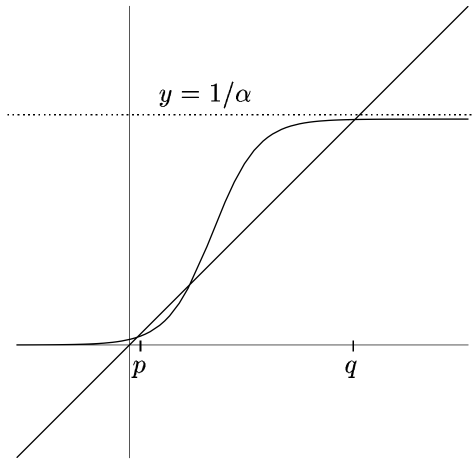

FIGURE 6. The graphs of $F_{\alpha,\beta}$ for $\alpha = 0.1$ and $\beta = 4$.

How does this Cantor set approach the Cantor bouquet as $\alpha \to 0$? Well, in the obvious way. One may attach an itinerary to each hair in the Cantor bouquet. Then the points with the corresponding itinerary in $J(F_{\alpha,\beta})$ tend to the endpoint of the hair with this itinerary as $\alpha \to 0$. The prepoles in $J(F_{\alpha,\beta})$ tend to ∞ as $\alpha \to 0$, as do certain other points whose itineraries do not exist in the symbolic dynamics for the entire map.

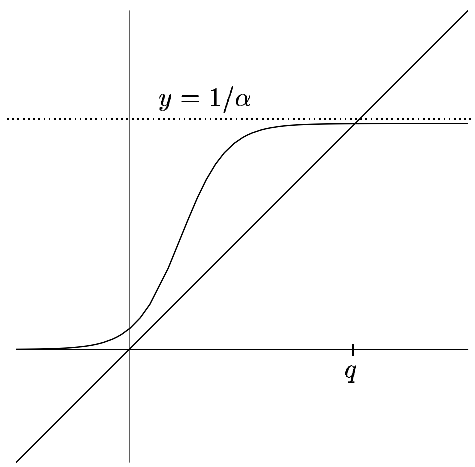

FIGURE 7. The graphs of $F_{\alpha,\beta}$ for $\alpha = 0.1$ and $\beta = 1$.

When $\alpha > 0$ a different picture emerges. The graph of $F_{\alpha,\beta}$ shows that a new attracting fixed point $q = q(\alpha)$ bifurcates away from ∞ on the positive real axis, yielding two attracting fixed points on $\mathbb{R}$. See Figure 6. The asymptotic value at $1/\alpha$ now tends to this fixed point. As before, the far left half plane now lies in the basin of p whereas the far right half plane now lies in the basin of q. Using symbolic dynamics, one can show that these two basins are separated by a simple closed curve passing through both ∞ and the repelling fixed point in $\mathbb{R}$. This simple curve becomes "wilder and wilder" and accumulates on the Cantor bouquet as $\alpha \to \infty$. See [4] for more details.

5. Accumulation on the Complex Plane

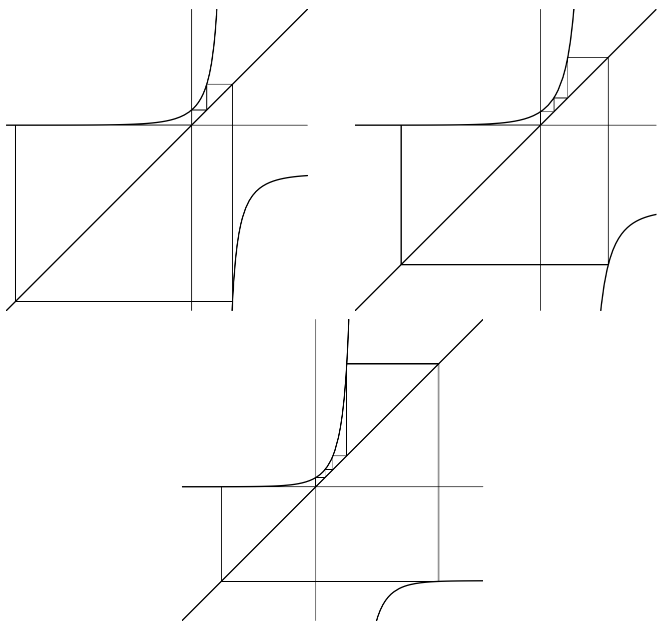

FIGURE 8. The graphs of various $F_{\alpha,\beta}$ with $\alpha < 0$ and $0 < \beta < e$ showing an attracting n-cycle for $n = 4, 5, 6$.

Now fix β with $0 < \beta < e$. We know that $J(F_{0,\beta}) = \mathbb{C}$ in this case. We address here how the Julia set approaches $\mathbb{C}$ as $\alpha \to 0$.

When $\alpha > 0$ we have a single attracting fixed point in $\mathbb{R}$ and the orbits of both asymptotic values tend to this point. The entire real axis lies in the basin of this fixed point. So do the lines $y = 2k\pi$ for each integer k. And so do the far right and left half planes. Call this portion of the basin $\mathcal{B}$ as before. Hence we find infinitely many closed disks V_j which are mapped onto $\overline{\mathbb{C}} - F_{\alpha,\beta}(\mathcal{B})$ as in the previous section, and so we see that $J(F_{\alpha,\beta})$ is again a Cantor set. This time the Cantor set spreads out to cover all of $\mathbb{C}$ as $\alpha \to 0$.

The case when $\alpha < 0$ is now quite a bit more complicated. The point $(0, \beta)$ with $0 < \beta < e$ is the accumulation point of infinitely many parameter values (α, β) for which

(1) $F_{\alpha,\beta}$ has an attracting cycle of arbitrary large period which attracts both asymptotic values.
(2) Both asymptotic values lie on a prepoles, so $J(F_{\alpha,\beta}) = \mathbb{C}$.

Several graphs showing attracting cycles are displayed in Figure 8. The full picture of the parameter plane in this case is not well understood, though it appears to be similar in many respects to the bifurcation plane for the family λe^z near the positive real axis. See [9].

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