The Escape Trichotomy for Singularly Perturbed Rational Maps

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0 Introduction

Our goal in this paper is to describe certain Julia sets of functions in the family of rational maps of the Riemann sphere given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

where $n, d \in \mathbb{Z}$ and $n \geq 2, d \geq 1$. When $\lambda = 0$, these maps reduce to $z \mapsto z^n$ and the dynamical behavior in this case is well understood: the Julia set of F_λ is just the unit circle and all other orbits tend either to ∞ or to the superattracting fixed point at 0.

When $\lambda \neq 0$, several things happen. First of all, the map F_λ now has degree $n+d$ rather than n. Secondly, the origin is a pole rather than a fixed point. And, finally, there are $n+d$ new critical points, in addition to the original critical points at 0 and ∞. The orbits of the critical points at ∞ and the origin are fixed and eventually fixed respectively, so their behavior is completely determined. As we show below, the orbits of all of the other critical points behave symmetrically with respect to rotation through angle $2\pi/n$, so we essentially have only one additional "free" critical orbit for each of these maps. As is well known in complex dynamics, the fate of this critical orbit plays a large role in determining the structure of the Julia sets of these maps. In this paper we shall describe a trichotomy in the topological structure of the Julia sets that arises when the free critical points have orbits that tend to ∞.

For comparison, we first recall the dichotomy that occurs for the well-studied family of quadratic polynomials, $Q_c(z) = z^2 + c$. As in our family, there is only one critical orbit for Q_c, namely the orbit of the critical point at 0. The following facts are well known (see [12]):

1. If the critical orbit for Q_c tends to ∞, then the Julia set of Q_c is a

Cantor set and Q_c is conjugate on the Julia set to the one-sided shift of two symbols.

2. If the critical orbit does not tend to ∞, then the Julia set of Q_c is a connected set.

In the quadratic polynomial case, the point at ∞ is a superattracting fixed point and so this point is surrounded by an immediate basin of attraction. If the critical orbit tends to ∞, then it is known that the critical point must lie in this basin and consequently, the entire forward orbit lies in this basin.

For the family F_λ, the point at ∞ is still a superattracting fixed point and so we still have an immediate basin of attraction which we denote by B. However, unlike the quadratic polynomial case, the full basin of attraction may consist of infinitely many disjoint preimages of the immediate basin of ∞. In particular, the component of the basin that contains 0 may be disjoint from B. If this is the case, then we denote this component by T. Since F_λ is n-to-1 on B and d-to-1 on T, it follows that the only two preimages of B are B and T. Then T must have disjoint preimages under F_λ^n for $n = 1, 2, 3 \ldots$, and so the basin of attraction of ∞ must have infinitely many distinct components.

Our goal in this paper is then to prove:

Theorem (The Escape Trichotomy). *Suppose the orbits of the free critical points of F_λ tend to ∞. Then*

1. *If one of the critical values lies in B, then $J(F_\lambda)$ is a Cantor set and $F_\lambda \mid J(F_\lambda)$ is a one-sided shift on $n + d$ symbols. Otherwise, $J(F_\lambda)$ is connected and the preimage T is disjoint from B.*

2. *If one of the critical values lies in T, then $J(F_\lambda)$ is a Cantor set of simple closed curves (quasicircles).*

3. *If one of the critical values lies in a preimage of T, then $J(F_\lambda)$ is a Sierpinski curve.*

We remark that case 2 was first observed by McMullen for small λ when $1/n + 1/d < 1$ (see [10]). The only cases in our family not covered by the McMullen result occur when $n = d = 2$ or n is arbitrary and $d = 1$. We show below that, in fact, case 2 in the theorem does not occur for these special values of n and d.

We also remark that the Julia sets in case 3, namely Sierpinski curves, are quite interesting sets from the topological as well as the dynamical systems point of view. A *Sierpinski curve* is a planar set that is homeomorphic to the well known Sierpinski carpet fractal. By a result of Whyburn [18], it is known that any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by disjoint simple closed curves is homeomorphic to a Sierpinski curve. Moreover, these sets are known to contain a homeomorphic copy of every one-dimensional plane continuum. However, as shown in [1], [5], despite the fact that the Sierpinski curve Julia sets in the Theorem are always homeomorphic, there are infinitely many of them on which the maps F_λ have non-conjugate dynamics. We also remark that the existence of Sierpinski curves as Julia sets has been observed before, notably in the work of Milnor and Tan Lei [13] and Ushiki [17].

In Figure 1, we illustrate these three cases with pictures of Julia sets drawn from the family $z^4 + \lambda/z^4$.

Since there is only one free critical orbit for these families, the λ-plane is the natural parameter plane for these maps. In Figure 2, we display the parameter plane for the family when $n = d = 4$ together with a magnification. The white regions in this figure correspond to parameters described in the escape trichotomy. The outside white region corresponds to the Cantor set

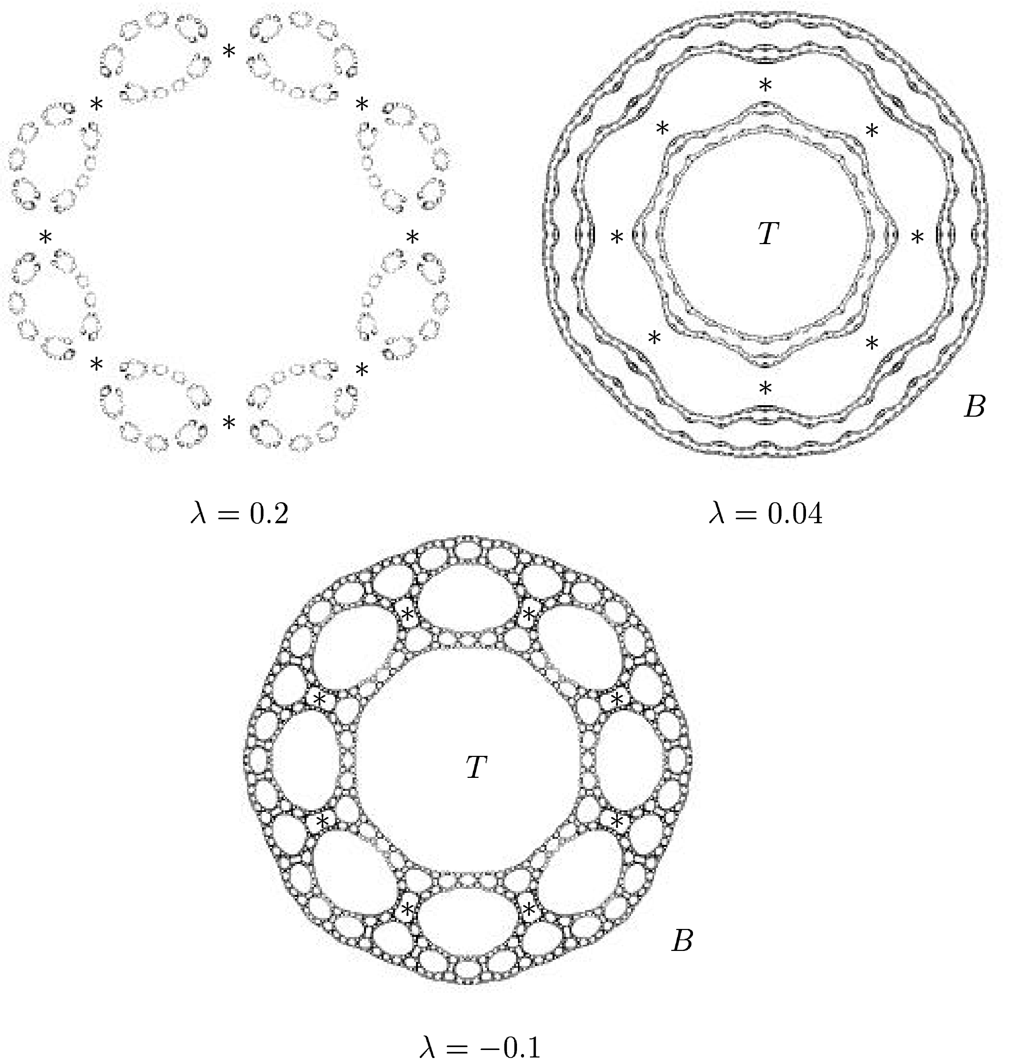

Figure 1: Some Julia sets for $z^4 + \lambda/z^4$: if $\lambda = 0.2$, $J(F_\lambda)$ is a Cantor set; if $\lambda = 0.04$, $J(F_\lambda)$ is a Cantor set of circles; and if $\lambda = -0.1$, $J(F_\lambda)$ is a Sierpinski curve. Asterisks indicate the location of critical points.

locus. The central white region contains parameters for which the Julia set is a Cantor set of closed curves; we call this region the *McMullen domain.* All other white regions correspond to *Sierpinski holes*, i.e., to parameters for which the Julia set is a Sierpinski curve.

These parameter planes contain a wealth of interesting structures. For example, it is known that, when $n = d \geq 3$, there is a single McMullen domain surrounding 0. This domain is surrounded by infinitely many disjoint closed curves S^j converging to the boundary of the McMullen domain and having the property that each S^j contains the centers of $(n-2)n^{j-1}+1$ Sierpinski holes as well as the same number of superstable parameter values that lie in the centers of baby Mandelbrot sets (unless $j = 2$) . See [6].

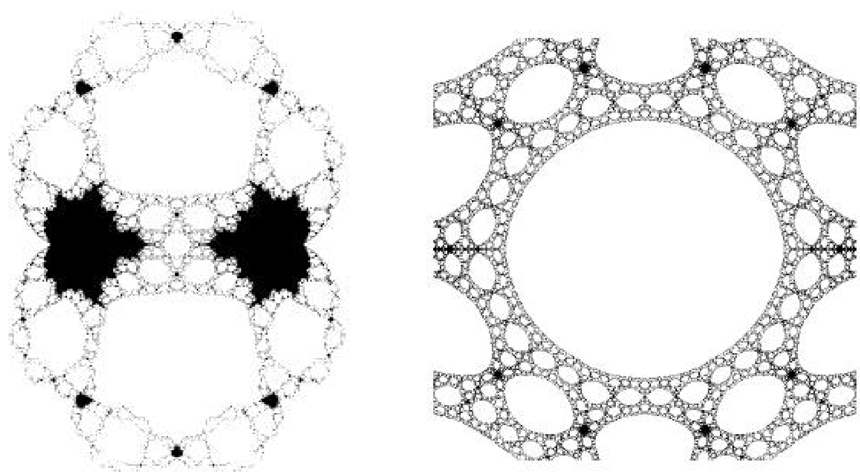

Figure 2: The parameter plane for the family $z^3 + \lambda/z^3$ and a magnification around the McMullen domain.

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1 Preliminaries

We consider the maps

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

where $n, d \in \mathbb{Z}^+$ and $n \geq 2$. The *Julia set* of F_λ, $J(F_\lambda)$, is defined to be the set of points at which the family of iterates of F_λ fails to be a normal family in the sense of Montel. Equivalently, the Julia set is the closure of the set of repelling periodic points for F_λ or, alternatively, as the set of points on which F_λ behaves chaotically. The complement of the Julia set is called the *Fatou set.*

There are $n + d$ critical points for F_λ and all are of the form $\omega^k c_\lambda$ where c_λ is one of the critical points and $\omega^{n+d} = 1$. Similarly, the critical values are arranged symmetrically with respect to $z \mapsto \omega z$, though there need not be $n + d$ of them. There are $n + d$ prepoles at the points $(-\lambda)^{1/(n+d)}$.

The proof of the following Proposition is straightforward.

Proposition (Dynamical Symmetry). *Suppose ω satisfies $\omega^{n+d} = 1$. Then $F_\lambda(\omega z) = \omega^n F_\lambda(z)$.*

As a consequence of this result, the orbits of points of the form $\omega^j z$ all behave "symmetrically" under iteration of F_λ. For example, if $F_\lambda^i(z) \to \infty$, then $F_\lambda^i(\omega^k z)$ also tends to ∞ for each k. If $F_\lambda^i(z)$ tends to an attracting cycle, then so does $F_\lambda^i(\omega^k z)$. We remark, however, that the cycles involved may be different depending on k and, indeed, they may even have different periods. Nonetheless, all points lying on these attracting cycles are of the form $\omega^j z_0$ for some $z_0 \in \mathbb{C}$. For example, when $n = 2, d = 1$, there are parameters for which some of the critical points tend to an attracting fixed point z_0 on the real line, whereas ωz_0 and $\omega^2 z_0$ lie on an attracting 2-cycle which attracts other critical points. See [4].

The point at ∞ is a superattracting fixed point for F_λ and we have that F_λ is conjugate to $z \mapsto z^n$ in a neighborhood of ∞, so we have an immediate basin of attraction B at ∞. Since F_λ has a pole of order d at 0, there is an open neighborhood of 0 that is mapped d to 1 onto a neighborhood of ∞ in B. If B does not contain this neighborhood, then there is a disjoint open set T about 0 that is mapped d to 1 onto B. We call T the *trap door* since any point whose orbit eventually enters B must pass through T enroute to B. Since the degree of F_λ is $n + d$, all points in the preimage of B lie either in B or in T.

Proposition ($(n + d)$-fold Symmetry). *Both B and T have $(n + d)$-fold symmetry, i.e., if $z \in B$, then $\omega z \in B$ as well, where $\omega^{n+d} = 1$.*

Proof: Let $U \subset B$ be the set of points z in B that have the property that the point ωz also lies in B. U is an open, nonempty set since B contains an open neighborhood around ∞. If $U \neq B$, we may choose a point $z_0 \in B \cap \partial U$, where ∂U denotes the boundary of U. So $z_0 \in B$ but $\omega z_0 \notin B$. Hence $\omega z_0 \in \partial B$. Therefore $F_\lambda^i(z_0) \to \infty$ whereas $F_\lambda^i(\omega z_0) \not\to \infty$. But by the Dynamical Symmetry Proposition, we have

$$F_\lambda^i(\omega z_0) = \omega^{ni} F_\lambda^i(z_0) \to \infty.$$

This gives a contradiction.

Since T surrounds the origin, the proof in this case is similar.

□

In particular, since the critical points are arranged symmetrically about the origin, it follows that, if one of the critical points lies in B (resp. T), then all of the critical points lie in B (resp. T).

For other components of the Fatou set, the symmetry situation may be somewhat different: if z_0 belongs to such a component, then either $\omega^j z_0$

belongs to this component for each j (as in the case of B and T), or $\omega^j z_0$ lies in a disjoint component whenever $\omega^j \neq 1$.

Symmetry Lemma. *Suppose U is a connected component of the Fatou set of F_λ. Suppose also that both z_0 and $\omega^j z_0$ belong to U, where $\omega^j \neq 1$. Then in fact, $\omega^i z_0$ belongs to U for all i and, as a consequence, U has $(n+d)$-fold symmetry and surrounds the origin.*

Proof: Suppose that z_0 and $\omega^j z_0$ lie in U but $\omega^i z_0$ does not lie in U. Let α_1 be a continuous curve in U that connects z_0 to $\omega^j z_0$. Define a second curve α_2 by $\omega^j \alpha_1$. By symmetry, α_2 also lies in a component of the Fatou set, but since $\omega^j z_0$ lies on α_2, it follows that α_2 also lies in U and so U also contains $\omega^{2j} z_0$. Continuing in this fashion, we see that U contains $\omega^{\ell j} z_0$ for all ℓ and that the analogous curve α_ℓ also lies in U.

Now suppose $\omega^{kj} = 1$. Then the union of the curves $\alpha_1, \ldots, \alpha_k$ forms a closed curve that lies in U and surrounds the origin. Call this curve α. By assumption, $\omega^i z_0$ does not lie on α. If we set $\omega^i \alpha_l = \beta_l$ for each l, we get another closed curve, call it β, that surrounds the origin and is contained in $\omega^i U$. Since $\omega^i z_0 \in \omega^i U$ but $\omega^i z_0 \notin U$ we know that $\omega^i U \neq U$. In fact, since U is a Fatou component and $F_\lambda(\omega z) = \omega^n F_\lambda(z)$ we get that $\omega^i U$ is also a Fatou component and hence $\omega^i U \cap U = \emptyset$. Since $\beta \subset \omega^i U$ and $\alpha \subset U$ we see that $\alpha \cap \beta = \emptyset$. However, α and β are both curves that surround the origin and $\beta = \omega^i \alpha$, implying that α and β must cross. This implies that α and β lie in the same Fatou component, yielding a contradiction.

2 The Cantor Set Case

In this section we prove that, if one of the critical values lies in B, then the Julia set of F_λ is a Cantor set. This result follows from several propositions.

Proposition. *Suppose some critical value v_λ of F_λ lies in B. Then all of* *the critical points of F_λ also lie in B.*

Proof: Recall that, by $(n+d)$-fold symmetry, if one of the critical points of F_λ lies in B, then all of the critical points must lie in B. So we assume for the sake of contradiction that none of the critical points lies in B. Since ∞ is attracting, we know there exists an analytic homeomorphism, ϕ_λ, which is defined in a neighborhood N_0 of ∞ and conjugates F_λ on N_0 to z^n on a disk of radius $r < 1$. Since no critical points lie in B we can pull this neighborhood back by F_λ^{-1} so that the larger neighborhood N includes one, and hence all (again by symmetry), of the critical values of F_λ. Now there exists a level set, γ, of the Green's function associated to ϕ_λ which bounds a simply connected open set containing N; call this set G. Let G^{-1} denote the preimage of G under F_λ that contains zero. We observe that:

1. G is simply connected.
2. G^{-1} contains $n+d$ nonzero critical points of multiplicity 1.
3. G^{-1} contains the critical point, 0, which has multiplicity $d-1$.
4. Since G^{-1} contains a neighborhood of the origin, F_λ has degree d on G^{-1}.

Therefore, by the Riemann-Hurwitz formula, we have

$$\text{boun}\,(G^{-1}) = d(\text{boun}(G) - 2) + n + 2d - 1 + 2 = n + d + 1$$

where boun (G^{-1}) denotes the number of boundary components of the region G^{-1}. Hence, G^{-1} has $n+d+1$ distinct boundary components. Now all of the boundary components of G^{-1} are mapped into γ. Since γ is a simple closed curve and there are no critical points on ∂G^{-1} (the critical values are strictly inside G), we see that the boundary components of G^{-1} are actually mapped onto γ. But this implies that γ has at least $n+d+1$ preimages, giving us a contradiction. Therefore all of the critical points lie in B.

□

Proposition. *For each critical point c_λ of F_λ, there is a curve $\gamma(c_\lambda)$ lying in B and extending from 0 to ∞. Moreover, these curves may be chosen so that they do not intersect and are symmetric under $z \mapsto \omega z$.*

Proof: It is known that the conjugacy ϕ_λ may be written

$$\phi_\lambda(z) = \lim_{m\to\infty} \left(F_\lambda^m(z)\right)^{1/n^m}.$$

Using this, we see that $\phi_\lambda(\omega z) = \omega\phi_\lambda(z)$. Hence the external rays in B are symmetric with respect to $z \mapsto \omega z$.

Given a particular critical point c_λ, suppose $v_\lambda = F_\lambda(c_\lambda)$. The preimages of the straight rays in $\overline{\mathbb{C}} - \overline{\mathbb{D}}$ are known as the external rays for F_λ. Let η be the portion of the external ray through v_λ that connects v_λ to ∞ in B. Let γ_1 be a preimage of η that lies in the external ray through c_λ. Since c_λ is a critical point, there is a second curve γ_2 containing c_λ and mapped by F_λ onto η. Now γ_2 connects c_λ to either 0 or ∞. We claim that the latter cannot occur. If this were the case, both γ_1 and γ_2 would be external rays that meet at a common point c_λ. Then these two curves would bound a connected open set $\mathcal{O}$ in B. Since each of these curves is mapped to η, the image of $\mathcal{O}$ would contain a neighborhood of ∞. Continuing, we can use symmetry to make the same construction for each of the critical points, and the corresponding regions cannot intersect since the external rays are symmetric with respect to $z \mapsto \omega z$. But then we have produced $n+d$ disjoint open sets tending to ∞ in B that are each mapped to a neighborhood of ∞. But F_λ is only n to 1 near ∞, so this cannot happen. Therefore we conclude that γ_2 must connect c_λ to 0.

Now let $\gamma = \gamma_1 \cup \gamma_2$. We may produce a similar curve for each of the critical points and, as above, each of these curves is disjoint from the curves passing through the other critical points. Hence these $n+d$ curves separate the plane into $n+d$ disjoint open subsets, and each of these sets is mapped univalently over the entire plane minus a portion of two external rays connecting ∞ to the two critical values. (These critical values are the images of the critical points that lie on the boundary of the region.) Standard arguments then show that the Julia set is a Cantor set and $F_\lambda \mid J(F_\lambda)$ is conjugate to a shift on $n+d$ symbols. This proves part 1 of the Escape Trichotomy Theorem.

3 The Cantor Set of Circles Case

In this section we generalize a result of McMullen [10] to show that, if a critical value of F_λ lies in T, then the Julia set of F_λ is a Cantor set of simple closed curves surrounding the origin. Recall that, when we say T, we are implicitly assuming that T and B are disjoint sets.

Proposition. *If some critical value v_λ lies in T, then the preimage of T is a connected set that contains all of the critical points.*

Proof: We first observe that we must have at least two critical points in some component of $F_\lambda^{-1}(T)$ and so, by symmetry, in each component of $F_\lambda^{-1}(T)$. If this were not the case, we would have $n+d$ components in $F_\lambda^{-1}(T)$, each of which is mapped with degree two onto T. But this means that each component would contain two prepoles. Then there would be at least $2(n+d)$ prepoles, which is not true. Applying the Symmetry Lemma, we see that in fact all of the critical points must lie in the same component of $F_\lambda^{-1}(T)$ and that this component is connected and surrounds the origin.

□

Proposition. *If v_λ lies in T, then the preimage of $\overline{T}$ is an annulus that* *divides the region between $\overline{B}$ and $\overline{T}$ into two open subannuli that are each mapped onto $\mathbb{C} - (\overline{B} \cup \overline{T})$.*

Proof: Note first that T is simply connected. This follows since the only critical point in T is the pole at the origin. Now suppose that $U = F_\lambda^{-1}(T)$ has ℓ boundary components. Since F_λ maps U with degree $n + d$ onto a simply connected set, and there are exactly $n + d$ critical points in U, the Riemann-Hurwitz formula gives

$$2 - \ell = (n + d)(2 - 1) + (n + d) = 0.$$

Hence U has connectivity 2 and so is an annulus. By the Symmetry Lemma, this annulus must surround the origin.

Note that ∂U does not meet ∂T and ∂B since ∂U is mapped to ∂T, whereas the boundaries of both T and B are mapped to ∂B and U separates B from T. Consequently, $\overline{U}$ divides the region between $\overline{B}$ and $\overline{T}$ into a pair of disjoint open annuli.

□

Proposition. *The boundaries of B, T, and all of the preimages of T are simple closed curves surrounding the origin.*

Proof: Let A denote the open set between $\overline{B}$ and $\overline{T}$. We have $\overline{U} \subset A$. So if we remove $\overline{U}$ from A, then the remaining open set has two components, one of which, A_{in}, abuts T and is mapped d to 1 onto A, the other of which, A_{out}, abuts B and is mapped n to 1 onto A.

Now let γ be a simple closed curve surrounding the origin in A_{in} and let ξ be the preimage of γ in A_{out}. The region between ξ and γ is an annulus. F_λ maps ξ in n to 1 fashion onto γ, and F_λ is also an n to 1 covering map on the open set in the exterior of ξ. Hence we can use quasiconformal surgery to construct a new map that

1. agrees with F_λ on and outside ξ;

2. is conjugate to the map $z \mapsto z^n$ defined on a disk of radius $r < 1$ in the simply connected region inside ξ.

See [7] or [12] for details on this surgery construction.

Since F_λ is an n to 1 covering outside ξ and maps ξ strictly inside itself, the new map is conjugate to $z \mapsto z^n$ everywhere. Hence the Julia set for this map is a quasicircle. But this Julia set is just the boundary of B, since the boundary of B is invariant under F_λ and lies in the exterior of ξ. This shows that the boundary of B is a simple closed curve. Hence so too is the boundary of T and all of its preimages.

□

To complete the proof of part 2 of the Escape Trichotomy Theorem, we note that F_λ is a d to 1 covering map taking the closed annulus $\overline{A}_{in}$ onto the closed annulus $\overline{A}$ between B and T, while F_λ is an n to 1 covering map taking $\overline{A}_{out}$ onto the same annulus. Hence the Julia set is given by a nested set of closed annuli and the result follows exactly as in the case described by McMullen in [10].

Note that, by the covering properties of F_λ on A_{in} and A_{out}, we must have

$$\operatorname{mod} A > \operatorname{mod} A_{in} + \operatorname{mod} A_{out} = \left(\frac{1}{d} + \frac{1}{n}\right) \operatorname{mod} A$$

where $\operatorname{mod} A$ denotes the modulus of A. Hence, as in the McMullen result, we must have $1/d + 1/n < 1$ in order for v_λ to lie in the trap door.

Corollary. *If $1/d + 1/n \geq 1$, then v_λ cannot lie in the trap door, so part 2 of the Escape Trichotomy Theorem cannot occur if $d = n = 2$ or $d = 1$ and n is arbitrary.*

To see that there are actual λ-values for which $v_\lambda \in T$, we need several facts. The proof of the following lemma is straightforward.

Lemma. *Suppose* $|z| \geq \max\{|\lambda|, 2\}$. *Then* $|F_\lambda^n(z)| \geq (1.5)^n|z|$, *so that* $z \in B$.

Lemma. *Suppose* $1/d + 1/n < 1$. *Then* $|F_\lambda(v_\lambda)| \to \infty$ *as* $\lambda \to 0$ *whereas* $v_\lambda \to 0$ *as* $\lambda \to 0$.

Proof: A simple computation shows that

$$F_\lambda(v_\lambda) = c_1 \lambda^{n - \frac{nd}{n+d}} + c_2 \lambda^{1 - \frac{nd}{n+d}}$$

where c_j are constants (depending only on d and n). The result then follows since $n > nd/(n+d) > 1$.

□

To see that v_λ actually lies in the trap door when λ is small, note that, for sufficiently small λ, the annulus $\mathcal{A}$ given by $1/2 \leq |z| \leq 2$ is mapped strictly outside itself as an n to 1 covering of its image. This follows since we may choose λ small enough so that the term λ/z^d in the expression for F_λ is arbitrarily small. Hence there is an invariant circle inside $\mathcal{A}$. Since each critical value v_λ lies inside $\mathcal{A}$ while $F_\lambda(v_\lambda)$ lies outside $\mathcal{A}$, it follows that v_λ lies inside T.

4 The Sierpinski Curve Case

In this section we discuss the case where the critical points have orbits that eventually escape through the trap door, but the critical values do not lie in the trap door. In this case, we prove that the Julia set of F_λ is a Sierpinski curve.

Proposition. *If* $v_\lambda \notin B \cup T$, *then the set* $\mathbb{C} - \overline{B}$ *has a single open, connected component.*

Proof: Suppose first that $\mathbb{C} - \overline{B}$ has more than one connected component. Let W_0 be the component of $\mathbb{C} - \overline{B}$ that contains the origin. Note that all of T must lie in W_0. We claim that at least one of the prepoles also lies in W_0. Suppose this is not the case. By the Symmetry Lemma, all of the prepoles either lie in the same component of the Fatou set or else they all lie in distinct components. In the latter case, this means that each Fatou component containing a prepole is mapped one-to-one onto W_0. Therefore there must be $n + d$ of these components. Thus every point in the boundary of W_0 has $n+d$ preimages, one in each of the boundaries of these components. But there are also d preimages of any such point in the boundary of the trap door which is contained inside $\overline{W_0}$. Since the boundary of T cannot equal the boundary of W_0, this yields too many preimages for any point in the boundary of W_0. Therefore all of the prepoles lie in the same component of the Fatou set, and this component must surround the origin and separate B from $\overline{W_0}$. This, however, is impossible, since the boundary of W_0 is contained in the boundary of B. Hence one and therefore all $n + d$ of the prepoles lie in W_0 and so F_λ is $n+d$ to 1 on W_0. Therefore all of the preimages of points in W_0 must also lie in W_0.

Now suppose that there is a second component W_1 in $\mathbb{C} - \overline{B}$. There are no points in W_1 that map into W_0. Consider a point of the boundary of W_1 that does not also lie on the boundary of W_0 and choose a neighborhood of this point that does not meet W_0. By Montel's Theorem, the forward images of this neighborhood map over points in W_0. But this cannot happen, since all preimages of points in W_0 lie in W_0. This proves that W_1 does not exist.

□

Proposition. *The Julia set of F_λ is compact, connected, locally connected, and nowhere dense.*

Proof: Since we are assuming that all of the critical orbits eventually enter the basin of ∞, we have that the Julia set is given by $\mathbb{C} - \cup F_\lambda^{-j}(B)$. That is, $J(F_\lambda)$ is $\mathbb{C}$ with countably many disjoint, open, simply connected sets removed. Hence $J(F_\lambda)$ is compact and connected. Since $J(F_\lambda) \neq \mathbb{C}$, $J(F_\lambda)$ cannot contain any open sets, so $J(F_\lambda)$ is also nowhere dense. Finally, since the critical orbits all tend to ∞ and hence do not lie in or accumulate on $J(F_\lambda)$, standard arguments show that $J(F_\lambda)$ is locally connected (see [12], Theorem 19.2). In particular, since B is a simply connected component of the Fatou set, it follows that ∂B is locally connected.

Proposition. *The boundary of B as well as all of the preimages of B are simple closed curves. These boundary curves are pairwise disjoint.*

Proof: Recall that, near ∞, F_λ is analytically conjugate to $z \mapsto z^n$. That is, there exists an analytic homeomorphism $\phi_\lambda : B \to \overline{\mathbb{C}} - \overline{\mathbb{D}}$ where $\mathbb{D}$ is the open unit disk in the plane. The map ϕ_λ satisfies

$$\phi_\lambda \circ F_\lambda(z) = (\phi_\lambda(z))^n.$$

The preimage under ϕ_λ of the straight ray with argument θ in $\overline{\mathbb{C}} - \overline{\mathbb{D}}$ is called the external ray of angle θ and denoted by $\gamma(\theta)$. Since the boundary of B is locally connected, it is known [2] that all of the external rays land at a point in the boundary of B. Thus, to show that this boundary is a simple closed curve, it suffices to prove that no two external rays land at the same point.

To see this, first recall that W_0 denotes the component of $\mathbb{C} - \overline{B}$ that contains the origin, and that W_0 is both connected and simply connected. Suppose that there exists $p \in \partial B$ such that $\gamma(t_1)$ and $\gamma(t_2)$ both land on p. Since these rays together with the point p form a Jordan curve, we have that W_0 lies entirely within one of the two open components created by this Jordan curve. Let $\gamma(t_1, t_2)$ denote the union of all of the external rays whose angles lie between t_1 and t_2 (where we assume that the angle between these two rays is smaller than π). Without loss of generality, assume that W_0 is such that $W_0 \cap \gamma(t_1, t_2) = \emptyset$ (so W_0 is "outside" the sector $\gamma(t_1, t_2)$ between $\gamma(t_1)$ and $\gamma(t_2)$).

We claim that there exist positive integers q and m such that the region

$$\gamma\left(\frac{q}{m}, \frac{q+1}{m}\right) \subset \gamma(t_1, t_2)$$

and neither of the external rays q/m nor $(q+1)/m$ land on ∂W_0. If this were not possible, then all rays in $\gamma(t_1, t_2)$ would land at p. This gives a contradiction because the set of angles $\theta \in \mathbb{R}/\mathbb{Z}$ such that the landing point of the ray with angle θ is p has measure 0 ([12], Theorem 17.4).

So suppose we have such q and m. As above, let $\gamma(q/m, (q+1)/m)$ denote the union of the external rays contained between q/m and $(q+1)/m$. After m iterations $\gamma(q/m, (q+1)/m)$ is mapped over all of B. In particular, if the external ray of angle θ lands on $\partial(\overline{\mathbb{C}} - \overline{B})$, then there is a ray of angle $\phi \in \gamma(q/m, (q+1)/m)$ whose image under F_λ^m is $\gamma(\theta)$. Since $\phi \in \gamma(q/m, (q+1)/m)$ we have that $\gamma(\phi)$ does not land on ∂W_0. Hence there exists a neighborhood N_ϕ of $\gamma(\phi)$ such that $N_\phi \cap W_0$ is empty. However, since $F_\lambda^m(\gamma(\phi))$ lands on the boundary of W_0 we know that $F_\lambda^m(N_\phi) \cap W_0$ is not empty. This is a contradiction since points not in W_0 never enter W_0. Hence, we can never have two rays landing at the same point on ∂B, implying that ∂B is a simple closed curve.

It follows that all of the preimages of B are also bounded by simple closed curves. We claim that no two of these curves can intersect. To see this, suppose first that there exists a point $z_0 \in \partial B \cap \partial T$. Then there exists an external ray γ in B landing at z_0. In T, there also exists a preimage, η, of an external ray that connects 0 to z_0. But the images of η and γ are the same external ray, and so it follows that z_0 is a critical point. But this contradicts our assumption that all critical orbits tend to ∞. So ∂B and ∂T are disjoint simple closed curves.

Now suppose that two earlier preimages of ∂B intersect, say one preimage in $F_\lambda^{-n}(\partial B)$ and one in $F_\lambda^{-m}(\partial B)$. If $n \neq m$, then by mapping these preimages forward, we see that ∂B and ∂T also meet, which cannot happen. If $n = m$, then this intersection point must again be a critical point, so this cannot occur either.

□

This completes the proof of part 3 of the Escape Trichotomy Theorem.

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