Open Problems in Complex Dynamics and “Complex” Topology

Robert L. Devaney *
Boston University

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*Please address all correspondence to Robert L. Devaney, Department of Mathematics, Boston University, 111 Cummington Street, Boston MA 02215, or e-mail bob@bu.edu.

Complex dynamics is a field in which a large number of captivating structures from planar topology occur quite naturally. Of primary interest in complex dynamics is the Julia set of a complex analytic function. As we discuss below, these are the sets that often are quite interesting from a topological point of view. For example, we shall describe examples of functions whose Julia sets (or invariant subsets of the Julia sets) are Cantor bouquets, indecomposable continua, and Sierpinski curves. Because both the topology of and the dynamics on these Julia sets is so rich, it is little wonder that there are many open problems in this field. Our goal in this paper is to describe several of these problems. To keep the exposition accessible, we shall restrict attention to two very special families of functions, namely the complex exponential function and a particular family of rational maps. However, the problems and topological structures encountered in these families occur for many other types of complex analytic maps.

1 Cantor Bouquets and Indecomposable Continua

In this section we consider the dynamics of the complex exponential family $E_\lambda(z) = \lambda e^z$ where, for simplicity, λ is for the most part chosen to be real and positive. The Julia set for such an entire transcendental map has several equivalent definitions. For example, the Julia set may be defined as the closure of the set of points whose orbits escape to ∞ under iteration of E_λ. (Note that this is different from the definition of polynomial Julia sets, where it is the boundary and not the closure of the set of escaping points that forms the Julia set.) Equivalently, the Julia set is also the closure of the set of repelling periodic points. These two defintions show that the Julia set of E_λ is home to chaotic behavior: arbitrarily close to any point in the Julia set are points whose orbits tend off to ∞ as well as other points whose orbits are not only bounded, but in fact periodic. So the map depends quite sensitively on initial conditions near any point in the Julia set. In fact, much more can be said since the Julia set may also be defined as the set of points at which the family of iterates of E_λ fails to be a normal family. By Montel's Theorem, it then follows that, for any neighborhood U of a point in the Julia set, the union of the sets $E_\lambda^n(U)$ covers all of $\mathbb{C} - \{0\}$. So arbitraily close to any point in the Julia set are points whose orbits visit any region whatsoever in $\mathbb{C}$. We denote the Julia set of a function F by $J(F)$.

The complement of the Julia set is called the Fatou set. Here the situation is quite different: the dynamics on the Fatou set is essentially completely understood. For example, all points in the basin of attraction af an attracting cycle clearly lie in the Fatou set: the orbits of all nearby points to a point in such a basin behave similarly. No nearby orbits tend to ∞ and none lie on repelling periodic cycles. There are a few other possible types of behavior in the Fatou set, but none of these behaviors involve anything chaotic.

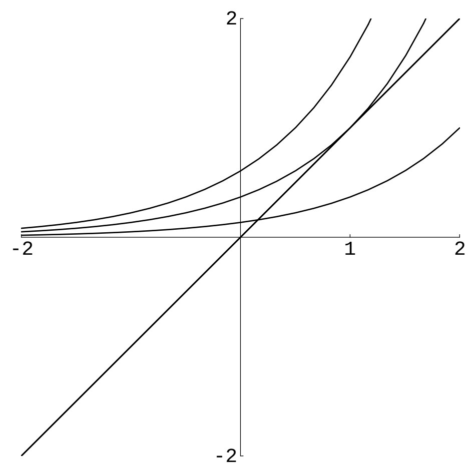

Figure 1: The graphs of E_λ for several λ-values.

The graphs of E_λ on the real line (see Figure 1) show that there are two different types of dynamical behavior depending upon whether $\lambda < 1/e$ or $\lambda > 1/e$. When $\lambda < 1/e$, there are two fixed points on the real line, an attracting fixed point at $q = q_\lambda$ and a repelling fixed point at $p = p_\lambda$. All orbits to the right of p tend to ∞, so these points are in the Julia set, as is p. All points to the left of p are in the basin of attraction of q, so these points are not in the Julia set. In fact, let x be any point in $\mathbb{R}$ with $q < x < p$. Then one checks easily that the entire half plane $H_x = \operatorname{Re} z < x$ is wrapped infinitely often around a disk minus the origin, and this disk lies strictly inside the half plane H_x. By the Schwarz Lemma, all points in any of these half planes therefore have orbits that simply tend to q and hence lie in the Fatou set. So the Julia set must lie in the half-plane $\operatorname{Re} z \geq p$. This is essentially true when $\lambda = 1/e$, though now all orbits in the half-plane $\operatorname{Re} z < p$ now tend to the neutral fixed point at $p = q$.

To get a feeling for the structure of the Julia set when $\lambda \leq 1/e$, we paint the picture of its complement. Consider the preimage of H_x. This preimage must contain the lines $y = (2n+1)\pi$ for each $n \in \mathbb{Z}$, since these lines are mapped to the negative real axis. Hence there are open neighborhoods of each of these lines extending from H_x to ∞ in the right half plane and mapped onto H_x. This means that the Julia set is contained in infinitely many symmetrically located, simply connected, closed sets that extend to ∞ in the right half plane. Each of these sets is mapped one-to-one onto the entire half plane $\operatorname{Re} z \geq x$. As a consequence, there are points in each of these regions that map into each of the neighborhoods of the lines $y = (2n+1)\pi$ and hence these points are also in the Fatou set. So this breaks each of these complementary domains into infinitely many more sets, each of which extend off to ∞ to the right. And so the Julia set must lie in these regions. Continuing in this fashion, one can show that the Julia set is actually an uncountable collection of curves (called "hairs") that extend to ∞ in the right half plane, and each of these hairs has a distinguished endpoint [DK]. The set of all such hairs forms the Julia set and is an example of a *Cantor bouquet*. So each of these hairs consists of two subsets: the endpoint and the remainder of the hair that we call the stem. For example, one such hair is the half-line $[p, \infty) \subset \mathbb{R}$. The point p is the endpoint, which is fixed, and as we saw earlier, all points to the right of p simply tend to ∞. In general, it is known that, if a point lies on the stem, then, as in the case of (p, ∞), the orbit of this point necessarily tends to ∞ (though it usually jumps around between different hairs). Hence all of the bounded orbits must lie in the set of endpoints. But the repelling periodic points are bounded and hence they must lie in the set of endpoints. But this means that the set of endpoints is dense in this entire set, and so they accumulate on each point on any given stem.

Because of this, a Cantor bouquet has some very interesting topological properties. For example, Mayer [Ma] has shown that, in the Riemann sphere, the set of endpoints together with the point at ∞ forms a connected set, whereas the set consisting of just the endpoints (i.e., remove just one point from the previous set) is not just disconnected but totally disconnected. Moreover, Karpinska [Ka] has shown that the Hausdorff dimension of the set of stems is 1, whereas the Hausdorff dimension of the "much" smaller set of endpoints is actually 2.

When λ passes through $1/e$, E_λ undergoes a simple saddle node bifurcation in which the two fixed points q_λ and p_λ coalesce when $\lambda = 1/e$ and then reappear for $\lambda > 1/e$ above and below the real axis. Meanwhile, all points on the real axis now tend to ∞, so the entire real axis suddenly lies in the Julia set. But much more is happening in the complex plane.

The origin is what is known as an asymptotic value. It is the omitted value for E_λ. As such, it plays the same role as the critical values do in polynomial dynamics. In particular, via a result of Sullivan [Su], as extended to the entire case by Goldberg and Keen [GK], if the orbit of 0 tends to ∞, then the Julia set of E_λ must be the entire plane. Hence, when $\lambda \leq 1/e$, all of the repelling periodic points are constrained to lie in the half plane $\text{Re}\, z \geq p$, whereas these points become dense in $\mathbb{C}$ for any $\lambda > 1/e$. Now no new repelling cycles are born as λ passes through $1/e$; all of these cycles simply move continuously, but the set of them migrates from occupying a small portion of the right half plane to suddenly filling all of $\mathbb{C}$.

However, even more is happening in this bifurcation. For example, consider what happens to the hair $[p, \infty)$ as soon as λ increases past $1/e$. Suddenly this hair is much longer: it becomes the entire real axis. But, in fact, it is longer still. Consider the set of points in the strip S defined by $0 \leq \text{Im}\, z \leq \pi$ that eventually map onto $\mathbb{R}$. Clearly, the line $y = \pi$ maps into $\mathbb{R}$ after one iteration. So we can think of this hair through the origin as being extended by adjoining the point at $-\infty$ to the real axis and the line $y = \pi$. Now E_λ maps S one-to-one onto the upper half plane. So there is a unique curve in S that is mapped to $y = \pi$ and hence into $\mathbb{R}$ after two iterations. This curve actually tends to ∞ in the right half plane in both directions. So we can similarly adjoin a point at ∞ to the upper end of this preimage and the right end of $y = \pi$. Then the preimage of this curve in S is another curve that also extends to ∞ in the right half plane in both directions. In fact, all of the subsequent preimages of $y = \pi$ have this property. If we successively adjoin one endpoint of each curve with the corresponding endpoint of its preimage, we get a curve in S that can be shown to accumulate everywhere upon itself. If we compactify this picture by contracting S to the strip $-1 \leq \text{Re}\, z \leq 1$ and again making these identifications, then this curve does not separate the plane. Using a result of Curry [Cu], the closure of this set can be shown to be an indecomposable continuum [D1]. That is, as soon as the bifurcation occurs, the hair $[p, \infty)$ suddenly explodes into an indecomposable continuum.

Here is where a number of open problems arise. Let C_λ denote the indecomposable continuum in $J(E_\lambda)$ in S.

Problem 1. *Suppose $\lambda, \mu > 1/e$. Are C_λ and C_μ homeomorphic?*

It is known that each of the maps E_λ and E_μ have the "same" symbolic dynamics on their Julia sets [DK], but the maps themselves are not topologically conjugate [DG]. This latter fact was proved by showing that certain collections of periodic points accumulate onto dynamically different points when $\lambda \neq \mu$. A more topological proof of this fact would ensue if Problem 1 were shown to be true.

The exact topology of these indecomposable continua is not known. There have been some piecewise linear models proposed [DM], but so far a complete topological description of these sets has not been given.

Problem 2. *Find a topological model for the sets C_λ.*

In contrast to the rich topology of these sets, the dynamical behavior on these sets is fairly well understood. There are only three types of orbits:

1. The fixed point (which moves upward off the real axis after q and p merge);

2. The points on any of the preimages of $\mathbb{R}$ whose orbits simply tend to ∞;

3. The orbits of all other points which accumulate on the orbit of 0 together with the point at ∞.

In line with this, there are many other questions having to do with the relation between the dynamics and the topology of C_λ. For example:

Problem 3. *What is the structure of the composant that contains the unique fixed point in C_λ?*

There are other indecomposable continua in the Julia set of E_λ. For example, one can associate an itinerary to any point in $J(E_\lambda)$ by watching how the orbit passes through the strips $S_n = \{z \mid (2n-1)\pi < \operatorname{Im} z < (2n+1)\pi\}$ at each iteration. Then we associate the infinite sequence of integers $s = (s_0 s_1 s_2 \ldots)$ to z if $E_\lambda^j(z) \in S_{s_j}$ for each j. Then, for $\lambda > 1/e$, consider the set of points whose itinerary is a given sequence s. For most sequences, this set of points remains a hair. However, if s terminates in all 0's, then this set is just a preimage of the indecomposable continuum (or its complex conjugate) constructed above and hence is homemorphic to this set. If the itinerary consists of blocks of 0's separated by non-zero entries and having the property that the lengths of the blocks of 0's goes to ∞ sufficiently quickly, then the corresponding set of points is also an indecomposable continuum which is presumably topologically diferent from the one constructed above. See [DJ]. A natural question is what other types of sets of points can correspond to a given itinerary.

Problem 4. *Identify which itineraries correspond to indecomposable continua when $\lambda > 1/e$ and which yield hairs. Are there any other possibilities for the types of sets corresponding to a given itinerary? And how does all of this depend on λ?*

Along this line, when λ is allowed to be complex and the orbit of 0 eventually lands on a repelling periodic orbit (as is the case when $\lambda = k\pi i$ with $k \neq 0$), then it is known that set of points corresponding to certain itineraries may be an indecomposable continuum together with a finite collection of curves that accumulates on the indecomposable continuum. But this is the only other type of set that is known to correspond to a given itinerary. See [DJM]. It seems strange that there is nothing "in-between": either such a set is a simple curve or it is (or contains) an indecomposable continuum.

Problem 5. *Identify the types of sets of points that can correspond to a given itinerary under a complex exponential map.*

We have restricted to the complex exponential in this section for several reasons. First of all, this has been the most widely studied example of an entire transcendental dynamical system. Secondly, the corresponding results for other functions seem much more difficult. For example, consider the simple cosine family $i\mu \cos z$ where $\mu > 0$. It is known that, if $\mu \approx 0.67$, the cosine function undergoes a similar bifurcation as the exponential does when $\lambda = 1/e$. The Julia set is a pair of Cantor bouquets (one in the upper and one in the lower half plane) when $\mu < 0.67$, whereas the Julia set explodes to become $\mathbb{C}$ as soon as μ increases beyond 0.67. How this occurs is still a mystery. The hairs forming the Cantor bouquet do change after the bifurcation, but do they become indecomposable continua? The difficulty arises because the cosine function has critical points and not asymptotic values. This seems to cause a very different structure in the hairs when the critical points suddenly escape to ∞.

Problem 6. *Explain the bifurcation at $\mu = 0.67$ in the family $i\mu \cos z$. In particular, do hairs suddenly become indecomposable continua?*

Of course, there are many other instances of similar (and more complicated) bifurcations in transcendental dynamics. Perhaps other exotic topological structures arise in these bifurcations as well. Along these lines, there are examples of simple bifurcations in which the Julia set of an entire map migrates from a Cantor bouquet to a simple closed curve (passing through ∞) and also from a Cantor bouquet to a Cantor set. See [D3].

2 Sierpinski Curve Julia Sets

In this section we turn to a very different type of topological structure that occurs often in complex dynamics, Sierpinski curves. A Sierpinski curve is any planar set that is homeomorphic to the well-known Sierpinski carpet fractal. This set is important in topology for several reasons. First, thanks to a result of Whyburn [Why], there is a topological characterization of any set that is homeomorphic to the carpet. Any planar set that is compact, connected, locally connected, nowhere dense, and has the property that each complementary domain is bounded by a simple closed curve, any pair of which are disjoint, is homeomorphic to the Sierpinski carpet (and thus called a Sierpinski curve). More importantly, Sierpinski curves are universal plane continua since any planar, one-dimensional, compact, connected set may by embedded homeomorphically in a Sierpinski curve.

To see these sets in complex dynamics, we now turn to the family of rational maps given by $F_\lambda(z) = z^n + \lambda/z^n$ where $n \geq 2$ and $\lambda \in \mathbb{C} - \{0\}$, although these types of sets occur in many other families of rational maps. For these maps, the definition of the Julia set is slightly different. The point at ∞ is no longer an essential singularity as in the case of the exponential map. Rather, since $n \geq 2$, the map F_λ is essentially given by z^n near ∞, so ∞ is an attracting fixed point for these maps and we have a basin of ∞ that we denote by B_λ. $J(F_\lambda)$ is still the closure of the set of repelling periodic points, but now it is the boundary of, not the closure of, the set of points whose orbits escape to ∞. Note that the origin is a pole and there is a neighborhood of 0 that is mapped n-to-1 onto a neighborhood of ∞ in B_λ. If this neighborhood of 0 does not intersect B_λ, then there is an open set containing 0 that is mapped n-to-1 onto the entire set B_λ. We then call this set the trap door and denote it by T_λ. T_λ is the trap door since any orbit that eventually reaches B_λ must in fact pass through T_λ exactly once.

These maps are special because, despite the high degree of the maps, there really is only one "free" critical orbit. Indeed the $2n$ critical points are given by $\lambda^{1/2n}$, but they are each mapped to one of the critical values $\pm 2\sqrt{\lambda}$ by F_λ. After that, the two critical values are mapped onto the same point (if n is even) or the orbits of these two points are arranged symmetrically under $z \mapsto -z$ (if n is odd). In either case, all the critical orbits behave in the same manner, so there is essentially only one critical orbit.

If one and hence all of the critical orbits end up in the basin of ∞, then the topology of the Julia set is completely determined. There are thre different ways that these orbits can reach B_λ. The following result is proved in [DLU]. Suppose the critical orbit tends to ∞.

1. If the critical values lie in B_λ, the Julia set is a Cantor set;

2. If the critical values lie in T_λ, the Julia set is a Cantor set of simple closed curves;

3. If the crtical values do not lie in B_λ or T_λ but some subsequent iterate of these points does so, then the Julia set is a Sierpinski curve.

As a remark, case 2 of this result was proved by McMullen [McM]. This case cannot occur when $n = 2$. In Figure 2, we display several examples of Sierpinski curve Julia sets drawn from the family when $n = 2$.

The fact that there is essentially only one critical orbit for maps in these families says that the λ-plane is the natural parameter plane for these families. In Figure 3 we have displayed the parameter planes for the families when $n = 3$ and $n = 4$. The external white region consists of parameters for which the Julia set is a Cantor set; the central white region is the "McMullen domain" where the Julia set is a Cantor set of simple closed curves; and all of the other white regions contain parameters for which the Julia set is a Sierpinski curve. These regions are called Sierpinski holes. The region in parameter plane that excludes the Cantor set locus and the McMullen domain is called the connectedness locus; Julia sets whose parameters lie in this region are known to be connected sets.

For a parameter drawn from a Sierpinski hole, the complementary domains consist of B_λ and all of its preimages. It is known that if two parameters, λ and μ, lie in the same Sierpinski hole, then F_λ and F_μ are dynamically the same, i.e., F_λ is topologically conjugate to F_μ on their Julia sets. In particular, the critical orbits all land in B_λ under the same number of iterations under both of these maps. But if λ and μ are drawn from holes for which the

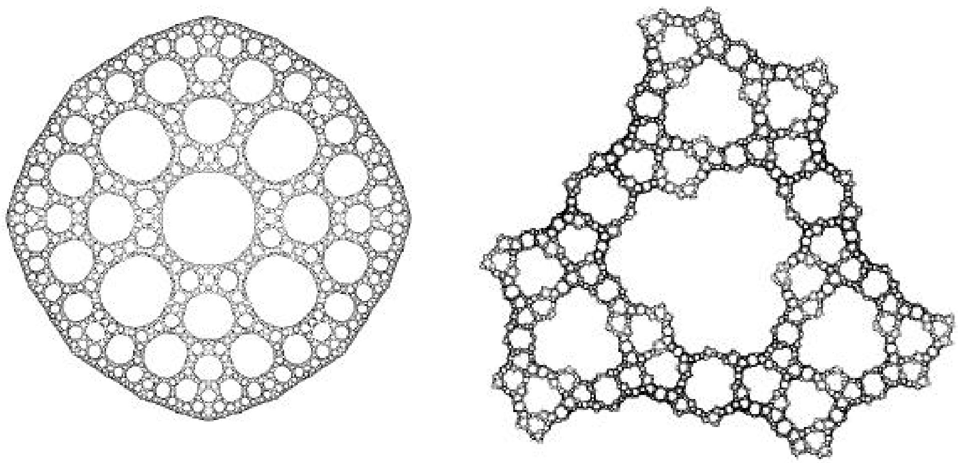

Figure 2: The Julia sets for (a) $z^2 - 0.06/z^2$, and (b) $z^2 + (-0.004 + 0.364i)/z$ are Sierpinski curves.

number of iterations that it takes for the critical orbit to reach B_λ is different, then these maps are not conjugate. However, there are many different holes for which the critical values take the same number of iterations to reach B_λ. For example, when $n = 3$, it is known [D2] that there are exactly $2 \cdot 6^j$ holes for which it takes the critical values $j + 2$ iterations to reach B_λ. This leads to a more dynamical type of problem:

Problem 7. *Determine whether the dynamical behavior that occurs for parameters drawn from two different Sierpinski holes with the same escape time is the same or different.*

There are many types of parameters for which the corresponding Julia sets are Sierpinski curves. For example, a magnification of the parameter plane for $n = 2$ shown in Figure 4 shows that there are (in fact, infinitely many) "buried" small copies of Mandelbrot sets contained in the parameter plane. These are the Mandelbrot sets that do not touch the outer boundary of the connectedness locus. It is known that if λ lies in the main cardioid of such a Mandelbrot set, then again the Julia set is a Sierpinski curve. The dynamics on these types of sets are again different from the dynamics of maps drawn from Sierpinski holes, since there is an attracting cycle for such a map. So the complementary domains for these maps consist of all the preimages of this

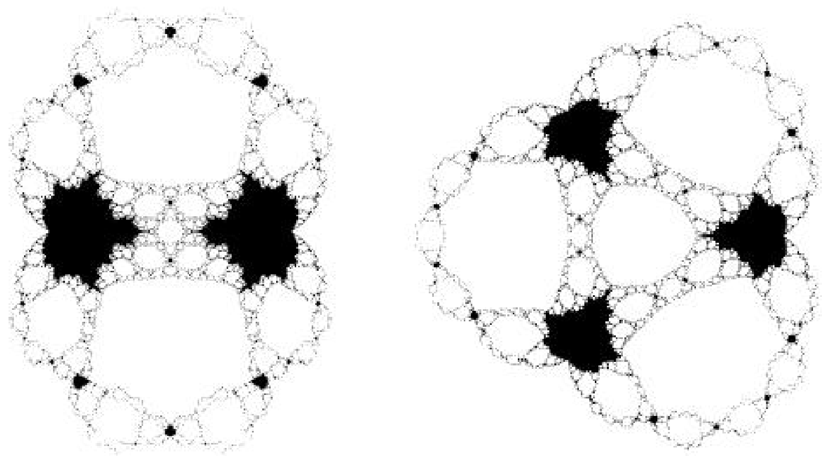

Figure 3: The parameter planes for the cases $n = 3$ and $n = 4$.

attracting basin as well as the preimages of B_λ. And there are other types of Sierpinski curve Julia sets: for example, it is known that there is a Cantor set of simple closed curves in the parameter plane that do not pass through any Sierpinski holes, yet all of the Julia sets corresponding to parameters on these curves are Sierpinski curves. As before, all but finitely many of these maps are dynamically distinct. So we have a huge number of Julia sets that are all the same from a topological point of view, but dynamically very different. This leads to a natural question:

Problem 8. *Classify the dynamics of all the different types of Sierpinski curve Julia sets that arise in these families.*

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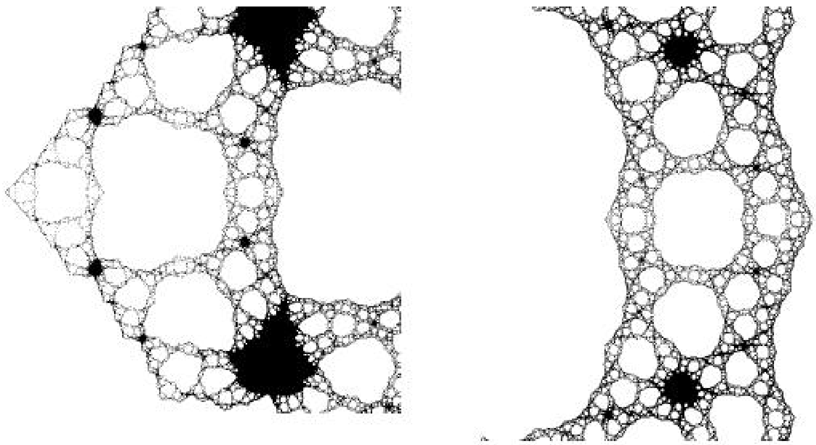

Figure 4: Two magnifications of the parameter plane for the family $z^2 + \lambda/z^2$ along the negative real axis. In the first image, $-0.4 \leq \mathrm{Re}\,\lambda \leq -0.06$ and, in the second, $-0.2 \leq \mathrm{Re}\,\lambda \leq -0.15$

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