Playing catchup with iterated exponentials

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1 Introduction

Suppose that we have two animals that make the same number of strides per minute, but one of them makes larger strides than the other. If the strides of the smaller animal (the prey) have length a, and those of the larger animal (the predator) have length b, it is easy to see that a persistent predator will always be able to catch up with its prey. Let us assume that the prey starts one step ahead of the predator. After n steps the distance between the two is

$$nb - (n+1)a = n(b-a) - a$$

and consequently, if $n > a/(b-a)$, the predator will have overtaken its prey.

Let us imagine a planet on which creatures move by jumps of increasing length. A creature on such a planet is at a distance a from where it started after one jump, and a distance a^n after n jumps. Let us also assume that $a > 1$ so that creatures move away from their starting point. We can again ask the question whether a small creature which starts one step ahead from a predator can escape from it. Let us assume that the initial step of the predator is of size $b > a > 1$, so that if $b^n > a^{n+1}$, the smaller creature is in the maw (or the extraterrestrial equivalent) of its predator. A simple calculation shows that this happens if the predator is sufficiently persistent to make

$$n > (\log a)(\log \frac{b}{a})^{-1}$$

steps. Of course, we can imagine an even stranger planet on which a creature makes an initial jump of size a, followed by a jump that moves it at distance a^a from its starting place, and another that brings it to a distance a^{a^a}, and so on. Thus the distance that such creatures travel is determined by *towers of a.*

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Definition 1.1 *Given any number a, define the* zero *tower of a by $T(a, 0) = a$. Recursively, for $n \geq 1$, define the nth tower of a by*

$$T(a, n) = a^{T(a,n-1)}.$$

Ackermann has introduced a natural way of ordering operations on real numbers [1], so that addition is an operation of type 1, multiplication is of type 2, exponentiation is of type 3, the operation $T(a, n)$ is of type 4, and so on. We therefore live on a type 2 planet, since our movement in space is determined by operations of type 2. The two imaginary planets we have described above are respectively of type 3 and 4.

Therefore, creatures on planets of type 2 and 3 cannot escape their predators, even if they have a head start. Is the same true for creatures on a planet of type 4? Surprisingly, if their step grows to a sufficient size, they will be able to escape faster predators, no matter how persistent. More precisely, we will prove:

Theorem 1.2 *If $a \leq e^{1/e}$, then there exists $n_0(a)$ such that $T(b, n) > T(a, n+1)$ for all $n > n_0(a)$. If $a > e^{1/e}$, then there exists $b_0(a) > a$ such that $T(b, n) < T(a, n + 1)$ for all n and $b \in (a, b_0(a)]$.*

Thus creatures of step size greater than $e^{1/e}$ can escape predators whose initial step is smaller than $b_0(a)$, while smaller creatures always get caught.

2 Proof of the Theorem

Note that if we define $F_a(x) = a^x$, and we denote the n-fold composition of F_a with itself by $F_a^n(x)$, then $F_a^n(a) = T(a, n + 1)$. The graph of $F_a(x)$ for $a < e^{1/e}$ intersects the line $y = x$ in two points $l(a)$ and $r(a)$ both of which are to the right of a. Under the iteration of $F_a(x)$, $l(a)$ is attracting, and hence the sequence $F_a^n(a) = T(a, n + 1)$ will approach it from the left. In other words, a creature whose initial step is of size $a < e^{1/e}$ tires quickly, takes progressively smaller steps, and never makes it past $l(a)$. It is easy to see that if $b > a$ then either $F_b(x)$ does not intersect $y = x$ or $l(b)$ is to the right of $l(a)$. In the first case the larger creature never tires, while in the second it will approach the point $l(b)$. In either case it must overtake the smaller creature eventually (see Figure 1). The case $a = e^{1/e}$ can be treated similarly.

On the other hand, when $a > e^{1/e}$, the graph of $F_a(x)$ does not cross the diagonal, and hence there are no fixed points. In this case we will show that the prey may escape to infinity and elude its predator.

Note that if the initial steps are slightly larger than $e^{1/e}$ the creature will initially slow down, until it makes it past the point $x = e$ after which it will catch a second wind, and make progressively larger leaps. Yet, it is not clear if the prey will be able to escape its predator.

To handle the case $a > e^{1/e}$, we will convert the problem of comparing towers of powers of different bases, to a problem of comparing iterates of an exponential map. We will need the following Lemma.

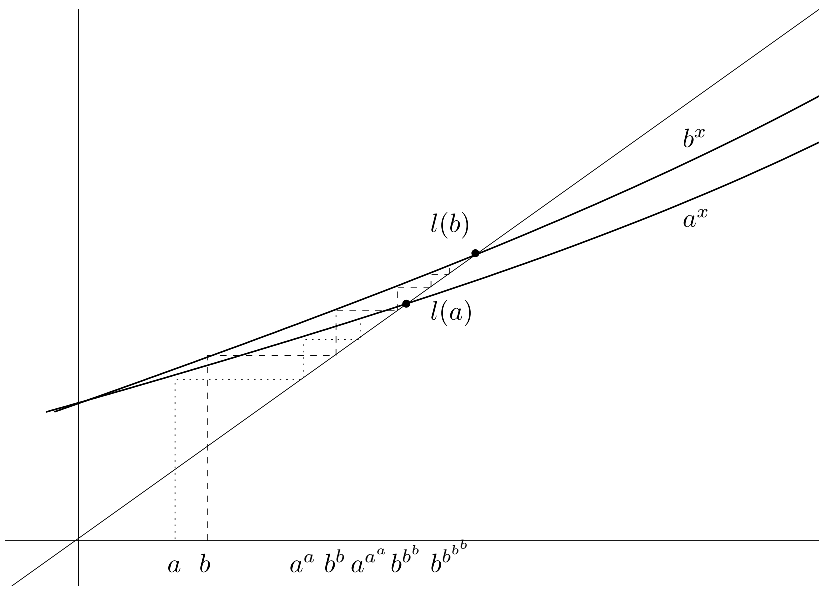

Figure 1: The towers of a and b converge to $l(a)$ and $l(b)$ respectively.

Lemma 2.1 *Fix $\lambda > \mu > 0$ and let $\eta > 0$. If $x > 0$ and $y - x > \log\big((\eta+1)\lambda/\mu\big)$, then:*

$$\mu e^y - \lambda e^x > \eta\lambda > 0.$$

Proof. We have

$$e^{y-x} > (\eta+1)\frac{\lambda}{\mu},$$

so that

$$\mu e^{y-x} - \lambda > \eta\lambda,$$

and therefore

$$\mu e^y - \lambda e^x > e^x\eta\lambda > \eta\lambda.$$

■

Let $E_\lambda(x) = \lambda e^x$ and $E_\mu(y) = \mu e^y$ with $\lambda > \mu > 1/e$ and x and y as in the Lemma. Choose η such that

$$\log\left(\frac{\eta+1}{\eta\mu}\right) < 1. \tag{1}$$

Again, E_λ^j denotes the $j-$th fold of the exponential map.

Corollary 2.2 *Under the assumptions of Lemma 2.1 for all $j \geq 1$,*

$$E_\mu^j(y) > E_\lambda^j(x).$$

Proof. Using the fact that $x \geq 1 + \log x$ for all $x > 0$, from the Lemma and equation (1) we have

$$\mu e^y - \lambda e^x > \eta\lambda > \log \eta\lambda + \log \frac{\eta+1}{\eta\mu} = \log \frac{(\eta+1)\lambda}{\mu},$$

so that $E_\lambda(x)$ and $E_\mu(y)$ satisfy the hypothesis of Lemma 2.1. The proof now follows by induction.

■

We can now return to the proof of the theorem. Fix $a > e^{1/e}$. Let $e^y = T(a, n_0 + 1)$ and $\mu = \log a$. Also, let $e^x = T(b_0, n_0)$ and $\lambda = \log b_0$, where $a < b_0 < a^a$ and n_0 are to be determined later on. For the given a, fix η so 1 holds. Note that $\lambda = \log b_0 > \log a = \mu > 1/e$. We have

$$E_\mu(y) = \mu e^y = T(a, n_0 + 1) \log a$$

and

$$E_\lambda(x) = \lambda e^x = T(b_0, n_0) \log b_0.$$

We will show that there exist n_0 and b_0 such that x and y satisfy the conditions of Lemma 2.1. By Corollary 2.2 it follows that $E_\mu^j(y) > E_\lambda^j(x)$ for all j. In terms of towers, we will have $T(a, n_0 + j) \log a > T(b_0, n_0 + j - 1) \log b_0$ for all j. Since $\log b_0 > \log a$ and using monotonicity of the towers, we will conclude

$$T(a, n) > T(b, n-1) \tag{2}$$

for all $b \in (a, b_0)$ and for all n. It suffices to find n_0 and b_0 such that $x > 0$ and $y - x > (\eta + 1)\lambda$.

The condition $x > 0$ follows automatically for any $b_0 > a$ and any n_0, as $a > 1/e$. Since $T(a, n) - T(a, n-1) \longrightarrow \infty$ as $n \longrightarrow \infty$, we can find n_0 such that

$$\big(T(a, n_0) - T(a, n_0 - 1)\big) \log a > (\eta + 3) \log a^a.$$

Therefore,

$$\big(T(a, n_0) - T(a, n_0 - 1)\big) \log a > (\eta + 3) \log b_0,$$

for any value of $a < b_0 < a^a$.

Let $b_1 > a$ be defined by

$$T(b_1, n_0 - 1) - T(a, n_0 - 1) = 1.$$

Similarly, we can choose b_2 close enough to a so that

$$T(a, n_0 - 1)(\log b_2 - \log a) < \log b_2.$$

Clearly, $a < b_1, b_2 < a^a$. Let $b_0 = \min\{b_1, b_2\}$. From the definition of x and y

$$y - x = T(a, n_0) \log a - T(b_0, n_0 - 1) \log b_0$$

or

$$\begin{aligned} y - x &= \big(T(a,n_0) - T(a,n_0-1)\big)\log a - T(a,n_0-1)(\log b_0 - \log a) \\ &\quad -\big(T(b_0,n_0-1) - T(a,n_0-1)\big)\log b_0. \end{aligned}$$

That is, we have

$$y - x > (\eta + 3)\log b_0 - 2\log b_0$$

or

$$y - x > (\eta + 1)\log b_0 = (\eta + 1)\lambda.$$

Hence, given any a we can produce b_0 and n_0 so that equation (2) holds.

3 Remarks

The smallest initial step a predator needs to take to catch a prey with initial step of size a has a sharp lower bound given by

$$\gamma(a) = \sup\{b \mid T(a,n+1) > T(b,n) \text{ for all } n\}.$$

We call γ the *catch-up* function. The previous theorem implies $\gamma(a) = a$ if $a \le e^{1/e}$, and $\gamma(a) > a$ if $a > e^{1/e}$. We can also define γ by letting $b_n(a)$ be the initial step size necessary to catch up in n steps, so that

$$T(b_n(a), n) = T(a, n+1).$$

Since n is the number of steps that a creature with initial step size $b_n(a)$ needs to take to catch the creature with initial step size a, it follows that $\gamma(a) = \lim_{n\to\infty} b_n(a)$.

The catch-up function has some interesting properties. Using estimates as in the previous section, one can show that γ is an increasing function. Other properties of the function $\gamma(a)$ are more difficult to establish. We conjecture that the function is smooth. It cannot be analytic at the point $a = e^{1/e}$, and we conjecture that at this point $\gamma(a)$ and the diagonal have a tangency of infinite order.

The catch-up problem has its origins in complex dynamics. The first and third authors have defined a piecewise semilinear family of continuous maps, h_λ, acting in the plane, which has dynamics and topology similar to that exhibited by the complex exponential family λe^z (see [2] and [3]). This family acts exponentially in the $x-$coordinate (the action is conjugate to λe^x), and essentially linearly in the $y-$coordinate. In [6], the third author has shown that for any pair of parameters λ and μ, the maps h_λ and h_μ are not topologically conjugate. The proof is based on the impossibility of catch-up as described above, but in the setting of h_λ.

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