Singular perturbations of quadratic maps.

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Abstract

We give a complete description of the dynamics of the mapping $f_\epsilon(z) = z^2 + \frac{\epsilon}{z}$ for positive real values of ϵ. We then consider two generalizations: the case of complex ϵ and the mapping $z \to z^n + \frac{\epsilon}{z^m}$, where ϵ is positive and real. In both of those cases we provide a full characterization of the map for a certain set of parameters, and give observations based on numerical evidence for all other parameter values. The dynamics of all maps that we consider bears striking resemblance to that of complex quadratic maps.

Introduction

In section 1, we describe the dynamics of $f_\epsilon(z) = z^2 + \frac{\epsilon}{z}$ in the complex plane for all positive values of ϵ. We show that there is only one bifurcation value ϵ_0 of ϵ in $\mathbf{R}_+$, and the family of maps $\{f_\epsilon | \epsilon > 0\}$ consists of only two equivalence classes with respect to conjugacy. In section 2, we generalize some of our results for the class of maps $z \to z^n + \frac{\epsilon}{z^m}$, where $\epsilon > 0$. It turns out that more than one bifurcation is present if and only if $m > 1$. If m=1, we give a complete description of the dynamics of the family. Otherwise, we point out the obstacles that prevent us from doing that. Finally, in section 3 we return to the map f_ϵ, and allow ϵ to be complex. The dependence of the map on ϵ turns out to be non-trivial in this case, and the set of the values of ϵ for which there is a finite attracting orbit has complicated structure. We give a picture of that set and prove some basic theoretical results on its structure.

Before we begin, let us list the most important theorems we rely on, and give references to the books that contain their proofs. First, we will make heavy use of the famous Julia theorem, which is proved, for example, in [3].:

Theorem 1 (Julia) *For any holomorphic map of the extended complex plane to itself, an attracting periodic cycle must attract at least one critical point.*

Another general result can be found in [1] (§9.8, Theorem 1).

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Theorem 2 *If a rational map R of $\mathbf{C}$ possesses a superattracting fixed point z_0 (i.e. $R'(z_0) = 0$), and all of the critical points of f are attracted by z_0, then the Julia set of R is totally disconnected, and the dynamics of R on that set is conjugate to full shift on n symbols, where n is the number of the critical points of R.*

Finally, we will rely on the following criterion of hyperbolicity for complex rational map. For its proof we refer to [5].

Theorem 3 *A rational map of degree $d \geq 2$ is hyperbolic if and only if the closure of the union of orbits of all critical point is disjoint from the Julia set.*

1 The dynamics of $f_\epsilon(z) = z^2 + \frac{\epsilon}{z}$, $\epsilon > 0$.

Before we treat the dynamics of f_ϵ in $\mathbf{C}$, let us briefly characterize its behavior in $\mathbf{R}$. The derivative of f is decreasing on $(-\infty, 0)$ and $(0, \sqrt[3]{\frac{\epsilon}{2}})$ and increasing on $(\sqrt[3]{\frac{\epsilon}{2}}, +\infty)$ (see Figure 1). Let us denote the critical point of f in $\mathbf{R}_+$, $\sqrt[3]{\frac{\epsilon}{2}}$, as z_c.

Since $f_\epsilon(\frac{1}{2})$ is less than $\frac{1}{2}$ for $\epsilon \ll 1$, all three fixed points of f_ϵ lie on the real line if ϵ is small. As ϵ increases, so does the value of $f_\epsilon(z)$ at each point of $\mathbf{R}_+$. At the bifurcation value ϵ_0 of ϵ the two positive fixed points collide in a saddle-node bifurcation, and for $\epsilon > \epsilon_0$ there are no fixed points for f_ϵ in $\mathbf{R}_+$. A computation shows that $\epsilon_0 = \frac{4}{27}$. The third fixed point remains in $\mathbf{R}_-$ for all values of ϵ. We denote that point z_-, and the other two fixed points, z_1 and z_2 (where $z_1 \leq z_2$ if both are real).

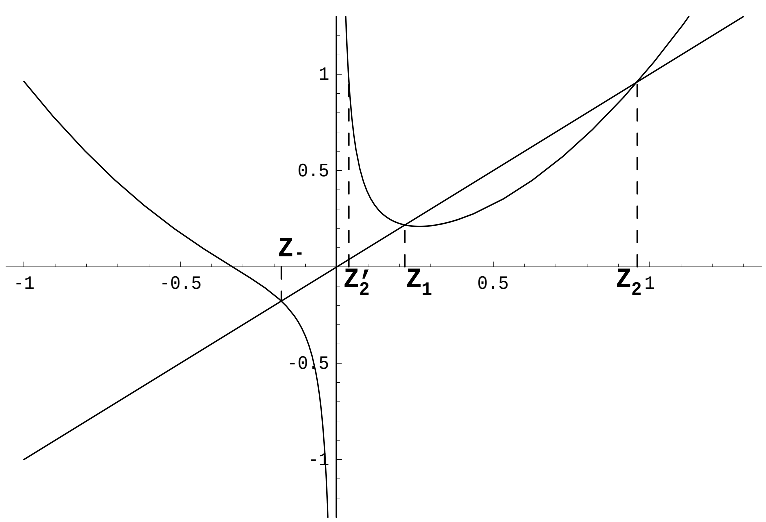

Figure 1: The graph of $f_\epsilon(z)$ for $\epsilon = \frac{1}{27}$.

Proposition 4 *Before the bifurcation value ϵ_0, z_- and z_2 are repelling, while z_1 is attracting. After the bifurcation value, all three fixed points are repelling.*

Proof Since $f'(z) = 2z - \frac{\epsilon}{z^2} = 3z - \frac{1}{z}(z^2 + \frac{\epsilon}{z})$, it follows that $f'(z) = 3z - 1$ at all fixed points of f. In particular, the value of $f'(z)$ at z_- is always less than -1. The value of $f'(z)$ at the positive fixed points is always greater than -1, and it is less than 1 only if the point is to the left of $\frac{2}{3}$. On the other hand, as ϵ grows, z_1 increases and z_2 decreases, and at the bifurcation value $\epsilon_0 = \frac{4}{27}$ both fixed points coincide at $\frac{2}{3}$. Therefore, before the bifurcation $z_1 < \frac{2}{3} < z_2$, and so z_1 is attracting, while z_2 is repelling.

After the bifurcation, the fixed points z_1 and z_2 become two complex conjugate numbers; their sum is twice their real part. Since the equation for the fixed points is $z^3 - z^2 + \epsilon = 0$, by Viet's theorem the sum of the fixed points must be 1. Since z_- decreases after the bifurcation value, the sum of the other two fixed points has to increase. Therefore, the real part of both z_1 and z_2 is greater than $\frac{2}{3}$ after the bifurcation. Hence neither of them lies in the circle of radius $\frac{1}{3}$ around $\frac{1}{3}$, and if $z = z_1$ or $z = z_2$, then $|f'(z)| = 3|z - \frac{1}{3}| > 1$, so both z_1 and z_2 are repelling. □

One checks easily that before the bifurcation z_c gets attracted to z_1 (we will prove that fact in general form in Section 2). It follows that, if $\epsilon \leq \epsilon_0$, all points in $[z_2', z_2)$ are attracted to z_1, where z_2' is the positive real preimage of z_2 different from z_2. All other points of the positive real line are attracted to infinity. It is also clear that, for $\epsilon > \epsilon_0$, all points of the positive real line are attracted to infinity. The dynamics of f_ϵ on the positive real line will play a crucial role in our analysis of the behavior of f_ϵ in the complex plane.

We note that f_ϵ possesses a remarkable symmetry (which remains valid even for complex ϵ). Let $\omega = \cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})$, so that $\omega^3 = 1$. Then

$$f_\epsilon(\omega z) = \omega^2 f_\epsilon(z) \tag{1}$$

(Obviously, that also means $f(\omega^2 z) = \omega f(z)$). In particular, the rays $\omega\mathbf{R}_+$ and $\omega^2\mathbf{R}_+$ are mapped into each other by f. In this section we will say that two points, z and v, are *symmetric* to each other if $z = \omega^k v$ for a positive integer k.

Before the bifurcation, f has one attracting fixed point, z_1, in $\mathbf{R}_+$. Due to the symmetry (1), the pair $(\omega z_1, \omega^2 z_1)$ is also an attracting periodic orbit of period 2. Since all three critical points of f, namely the three cubic roots of $\frac{\epsilon}{2}$, tend to one of these two periodic orbits, f cannot have other attracting periodic orbits in $\mathbf{C}$. It is clear that infinity is also an attracting fixed point (of order 2).

We shall describe the dynamics of f_ϵ in $\mathbf{C}$ completely by means of a geometric construction. Consider a circle C close to infinity. Its preimage consists of two closed curves. One of them is also close to infinity, and it is mapped onto C in a two-to-one fashion, making two twists counterclockwise around the origin. Let us call that curve C_1. The other piece of the preimage is close to zero, and it is mapped in a one-to-one fashion onto C, making one twist around the origin clockwise. We denote that curve C_2 (see Figure 2). All points outside C_1, as well as all points inside C_2, get attracted to infinity, and all interesting

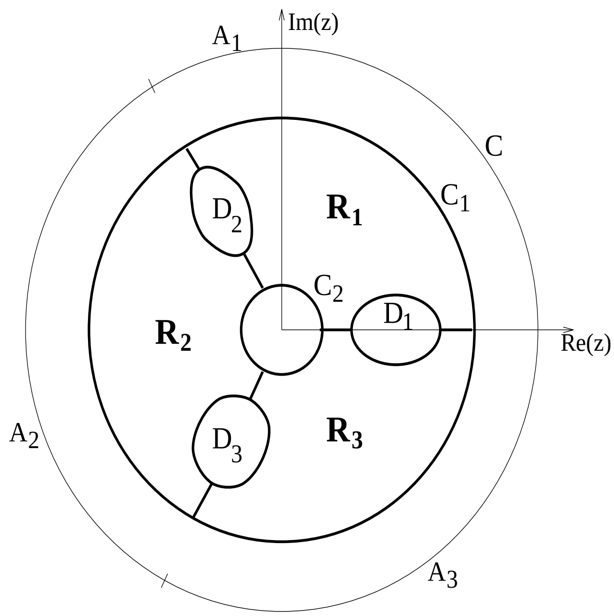

Figure 2: The division of the complex plane into regions for $\epsilon < 4/27$.

dynamics happens in the annular region R between C_2 and C_1. To make use of Theorem 3, we want to remove parts of that region that contain the orbits of the critical points.

Proposition 5 *Before the saddle-node bifurcation, there is a simply connected region $D_1 \subset R$ that includes z_1 and z_c, gets mapped into itself by f_ϵ, and consists only of points that are attracted to z_1.*

Proof Fix $\epsilon \in (0, \frac{4}{27})$. Let N be a small circular neighborhood of z_1 that is mapped into itself, and consists only of points that get attracted to z_1. If we take the connected component of the preimage of N containing N, and repeat this operation sufficiently many times, we will get the required region D_1, because the critical point z_c is attracted to z_1. □

The symmetry (1) produces a corollary:

Corollary 6 *Similar regions D_2 and D_3 can be constructed in such a way that D_2 contains ωz_1 and ωz_c, while D_3 contains $\omega^2 z_1$ and $\omega^2 z_c$. D_2 and D_3 are mapped into each other, and all points in both of them are attracted by the period two orbit $(\omega z_1, \omega^2 z_1)$.*

Remark 1. D_1 can be chosen in such a way that it maps precisely onto itself, and D_2 and D_3, onto each other; to get such D_1, we simply take the immediate basin of attraction of z_1.

Remark 2. D_1 lies entirely inside the region $-\frac{\pi}{3} < \arg(z) < \frac{\pi}{3}$, because all points of the rays $\omega\mathbf{R}_-$ and $\omega^2\mathbf{R}_-$ do not get attracted to z_1 (since $\omega\mathbf{R}$ gets mapped to $\omega^2\mathbf{R}$, and vice versa). Similar statements hold for D_2 and D_3.

Let us consider the region that is obtained from the annulus between C_2 and C_1 by excluding D_1, D_2 and D_3. We cut that region into three parts with the rays $\arg z = 0$, $\arg z = \frac{2\pi}{3}$ and $\arg z = \frac{4\pi}{3}$, and denote the parts R_1, R_2 and R_3. We also denote the arcs of C corresponding to those regions as A_1, A_2, and A_3 (see Figure 2).

By now we know these facts (see Figure 3):

1. The outer boundary of R_1 (curve 1 on the figure) is mapped in a one-to-one fashion onto A_1 and A_2 (by the definition of C_1).

2. The common boundary of R_1 and R_2 (curves 2 and 4) is mapped on the segment of $\omega^2\mathbf{R}_+$ that lies between C and D_3, and also inside D_3; this mapping is two-to-one.

3. The common boundary of R_1 and D_2 (curve 3) is mapped inside D_3, making one full turn around $\omega^2 z_1$. This is because D_2 contains one critical point of order 2.

4. The inner boundary of R_1 (curve 5) is mapped in a one-to-one fashion onto A_3.

5. The common boundary of R_1 and R_3 (curves 6 and 8) is mapped partly the segment of $\mathbf{R}$ between C and D_1, and also inside D_1, in a two-to-one fashion.

6. The common boundary of R_1 and D_1 (curve 7) is mapped inside D_1, making one full turn around z_1.

Therefore, the image of the region R_1 includes R_1, R_2, and R_3, as well as some points of D_1 and D_3, and the entire D_2. It follows by symmetry that the same can be said about the images of R_2 and R_3, except that the image of R_2 covers D_1 entirely and D_2 and D_3, partially, and the image of R_3 covers D_3 entirely, and D_1 and D_2, partially. We will use regions R_1, R_2 and R_3 to introduce symbolic dynamics on the Julia set J of f_ϵ. In the statement of the following theorem we include pieces of the rays $\arg z = 0$, $\arg z = \frac{\pi}{3}$ and $\arg z = -\frac{\pi}{3}$ in R_1, R_2 and R_3, correspondingly.

Theorem 7 *There is a semiconjugacy between the mapping f_ϵ on its Julia set J and the full shift on the space of sequences of three symbols. The semiconjugacy is given by associating each point of J with the sequence of the numbers of regions in which the iterates of that point lie.*

Proof The two things we need to prove are that each sequence corresponds to at least one point, and that no sequence corresponds to two different points. The first follows the fact that the image of each of the three regions covers all three. The second follows from the hyperbolicity of f_ϵ, which is established by

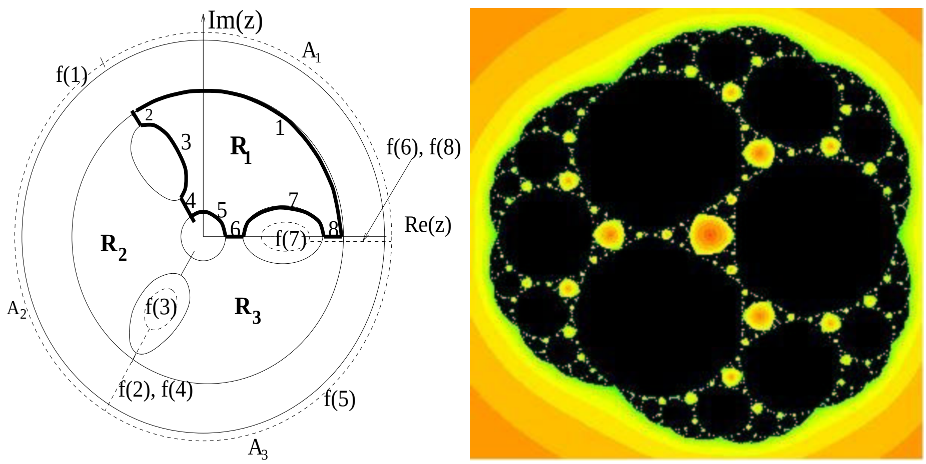

Figure 3: The image of R_1 includes all three regions R_1, R_2, and R_3. The boundary of R_1 is shown by the bold lines; the boundary of its image, by dashed lines. On the right is the set of points whose orbits do not tend to infinity.

Theorem 3, since the orbits of all three critical points are disjoint from the union of R_1, R_2 and R_3. □

We note that almost all points that lie on the boundaries between the regions R_1, R_2, and R_3 are attracted either to z_1, to $(\omega z_1, \omega^2 z_1)$, or to infinity. It follows from the symmetry (1) and from the dynamics of f on the real line that the only exception is z_2 and its symmetric images, ωz_2 and $\omega^2 z_2$. Thus there are two cases when different sequences correspond to one and the same point: a sequence ending with a series of 2's is equivalent to the same sequence ending with a series of 0's, and a sequence ending with $111111\ldots$ is identical to the same sequence ending with $020202\ldots$. All those sequences correspond to preimages of either z_2 or the period two orbit $(\omega z_2, \omega^2 z_2)$. If we exclude these cases, our semiconjugacy becomes a conjugacy.

Also, we can now describe the bifurcation at $\epsilon = 0$. In that case the Julia set is the unit circle, and the action of f_ϵ on it is semiconjugate to full shift on two symbols. For $\epsilon > 0$ the hyperbolicity of z^2 implies that there is still an invariant curve close to the unit circle which gets mapped onto itself in a two-to-one fashion. The points of that curve correspond to the three-symbol sequences that obey the following rules:

1. 0 can be followed either by 0 or by 1.

2. 1 can be followed either by 0 or by 2.

3. 2 can be followed either by 1 or by 2.

The shift map on such sequences is semiconjugate to the map $z \to z^2$ on the unit circle, because if we divide S^1 into three equal parts instead of two, we

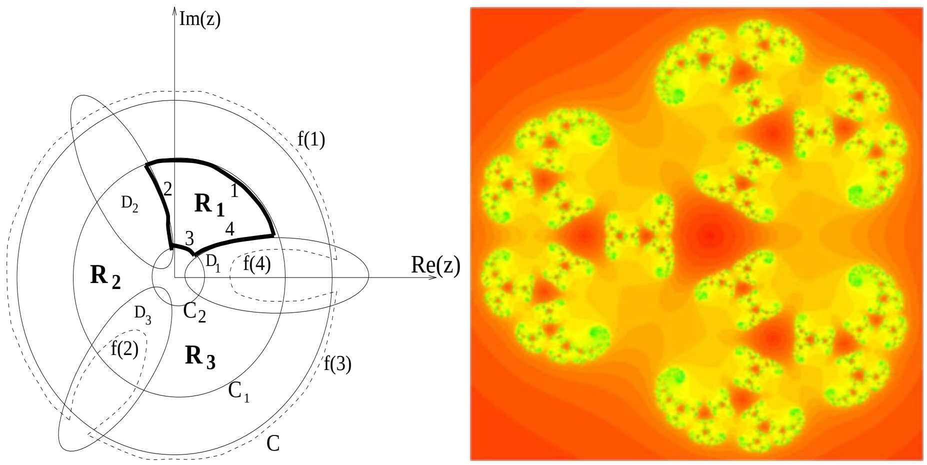

Figure 4: The left picture shows the division of $\mathbf{C}$ into regions and the image of R_1 after the bifurcation. On the right is a sketch of the Julia set of f_ϵ; the action of f_ϵ on it is conjugate to a shift on three symbols.

get precisely that subshift. Therefore, all the dynamics that existed at $\epsilon = 0$ remains there for small positive ϵ, though much more complexity is added.

Now let us consider the case $\epsilon > \epsilon_0$. As before, we can exclude the critical points from the annulus, and then divide it into R_1, R_2 and R_3 and introduce symbolic dynamics in the same way. The only difference is in the construction of D_1, D_2 and D_3. This time we want all points of D_1 to tend to infinity, and the whole D_1 to be mapped into the union of D_1 with the outside of C. That can be achieved in the same way we did it before the bifurcation (in the proof of Proposition 5), except that our initial open set, N_1, will be a tiny neighborhood of $[x, f_\epsilon(x)]$, where x is a very large positive number (x is much larger than the radius of C). Also, after we get a region that contains z_c, we take one extra connected preimage of that region and denote it as D_1. This will ensure that D_1 contains the entire intersection of $\mathbf{R}_+$ with the annulus between C_1 and C_2, and that, in turn, will show that the Julia set in this case is totally disconnected (see Figure 4).

After we construct D_1 in this way, we can construct D_2 and D_3 as its symmetric images, and then divide R into R_1, R_2 and R_3 as before. Again, we check easily that the image of each of the three regions covers the entire region R, and also some points of $\mathbf{C} - R$, which eventually escape to infinity. Therefore, we can introduce symbolic dynamics on J in the same way we did before the bifurcation.

We see that the structure of the map $f_\epsilon(z) = z^2 + \frac{\epsilon}{z}$ in the complex plane depends only on whether ϵ is less than $\frac{4}{27}$. If it is, there are fractal sets of points that get attracted to z_1, to the attracting period two orbit $(\omega z_1, \omega^2 z_1)$ and to infinity; also, the dynamics of f_ϵ on its Julia set is semiconjugate to a full shift on three symbols. If it is not, all points tend to infinity, except for the Julia set of f, on which the dynamics is fully conjugate to the full three-shift. It follows that if both ϵ_1 and ϵ_2 lie in $(0, \frac{4}{27})$, or they both lie in $(\frac{4}{27}, \infty)$, then the maps f_{ϵ_1} and f_{ϵ_2} are conjugate to each other.

2 Generalization for $z^n + \frac{\epsilon}{z^m}$, $n \geq 2$

In this section we show a way to extend our reasoning to the dynamics of $z \to z^n + \frac{\epsilon}{z^m}$ in the case $m = 1$, and to certain values of ϵ in other cases. First, let us explain what causes the dichotomy between $m = 1$ and other values of m. Let $h_\epsilon(z) = z^n + \frac{\epsilon}{z^m}$. As in section 1, one can see that h_ϵ has no more than two fixed points in $\mathbf{R}_+$, and for small values of ϵ it has exactly two. Let us denote those critical points z_1 and z_2, $z_1 < z_2$.

Proposition 8 *z_1 is attracting for all $\epsilon \in (0, \epsilon_0)$ if and only if $m = 1$.*

Proof The derivative of h_ϵ is $(n+m)z^{n-1} - \frac{m}{z}h_\epsilon(z)$, which is equal to $(n+m)z^{n-1} - m$ at the fixed points. Therefore, $h'_\epsilon(z_1)$ is greater than -1 for very small values of ϵ if and only if $m = 1$. ($h'_\epsilon(z_1)$ cannot be greater than 1 before the bifurcation, because $h_\epsilon(z) < z$ for z slightly greater than z_1.) □

The symmetry property has to be modified in the general case: it now states that

$$h_\epsilon(\alpha z) = \alpha^n h_\epsilon(z),$$

where $\alpha = \cos(\frac{2\pi}{n+m}) + i\sin(\frac{2\pi}{n+m})$ is an $(n+m)$-th root of unity. In this section we shall say that z is *symmetric* to w if $z = \alpha^k w$ for some positive integer k.

A simple computation shows that for any m and n the value of ϵ at which two fixed points coincide in a saddle-node bifurcation is

$$\epsilon_0 = \left(\frac{n-1}{n+m}\right)\left(\frac{m+1}{n+m}\right)^{\frac{m+1}{n-1}}$$

Since z_1 depends continuously on ϵ, and $h'_\epsilon(z_1)$, depends continuously on z_1 and ϵ, $h'_\epsilon(z_1)$ is a continuous function of ϵ. Therefore, since $h'_\epsilon(z_1) = 1$ for $\epsilon = \epsilon_0$, there is an interval of parameters (ϵ_1, ϵ_0) on which $h'_\epsilon(z_1)$ is greater than -1. For any ϵ from that interval z_1 is attracting.

If z_1 is attracting, or if all points of the real line are attracted to infinity, then we can describe the dynamics of f completely in the same fashion we did for $n = 2$, $m = 1$. Namely, we define the annular region containing the Julia set in the same way, exclude from it small regions containing the orbits of the critical points, and divide the remainder into $n + m$ symmetric parts, each of which is mapped over the entire region (see Figure 5). Before the bifurcation, this partition allows us to introduce a semiconjugacy between the action of h_ϵ on its Julia set and a full shift on $n+m$ symbols; the only points that correspond to more than one sequence are the preimages of the repelling fixed point and the points symmetric to it. After the bifurcation, there is an actual conjugacy between h_ϵ on its Julia set and the full shift on $n + m$ symbols, because the

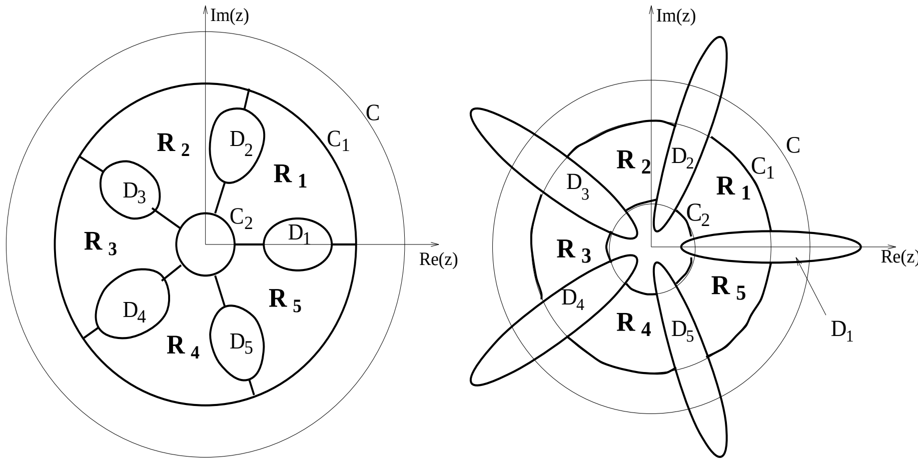

Figure 5: The partition of $\mathbf{C}$ into regions for $\epsilon < \epsilon_0$ (left) and $\epsilon > \epsilon_0$ (right), $n = 4$, $m = 1$.

Julia set does not intersect the positive real line (thus no point can correspond to two different sequences).

If $m = 1$, this completes the description of the dynamics of h_ϵ: the only bifurcation occurs at ϵ_0 (and it is of saddle-node type). Otherwise, as ϵ decreases below ϵ_1, the dynamics of h_ϵ becomes increasingly complicated.

Proposition 9 *Assume $n > 1$, $m > 1$, and m and n are not equal to two simultaneously. Then full period-doubling cascade can be observed in $\boldsymbol{R}_+$ as ϵ decreases from ϵ_0 to zero.*

Proof We will rely on a result from [2], which states that the full period-doubling cascade is present if there is a parameter value ϵ_c for which $h^2_{\epsilon_c}(z_c)$ coincides with the repelling fixed point. Our goal is thus to show that there is such an ϵ_c. We start by observing that for ϵ very close to ϵ_0, z_c is less than z_1 (because $h'_\epsilon(z_1)$ is positive). In that case h_ϵ increases on $[z_c, z_1]$, and $z_c < h_\epsilon(z_c) < h^2_\epsilon(z_c) < z_1 < z_2$. Therefore, for ϵ close to ϵ_0, $h^2_\epsilon(z_c)$ is less than z_2.

The next step is to show that if ϵ is very close to zero, then $h^2_\epsilon(z_c)$ is greater than z_2. Since h_ϵ decreases on $(0, z_c]$, it suffices to show that $h_\epsilon(z_c)$ is less than z'_2 for very small values of ϵ. Here z'_2 is, as before, the positive preimage of z_2 different from z_2.

Since z_2 is less than 1 (for positive ϵ), $\frac{\epsilon}{(z'_2)^m}$ is also less than 1, and thus z'_2 is greater than $\epsilon^{\frac{1}{m}}$. We also know that $z_c = \sqrt[m+n]{\frac{m\epsilon}{n}}$, and thus

$$h_\epsilon(z_c) = \epsilon^{\frac{n}{m+n}} \left(\sqrt[m+n]{\frac{m^n}{n^n}} + \sqrt[m+n]{\frac{n^m}{m^m}} \right)$$

If $\frac{n}{n+m}$ is greater than $\frac{1}{m}$, then for ϵ very close to zero $h_\epsilon(z_c)$ will become less than z_2' (because the former is order $\epsilon^{\frac{n}{m+n}}$, while the latter is at least order $\epsilon^{\frac{1}{m}}$).

Finally, a simple calculation yields

$$\frac{n}{n+m} > \frac{1}{m} \Leftrightarrow m(n-1) > n \Leftrightarrow m > 1 + \frac{1}{n-1}$$

which always holds if $m > 1$, $n > 1$ and m and n are not equal to two simultaneously. Therefore, by the Intermediate Value Theorem there is ϵ_c for which $h_{\epsilon_c}(z_c) = z_2$, and thus the full period-doubling cascade is present. □

We have just shown that if $n \geq 2$, $m \geq 2$, and m and n are not equal to two simultaneously, then for very small values of ϵ z_c is mapped outside of $[z_2', z_2]$, and so the dynamics of h_ϵ on $[z_2', z_2]$ is conjugate to a full shift on two symbols.

In the exceptional case $n = m = 2$, the critical point z_c never escapes $[z_2', z_2]$ (the proof is an easy computation). Nevertheless, it is likely that a period-doubling cascade is still present in this case. The existence of the first period-doubling bifurcation at $\epsilon = \frac{3}{256}$ can be easily proved analytically (by checking the non-degeneracy conditions), and further bifurcations have been observed numerically.

3 The case of complex ϵ.

We now consider again the map $f_\epsilon(z) = z^2 + \frac{\epsilon}{z}$, but this time we let ϵ be a non-zero complex number. Clearly the symmetry (1) remains valid. Since the three critical points are all symmetric to one another, they either all tend to infinity or all have bounded orbits. Therefore, it follows from Julia theorem that if at least one critical point escapes to infinity, then there are no attracting periodic cycles other than $\{\infty\}$ for f_ϵ. The set of the values of ϵ for which the orbits of the critical points are bounded has complicated structure; it is natural to give it a special name.

Definition 10 *The set $P = \{\epsilon|$ the orbits of the critical points are bounded for $f_\epsilon\}$ will be called the* pseudo-Mandelbrot set.

P is shown on Figure 6. Numerical observations show that it includes many parts homeomorphic to the Mandelbrot set; each of these parts corresponds to a period-doubling cascade in the z-plane. We will prove two theoretical results about the structure of P. The first describes the main figure-eight of the pseudo-Mandelbrot set, i.e. the set of the values of ϵ for which the map possesses an attracting fixed point (cf. the main cardioid of the Mandelbrot set).

Proposition 11 *The set of ϵ such that f_ϵ possesses an attracting fixed point is the image of the open disk $|z - \frac{1}{3}| < \frac{1}{3}$ under the map $\psi(z) = z^2 - z^3$.*

Proof. We note that the condition on the existence of an attracting fixed point z_0 is the following system of equations:

$$\begin{cases} f_\epsilon(z_0) = z_0 \\ |f_\epsilon'(z_0)| < 1 \end{cases} \Leftrightarrow \begin{cases} \epsilon = z_0^2 - z_0^3 \\ |2z_0 - \frac{\epsilon}{z_0^2}| < 1 \end{cases} \Leftrightarrow \begin{cases} |3z_0 - 1| < 1 \\ \epsilon = z_0^2 - z_0^3 \end{cases}$$

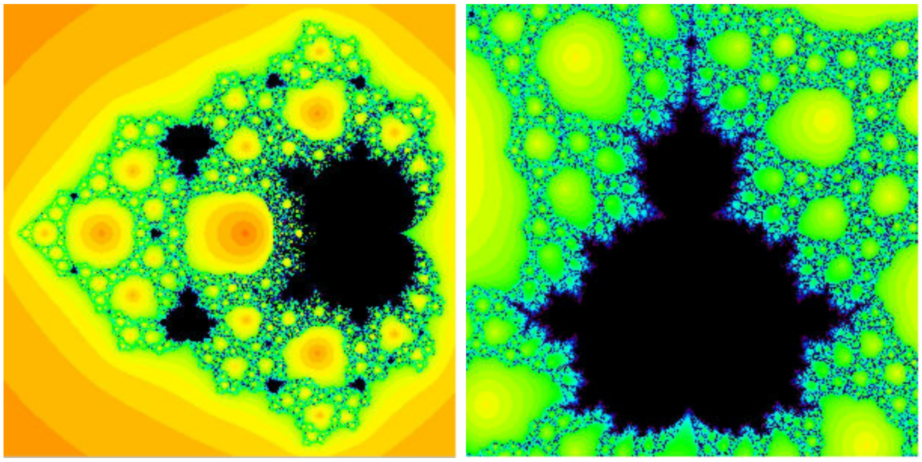

Figure 6: The pseudo-Mandelbrot set (left) includes infinitely many parts that are topologically equivalent to the Mandelbrot set. One of them is shown on the right.

which is what we need. □

In the second result we show that the pseudo-Mandelbrot set is contained within the open unit disk.

Theorem 12 *For any ϵ such that $|\epsilon| \geq 1$ infinity is a superattracting fixed point, and all three critical points lie in its immediate basin of attraction.*

Note that, by Theorem 2, it follows that the Julia set of f_ϵ is totally disconnected, and the action of f_ϵ on it is conjugate to the full shift on 3 symbols for any ϵ with $|\epsilon| \geq 1$.

To prove Proposition 12, let us first show that for any non-zero ϵ the immediate basin of attraction of infinity contains $\{z : |z| > 1 + |\epsilon|\}$.

Lemma 13 *If $|z| > 1 + |\epsilon|$, then $\lim_{n\to\infty} f_\epsilon^n(z) = \infty$.*

Proof. Suppose $|z| > 1 + |\epsilon|$. Then $|z^2| \cdot (|z| - 1)| > |\epsilon|$, because it must hold that $|z| > 1$. Therefore, $|z^2| - |z| > |\frac{\epsilon}{z}|$, and thus $|z^2| - |\frac{\epsilon}{z}| > |z|$. It follows that $|f_\epsilon(z)| \geq |z^2| - |\frac{\epsilon}{z}| > |z|$.

We see that $1 + |\epsilon| < |z| < |f_\epsilon(z)| < |f_\epsilon^2(z)| < \ldots$, i.e. the sequence of $|f_\epsilon^n(z)|$ is increasing. To show that it converges to infinity we need to prove that it does not have a finite limit. Indeed, assume the contrary is true, and $\lim_{n\to\infty} |f_\epsilon^n(z)| = l$, $|l| < \infty$. Then by the compactness of a circle the sequence $f_\epsilon^n(z)$ has an accumulation point z_0 on the circle $|z| = l$. It follows by continuity that $|f_\epsilon(z_0)| \leq |z_0|$, which is a contradiction, because $|z_0| > 1 + |\epsilon|$. Therefore, our assumption is false, and $\lim_{n\to\infty} |f_\epsilon^n(z)| = \infty$. □

Proof of Theorem 12 Fix ϵ with $|\epsilon| \geq 1$. To show that ∞ is a superattracting fixed point, we note that $\frac{1}{f_\epsilon(1/z)} = \frac{z^2}{z^3+\epsilon}$ and for that map 0 is a superattracting fixed point.

Now let z_c be a critical point of f_ϵ, i.e. $z_c = \sqrt[3]{\frac{\epsilon}{2}}$ (we do not specify which of the three cubic roots we consider, because our proof works for all three). We claim that all points of the ray $\{kz_c : k \in \mathbf{R}, k \geq 1\}$ get attracted to infinity. Once we prove that, it will follow that z_c lies in the immediate basin of attraction of infinity.

We fix $k \geq 1$ and get

$$f_\epsilon(kz_c) = \sqrt[3]{\epsilon^2}\left(\frac{k^2}{\sqrt[3]{4}} + \frac{\sqrt[3]{2}}{k}\right)$$

The function $F(k) = \frac{k^2}{\sqrt[3]{4}} + \frac{\sqrt[3]{2}}{k}$ reaches its minimum on $[1, \infty)$ at 1, and that minimum is equal to $\frac{3}{\sqrt[3]{4}}$. Therefore, $f_\epsilon(kz_c) = p\sqrt[3]{\epsilon^2}$, where $p = \frac{k^2}{\sqrt[3]{4}} + \frac{\sqrt[3]{2}}{k}$ is greater than or equal to $\frac{3}{\sqrt[3]{4}}$. Further,

$$f_\epsilon^2(kz_c) = p^2\epsilon\sqrt[3]{\epsilon} + \frac{1}{p}\sqrt[3]{\epsilon}$$

and from this and $|\epsilon| \geq 1$ we obtain

$$|f_\epsilon^2(kz_c)| \geq |p^2\epsilon + \frac{1}{p}| \geq p^2|\epsilon| - \frac{1}{p}$$

To complete the proof, we have to show that the number on the right is greater than $1+|\epsilon|$: it will then follow from Lemma 13 that $f^2(kz_c)$ lies in the immediate basin of attraction of infinity. On the other hand,

$$p^2|\epsilon| - \frac{1}{p} > 1 + |\epsilon| \Leftrightarrow |\epsilon| > \frac{1 + \frac{1}{p}}{p^2 - 1}$$

(because $|p| > 1$). The right-hand side of the second inequality reaches its maximum on $[\frac{3}{\sqrt[3]{4}}, \infty)$ at $\frac{3}{\sqrt[3]{4}}$, and that maximum is less than 0.6 . Therefore, the second inequality always holds if $|\epsilon| \geq 1$. □

References

[1] Beardon, Alan F.: *Iteration of Rational Functions.* Graduate Texts in Mathematics, vol. 132. Springer-Verlag, New York, 1991.

[2] Douady, Adrien, and Hubbard, John Hamal. *On the dynamics of polynomial-like maps.* An. scient. c. Norm. Sup., 4 srie, t. 18, 1985.

[3] Hasselblatt, Boris; and Katok, Anatole: *Modern Theory of Dynamical Systems.* Encyclopedia of Mathematics and Its Applications, vol. 54. Cambridge University Press, Cambridge, 1995.

[4] Kuznetsov, Yuri A. *Elements of Applied Bifurcation Theory.* 2nd ed. Springer-Verlag, New York, 1998.

[5] Milnor, John. *Dynamics in One Complex Variable: Introductory Lectures* 2nd ed. Vieweg Verlag, Braunschweig, 2000.