# Solutions Manual

to accompany

*A First Course in Chaotic Dynamical Systems: Theory and Experiment*

by

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# Contents

# List of Figures

# Preface

My hope is that this manual will benefit teachers and students alike, and so most of the solutions have been written in great detail, with many intermediate steps, thereby avoiding a common complaint among mathematics students. An instructor might use them to supplement his or her lecture notes, and students will certainly want to compare them with their own handwritten solutions.

The manual is reasonably complete. However, all of the chapter 8 exercises (most of which are experiments) have been neglected, as well as the material in chapter 18. Of the remaining exercises, approximately 330 have been solved. A few supplemental exercises have been added at the end of chapters 3 and 4, and to make the analysis of neutral points more straightforward, some results not in the text have been added to chapter 5. There are two appendices: a list of mathematical notation and a Map Index (that is, an index to all of the mappings in the text itself).

The solutions manual was typeset using OzTEX, Andrew Trevorrow's excellent freeware TEX implementation for the Macintosh. Most of the figures were done with *PSMathGraphsII*, and I'm grateful to the program's author, John Jacob, for adding an `Orbit` command at my request, without which many of the figures, especially those in chapter 4, would have been hopelessly difficult. I've also benefited from the productivity and generosity of many individuals, especially Larry Siebenmann, Pablo Straub, Pete Keleher, Steve Cochran, Rick Zaccone, Cameron Smith, Piet van Oostrum, and Paul Taylor, whose macro packages and applications I've come to depend on.

Most of what I know about dynamics was learned at the Regional Institute in Dynamical Systems at Boston University over the last three summers. I'd like to thank the conference organizers and the National Science Foundation for making it all possible. I'm particularly indebted to my



friends John Thoo, Larry Riddle, and Rod Shaughnessy for valuable discussions while at RIDS. But most of all, I'd like to thank Bob Devaney, who (although he may not realize it) continues to be a tremendous source of inspiration and support.

Tom Scavo
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# Chapter 3

# Orbits

## Exercises

**1.** Let $F(x) = x^2$. Compute the first five points on the orbit of 1/2.

$F^1(\frac{1}{2}) = F(\frac{1}{2}) = \frac{1}{4}$

$F^2(\frac{1}{2}) = F(\frac{1}{4}) = \frac{1}{16}$

$F^3(\frac{1}{2}) = F(\frac{1}{16}) = \frac{1}{256}$

$F^4(\frac{1}{2}) = F(\frac{1}{256}) = \frac{1}{65536}$

$F^5(\frac{1}{2}) = F(\frac{1}{65536}) = \frac{1}{4294967296}$

(Note how these fractions visually trace out a parabola!) It appears that

$$F^n\left(\frac{1}{2}\right) = \frac{1}{2^{2^n}}$$

in general, and this can be proved by induction (see Exercise 5).

**2.** Let $F(x) = x^2 + 1$. Compute the first five points on the orbit of 0.

$F^1(0) = F(0) = 1$

$F^2(0) = F(1) = 2$

$F^3(0) = F(2) = 5$

$F^4(0) = F(5) = 26$

$F^5(0) = F(26) = 677$



(See Figure 3.2 for the graph of $F$.)

**3.** Let $F(x) = x^2 - 2$. Compute $F^2(x)$ and $F^3(x)$.

The second iterate of $F$ is

$$F^2(x) = F(F(x)) = (x^2-2)^2 - 2 = x^4 - 4x^2 + 2$$

while the third iterate is given by

$$\begin{aligned} F^3(x) = F(F^2(x)) &= F(x^4 - 4x^2 + 2) \\ &= (x^4 - 4x^2 + 2)^2 - 2 \\ &= x^8 - 8x^6 + 20x^4 - 16x^2 + 2. \end{aligned}$$

$F^4$ is a lengthy 16th-degree polynomial with nine terms. It appears that $F^n$ is a $2^n$th degree polynomial with $(2^{n-1}+1)$ terms. What patterns do you see in the iterates of $F$?

**5.** Let $F(x) = x^2$. Compute $F^2(x)$, $F^3(x)$, and $F^4(x)$. What is the formula for $F^n(x)$?

We have that

$$F^2(x) = F(F(x)) = F(x^2) = (x^2)^2 = x^4$$

and

$$F^3(x) = F(F^2(x)) = F(x^4) = (x^4)^2 = x^8.$$

We also have

$$F^4(x) = F(F^3(x)) = F(x^8) = (x^8)^2 = x^{16}.$$

In general, it appears that

$$F^n(x) = x^{2^n} \tag{3.1}$$

which we will now show by induction (see Exercise 1 for a special case).

We've already verified that (3.1) holds for $n = 1, 2, 3,$ and 4, and so the base case of the inductive argument has been established. Now, suppose that the equation is true for $n := k$. Then

$$\begin{aligned} F^{k+1}(x) &= F(F^k(x)) \\ &= F(x^{2^k}) \qquad \text{by the inductive hypothesis} \\ &= (x^{2^k})^2 \\ &= x^{2\cdot 2^k} \\ &= x^{2^{k+1}} \end{aligned}$$

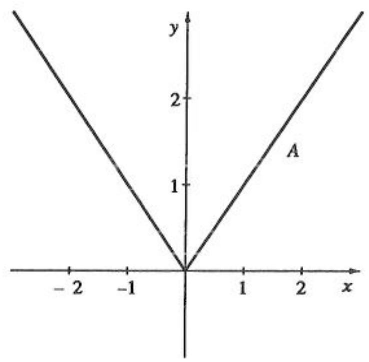



Figure 3.1: The absolute value function $A(x) = |x|$.

which proves that (3.1) holds for all $n$. Note how this argument mirrors the above computations of $F^2(x)$, $F^3(x)$, and $F^4(x)$, by the way.

**6.** Let $A(x) = |x|$. Compute $A^2(x)$ and $A^3(x)$.

By definition,

$$A(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases},$$

and so $A(x) \geq 0$ for all $x$ (see Figure 3.1). We also have that

$$A^2(x) = A(A(x)) = |A(x)| = A(x)$$

since $A(x) \geq 0$. Similarly,

$$A^3(x) = A(A^2)) = A(A(x)) = |A(x)| = A(x),$$

and in fact,

$$A^n(x) = A(x)$$

for $n \geq 1$. (Can you prove this by induction?) What does this mean? It implies that $A$ has no periodic points of prime period $n > 1$.

**7.** Find all real fixed points (if any) for each of the following functions:

7a) $F(x) = 3x + 2$

$3x + 2 = x \Rightarrow 2x = -2 \Rightarrow x = -1$. Therefore, $\text{fix}\, F = \{-1\}$.



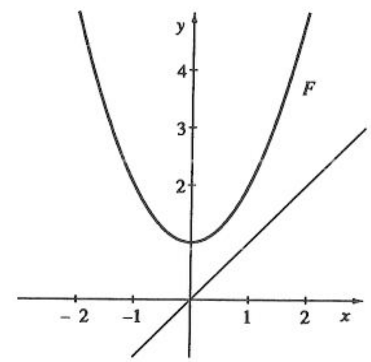



Figure 3.2: The graph of $F(x) = x^2 + 1$ is a parabola with no fixed points.

7b) $F(x) = x^2 - 2$

$x^2 - 2 = x \Rightarrow x^2 - x - 2 = 0$. Applying the quadratic formula to this second degree equation, we get

$$x = \frac{1 \pm \sqrt{1 - 4(1)(-2)}}{2} = \frac{1 \pm \sqrt{9}}{2},$$

and therefore, $\text{fix}\, F = \{-1, 2\}$.

7c) $F(x) = x^2 + 1$ (see Figure 3.2)

$x^2 + 1 = x \Rightarrow x^2 - x + 1 = 0$. Again applying the quadratic formula, we get

$$x = \frac{1 \pm \sqrt{1 - 4(1)(1)}}{2} = \frac{1 \pm \sqrt{3}\, i}{2}.$$

In this case, the fixed points are complex and the reader is referred to Chapter 15 for details concerning complex functions.

7d) $F(x) = x^3 - 3x$ (see Figure 3.3)

$x^3 - 3x = x \Rightarrow x^3 - 4x = 0 \Rightarrow x(x^2 - 4) = 0 \Rightarrow x = 0$ or $x = \pm 2$. Consequently, $\text{fix}\, F = \{0, \pm 2\}$.

7e) $F(x) = |x|$ (see Figure 3.1)

Since $|x| = x$ for nonnegative $x$, we have that $\text{fix}\, F = \{\, x \mid x \geq 0 \,\}$.

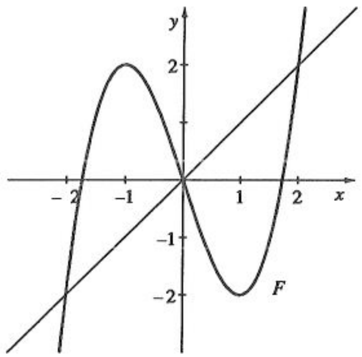



Figure 3.3: The graph of the cubic equation $F(x) = x^3 - 3x$.

7f) $F(x) = x^5$

$x^5 = x \Rightarrow x^5 - x = 0 \Rightarrow x(x^4 - 1) = 0$ which has the fairly obvious solutions $x = 0$ and $x = \pm 1$. But there are also two other (complex) solutions. Can you find them?

7g) $F(x) = x^6$

$x^6 = x \Rightarrow x^6 - x = 0 \Rightarrow x(x^5 - 1) = 0$ which has solutions $x = 0$ and $x = 1$, and four others which are not so easy to find (see Chapter 15). *Note:* The solutions to $x^n - 1 = 0$ are called the **$n$th roots of unity**.

7h) $F(x) = x \sin x$

Observe that $x \sin x = x \Rightarrow 0 = x - x \sin x = x(1 - \sin x)$. Thus, $x = 0$ or $1 - \sin x = 0$. Now, $\sin x = 1$ if $x = \pi/2$ or any $2\pi$-multiple of $\pi/2$. Thus,

$$\begin{aligned}\operatorname{fix} F &= \left\{\ldots, \frac{\pi}{2} - 4\pi, \frac{\pi}{2} - 2\pi, \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 4\pi, \ldots\right\} \cup \{0\} \\ &= \left\{\ldots, \frac{-7\pi}{2}, \frac{-3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots\right\} \cup \{0\} \\ &= \{\,(4k+1)\pi/2 \mid k \in \mathbb{Z}\,\} \cup \{0\}.\end{aligned}$$

*Note:* $F$ is an even function, that is, $F(x) = F(-x)$ for all $x$. See Figure 3.4.

**8.** What are the eventually fixed points for $A(x) = |x|$?



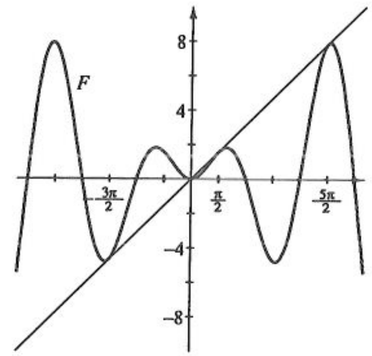



Figure 3.4: The graph of $F(x) = x \sin x$.

As shown above in Exercise 7e, $\operatorname{fix} A = \{\, x \mid x \geq 0\,\}$. But fixed points are eventually fixed with preperiod 0. The negative real numbers are also eventually fixed since each becomes positive after just one iteration. Thus, all real numbers are eventually fixed under iteration of $A$. We write

$$\overline{\operatorname{fix} A} = \mathbb{R}$$

to denote this fact.

**9.** Let $F(x) = 1 - x^2$. Show that 0 lies on a 2-cycle for this function.

This is most certainly true since $F(0) = 1$ and $F(1) = 0$.

**10.** Consider the function

$$F(x) = |x - 2| = \begin{cases} x - 2 \text{ if } x \geq 2 \\ 2 - x \text{ if } x < 2 \end{cases}.$$

See Figure 3.5 for the graph of $F$.[1]

10a) What are the fixed points for $F$?

If $x \geq 2$, then $|x - 2| = x \Rightarrow x - 2 = x$, which has no solution. On the other hand, if $x < 2$, we have that $|x - 2| = x \Rightarrow 2 - x = x \Rightarrow 2 = 2x \Rightarrow x = 1$. Therefore, $\operatorname{fix} F = \{1\}$.

[1] This is called the **slide-and-fold** dynamical system on pp. 141–142 of: Barnsley, Michael. *Fractals Everywhere*. Boston: Academic Press, 1988.

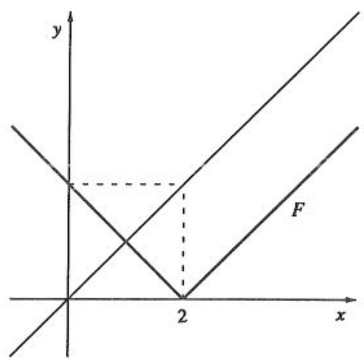



Figure 3.5: Graph of $F(x) = |x - 2|$.

10b) If $x$ is an odd integer, what can you say about the orbit of $x$?

Observe that if $x$ is an odd integer, then so is $x - 2$. Now, suppose $x$ is an odd positive integer greater than 2. Then $x - 2$ is an odd positive integer smaller than $x$, and repeated subtraction eventually produces a value of 1 which is fixed by $F$. In other words, all odd positive integers are eventually fixed. But what about odd negative $x$? Well, since $F(x) \geq 0$ for all $x$, all we need to do is apply this very same argument to $F(x)$.

10c) What happens to the orbit if $x$ is even?

Suppose $x$ is an even positive integer greater than or equal to 2. Then $x - 2$ is an even positive integer smaller than $x$. In this case, repeated subtractions eventually vanish, but $F(0) = |0 - 2| = 2$. And since $F(2) = 0$, we see that the orbit is eventually caught in a cycle of period 2. Similar arguments hold for even negative $x$.

The following four exercises deal with the doubling function $D$.

**11.** For each of the following seeds, discuss the behavior of the resulting orbit under iteration of $D$.

11a) $x_0 = 0.3$

Since $D(0.3) = 0.6$, $D(0.6) = 0.2$, $D(0.2) = 0.4$, $D(0.4) = 0.8$, and $D(0.8) = 0.6$, the orbit of 0.3 is eventually periodic with preperiod 1



and period 4. We write $0.3 \in \mathrm{per}_4^1 D$.

11b) $x_0 = 0.7$

Since $D(0.7) = 0.4$, and since $0.4 \in \mathrm{per}_4 D$ from Exercise 11a, it follows that $0.7 \in \mathrm{per}_4^1 D$.

11c) $x_0 = 1/8$

$1/8 \mapsto 2/8 \mapsto \underline{4/8 \mapsto} 1 \bmod 1 = 0$. But 0 is fixed by $D$. Therefore, $1/8 \in \mathrm{per}_1^3 D \subseteq \overline{\mathrm{fix}\, D}$.

11d) $x_0 = 1/16$

$1/16 \mapsto 2/16 \mapsto 4/16 \mapsto 8/16 \mapsto 0$. Therefore, $1/16 \in \mathrm{per}_1^4 D$.

11e) $x_0 = 1/7$

Since $D(1/7) = 2/7$, $D(2/7) = 4/7$, and $D(4/7) = 1/7$, we have that $1/7 \in \mathrm{per}_3 D$.

11f) $x_0 = 1/14$

Note that $D(1/14) = 2/14 = 1/7$. But it was shown in Exercise 11e that $1/7 \in \mathrm{per}_3 D$. Therefore, $1/14 \in \mathrm{per}_3^1 D$.

11g) $x_0 = 1/11$

Since $1/11 \mapsto 2/11 \mapsto 4/11 \mapsto 8/11 \mapsto 5/11 \mapsto 10/11 \mapsto 9/11 \mapsto 7/11 \mapsto 3/11 \mapsto 6/11 \mapsto 1/11$, we see that $1/11 \in \mathrm{per}_{10} D$.

11h) $x_0 = 3/22$

Since $D(3/22) = 6/22 = 3/11 \in \mathrm{per}_{10} D$ by Exercise 11g, we have that $3/22 \in \mathrm{per}_{10}^1 D$.

**12.** Give an explicit formula for $D^2(x)$ and $D^3(x)$. Can you write down a general formula for $D^n(x)$?

Recall that the doubling function $D$ is given by the equations

$$\begin{aligned} D(x) &= 2x \bmod 1 \\ &= \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2x - 1 & \text{if } 1/2 \leq x < 1 \end{cases} . \end{aligned}$$

See Figure 3.6 for the graph of $D$.

To derive a formula for $D^2(x)$, we partition the interval $[0, 1)$ into four parts

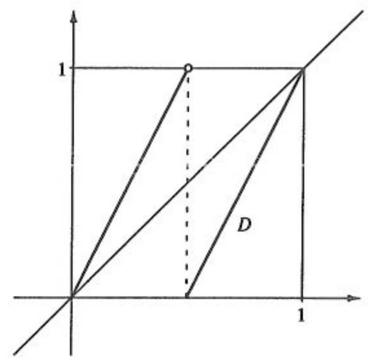



Figure 3.6: The doubling map $D(x) = 2x \bmod 1$.

and compute the image of each of these subintervals under $D^2$.

$$\begin{aligned}
0 \le x < 1/4 \quad &\Rightarrow \quad 0 \le D(x) < 1/2 \quad \Rightarrow \quad D^2(x) = D(D(x)) = D(2x) = 2(2x) = 4x\\
1/4 \le x < 1/2 \quad &\Rightarrow \quad 1/2 \le D(x) < 1 \quad \Rightarrow \quad D^2(x) = D(D(x)) = D(2x) = 2(2x) - 1 = 4x - 1\\
1/2 \le x < 3/4 \quad &\Rightarrow \quad 0 \le D(x) < 1/2 \quad \Rightarrow \quad D^2(x) = D(D(x)) = D(2x-1) = 2(2x-1) = 4x - 2\\
3/4 \le x < 1 \quad &\Rightarrow \quad 1/2 \le D(x) < 1 \quad \Rightarrow \quad D^2(x) = D(D(x)) = D(2x-1) = 2(2x-1) - 1 = 4x - 3
\end{aligned}$$

Thus, we have shown that

$$D^2(x) = \begin{cases} 4x & \text{if } 0 \le x < 1/4 \\ 4x-1 & \text{if } 1/4 \le x < 1/2 \\ 4x-2 & \text{if } 1/2 \le x < 3/4 \\ 4x-3 & \text{if } 3/4 \le x < 1 \end{cases}.$$



(See Figure 3.7a for the graph of $D^2$.) An expression for $D^3(x)$ is derived similarly, but this time we partition $[0, 1)$ into eight subintervals. The first couple of steps in the computation of $D^3(x)$ are given below.

$$\begin{aligned}
0 \le x < 1/8 \quad &\Rightarrow \quad 0 \le D(x) < 1/4 \text{ and } 0 \le D^2(x) < 1/2\\
&\Rightarrow \quad D^3(x) = D(D^2(x)) = D(4x) = 2(4x) = 8x\\
1/8 \le x < 1/4 \quad &\Rightarrow \quad 1/4 \le D(x) < 1/2 \text{ and } 1/2 \le D^2(x) < 1\\
&\Rightarrow \quad D^3(x) = D(D^2(x)) = D(4x) = 2(4x) - 1 = 8x - 1
\end{aligned}$$

Can you predict the outcome? The reader is encouraged to complete the remaining six cases. When the dust clears, you should get

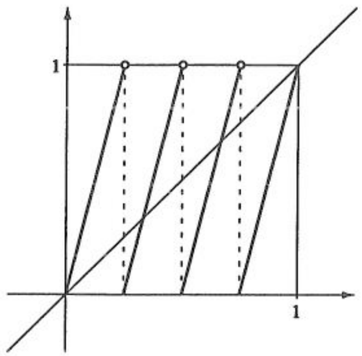
$$D^3(x) = \begin{cases} 8x & \text{if } 0 \le x < 1/8 \\ 8x-1 & \text{if } 1/8 \le x < 1/4 \\ 8x-2 & \text{if } 1/4 \le x < 3/8 \\ 8x-3 & \text{if } 3/8 \le x < 1/2 \\ 8x-4 & \text{if } 1/2 \le x < 5/8 \\ 8x-5 & \text{if } 5/8 \le x < 3/4 \\ 8x-6 & \text{if } 3/4 \le x < 7/8 \\ 8x-7 & \text{if } 7/8 \le x < 1 \end{cases}. \tag{3.2}$$

(See Figure 3.7b for the graph of $D^3$.) Observe the pattern in the expressions for $D^2(x)$ and $D^3(x)$, and then try to write down a comparable expression for $D^n(x)$. You'll find that

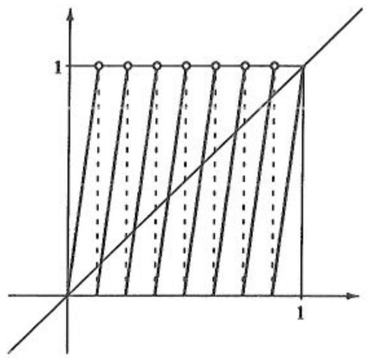
$$D^n(x) = \begin{cases} 2^n x & \text{if } 0 \le x < 1/2^n \\ 2^n x - 1 & \text{if } 1/2^n \le x < 2/2^n \\ 2^n x - 2 & \text{if } 2/2^n \le x < 3/2^n \\ \vdots & \vdots \\ 2^n x - (2^n - 1) & \text{if } (2^n-1)/2^n \le x < 1 \end{cases}.$$

Putting it more succinctly,

$$D^n(x) = 2^n x - k \quad \text{if } k/2^n \le x < (k+1)/2^n \tag{3.3}$$




(a) $D^2(x) = 4x \bmod 1$




(b) $D^3(x) = 8x \bmod 1$

Figure 3.7: The second and third iterates of the doubling map.

for $k = 0, 1, \ldots, 2^n - 1$. This is precisely the meaning of the abbreviated notation

$$D^n(x) = 2^n x \bmod 1,$$

by the way.

**13.** Sketch the graph of $D^2$ and $D^3$. What will the graph of $D^n$ look like?

See Figures 3.7a and 3.7b for the graphs of $D^2$ and $D^3$, respectively. In general, $D^n$ consists of $2^n$ linear pieces of slope $2^n$, with $2^n - 1$ discontinuities at each of the rational numbers in $\{\, p/2^n \mid 0 < p < n \,\}$.

**14.** Using your answer to Exercise 12, find all fixed points for $D^2$ and $D^3$? How many fixed points does $D^4$ and $D^5$ have? What about $D^n$?

We've already seen that $D$ itself has but a single fixed point. We will show below that $D^2$ has three fixed points, while $D^3$ has seven. Apparently, all we need to do is double the previous number of fixed points, and then add 1. Indeed, we can show (by induction, of course!) that $D^n$ has $2^n - 1$ fixed points.

From the graph of $D$, we see that $\operatorname{fix} D = \{0\}$.



Next we have that $\operatorname{fix} D^2 = \operatorname{per}_2 D = \{0, \frac{1}{3}, \frac{2}{3}\}$ since

$$\begin{aligned} 4x = x &\Rightarrow x = 0 \\ 4x - 1 = x &\Rightarrow x = 1/3 \\ 4x - 2 = x &\Rightarrow x = 2/3 \\ 4x - 3 = x &\Rightarrow x = 1 \end{aligned}$$

(Hmm... This seems to be saying that $x = 1$ is a fixed point of $D^2$. Is this true?) Similarly, we see that $\operatorname{fix} D^3 = \operatorname{per}_3 D = \{0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}\}$ since

$$\begin{aligned} 8x = x &\Rightarrow x = 0 \\ 8x - 1 = x &\Rightarrow x = 1/7 \\ 8x - 2 = x &\Rightarrow x = 2/7 \\ 8x - 3 = x &\Rightarrow x = 3/7 \\ 8x - 4 = x &\Rightarrow x = 4/7 \\ 8x - 5 = x &\Rightarrow x = 5/7 \\ 8x - 6 = x &\Rightarrow x = 6/7 \\ 8x - 7 = x &\Rightarrow x = 1 \end{aligned}$$

(Once again we arrive at the solution $x = 1$. Maybe we should define $D\colon [0,1] \to [0,1]$ with $D(1) = 1$ for consistency?[2]) From Equation 3.3, we see that

$$\operatorname{fix} D^n = \operatorname{per}_n D = \{\, k/(2^n - 1) \mid 0 \le k < 2^n - 1 \,\}.$$

For example, when $n = 4$, we obtain

$$\operatorname{per}_4 D = \left\{0, \frac{1}{15}, \frac{2}{15}, \ldots, \frac{14}{15}\right\},$$

but not all of these are of *prime* period 4. In fact, there are three distinct orbits of prime period 4. (Can you find them?) Also note that

$$\operatorname{fix} D = \operatorname{per}_1 D \subset \operatorname{per}_2 D \subset \operatorname{per}_4 D$$

which is not all that surprising when you stop to think about it.

[2] Another reason why we might want to define $D$ on the *closed* interval $[0, 1]$ is that later on we'll be able to show that $D$ is conjugate to the tent map $T$ via $T$ itself! See Exercises 15–19 for an introduction to the tent map.

But you may be surprised to learn that *every* rational number in $[0,1)$ is eventually periodic. And even more interesting is the fact that few computer programs are able to find them! See Experiment 3.6 in the text for an explanation of this strange phenomenon.

The following five exercises deal with the function

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2-2x & \text{if } 1/2 < x \le 1 \end{cases} .$$

The function $T$ is called a **tent map** because of the shape of its graph on the interval $[0,1]$. See Figure 3.8.

**15.** Find a formula for $T^2(x)$.

The trick is to partition $[0,1]$ into four equal length subintervals:

$$0 \le x < 1/4 \quad \Rightarrow \quad 0 \le T(x) \le 1/2 \quad \Rightarrow \quad \begin{aligned} T^2(x) &= T(T(x)) \\ &= T(2x) \\ &= 2(2x) \\ &= 4x \end{aligned}$$

$$1/4 \le x < 1/2 \quad \Rightarrow \quad 1/2 \le T(x) \le 1 \quad \Rightarrow \quad \begin{aligned} T^2(x) &= T(T(x)) \\ &= T(2x) \\ &= 2-2(2x) \\ &= 2-4x \end{aligned}$$

$$1/2 \le x < 3/4 \quad \Rightarrow \quad 1/2 \le T(x) \le 1 \quad \Rightarrow \quad \begin{aligned} T^2(x) &= T(T(x)) \\ &= T(2-2x) \\ &= 2-2(2-2x) \\ &= 4x-2 \end{aligned}$$

$$3/4 \le x < 1 \quad \Rightarrow \quad 0 \le T(x) \le 1/2 \quad \Rightarrow \quad \begin{aligned} T^2(x) &= T(T(x)) \\ &= T(2-2x) \\ &= 2(2-2x) \\ &= 4-4x \end{aligned}$$

Thus, we have shown that

$$T^2(x) = \begin{cases} 4x & \text{if } 0 \le x \le 1/4 \\ 2-4x & \text{if } 1/4 \le x \le 1/2 \\ 4x-2 & \text{if } 1/2 \le x \le 3/4 \\ 4-4x & \text{if } 3/4 \le x \le 1 \end{cases} .$$



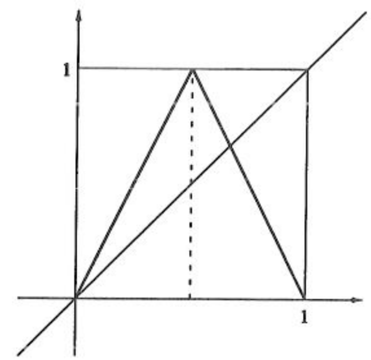
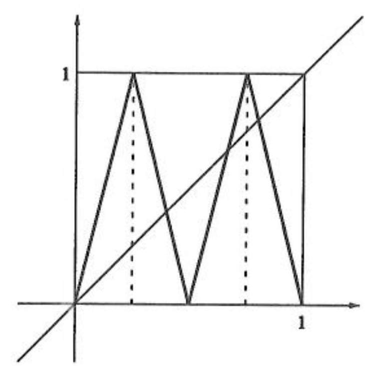
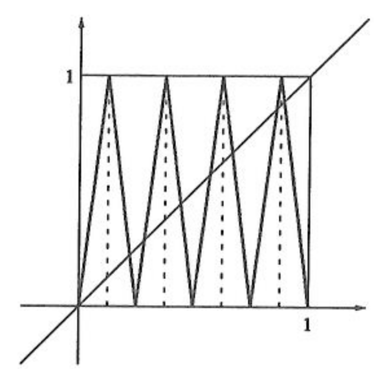



Figure 3.8: The tent map $T(x) = 1 - |2x-1|$.




(a) Graph of $T^2$.




(b) Graph of $T^3$.

Figure 3.9: The second and third iterates of the tent map.

**16.** Sketch the graphs of $T$ and $T^2$.

See Figure 3.8 for the graph of $T$, and Figure 3.9a for the graph of $T^2$.

**17.** Find all fixed points for $T$ and $T^2$.

To find the fixed points for $T$, simply set each piece of $T(x)$ equal to $x$ and solve:

$$\begin{aligned} & 2x = x \\ \Rightarrow\ & 2x - x = 0 \\ \Rightarrow\ & x = 0 \end{aligned} \qquad \begin{aligned} & 2 - 2x = x \\ \Rightarrow\ & 2 = 3x \\ \Rightarrow\ & 2/3 = x \end{aligned}$$

Check:

$$\begin{aligned} T(0) &= 0 \quad \checkmark \\ T(2/3) &= 2 - 2(2/3) = 2 - 4/3 = 2/3 \quad \checkmark \end{aligned}$$

Now set each piece of $T^2$ equal to $x$ and solve:

$$\begin{aligned} & 4x = x \\ \Rightarrow\ & 3x = 0 \\ \Rightarrow\ & x = 0 \end{aligned} \qquad \begin{aligned} & 2 - 4x = x \\ \Rightarrow\ & 2 = 5x \\ \Rightarrow\ & 2/5 = x \end{aligned}$$

$$\begin{aligned} & -2 + 4x = x \\ \Rightarrow\ & 3x = 2 \\ \Rightarrow\ & x = 2/3 \end{aligned} \qquad \begin{aligned} & 4 - 4x = x \\ \Rightarrow\ & 4 = 5x \\ \Rightarrow\ & 4/5 = x \end{aligned}$$

Check:

$$\begin{aligned} T(2/5) &= 2(2/5) = 4/5 \quad \checkmark \\ T(4/5) &= 2 - 2(4/5) = 2 - 8/5 = 2/5 \quad \checkmark \end{aligned}$$

The other two are actually fixed points for $T$ and were checked above.

**18.** Find an explicit formula for $T^3(x)$ and sketch the graph of $T^3$.



Half of the eight cases are worked out in detail below.

$$\begin{aligned}
0 \le x \le 1/8 \quad &\Rightarrow \quad 0 \le T(x) \le 1/4 \text{ and } 0 \le T^2(x) \le 1/2 \\
&\Rightarrow \quad T^3(x) = T(T^2(x)) \\
&\qquad\quad = T(4x) \\
&\qquad\quad = 2(4x) \\
&\qquad\quad = 8x \\
1/8 \le x \le 1/4 \quad &\Rightarrow \quad 1/4 \le T(x) \le 1/2 \text{ and } 1/2 \le T^2(x) \le 1 \\
&\Rightarrow \quad T^3(x) = T(T^2(x)) \\
&\qquad\quad = T(4x) \\
&\qquad\quad = 2 - 2(4x) \\
&\qquad\quad = 2 - 8x \\
1/4 \le x \le 3/8 \quad &\Rightarrow \quad 1/2 \le T(x) \le 3/4 \text{ and } 1/2 \le T^2(x) \le 1 \\
&\Rightarrow \quad T^3(x) = T(T^2(x)) \\
&\qquad\quad = T(2 - 4x) \\
&\qquad\quad = 2 - 2(2 - 4x) \\
&\qquad\quad = -2 + 8x \\
3/8 \le x \le 1/2 \quad &\Rightarrow \quad 3/4 \le T(x) \le 1 \text{ and } 0 \le T^2(x) \le 1/2 \\
&\Rightarrow \quad T^3(x) = T(T^2(x)) \\
&\qquad\quad = T(2 - 4x) \\
&\qquad\quad = 2(2 - 4x) \\
&\qquad\quad = 4 - 8x
\end{aligned}$$

The student should complete the remaining four cases. Ultimately, we get the following monstrous expression for $T^3(x)$:

$$T^3(x) = \begin{cases} 8x & \text{if } 0 \le x < 1/8 \\ 2 - 8x & \text{if } 1/8 \le x < 1/4 \\ 8x - 2 & \text{if } 1/4 \le x < 3/8 \\ 4 - 8x & \text{if } 3/8 \le x < 1/2 \\ 8x - 4 & \text{if } 1/2 \le x < 5/8 \\ 6 - 8x & \text{if } 5/8 \le x < 3/4 \\ 8x - 6 & \text{if } 3/4 \le x < 7/8 \\ 8 - 8x & \text{if } 7/8 \le x < 1 \end{cases}.$$

Notice how every other piece of this multi-part function agrees exactly with the corresponding piece of $D^3(x)$ computed in Exercise 12 (see Equation 3.2). See Figure 3.9b for the graph of $T^3$.

**19.** What does the graph of $T^n$ look like?

The graph of $T^n$ has $2^{n-1}$ symmetric "spikes" with peaks at $x = p/2^n$ where $p$ is an odd integer. The sides of each spike have slope $\pm 2^n$. Finally, note that $T^n$ is not differentiable at the $2^{n+1}$ points contained in $\{ p/2^n \mid 0 \le p \le 2^n \}$.

The following additional exercises suggest a relationship between the doubling map $D$ and the tent map $T$.

**20.** Compute $T \circ D(x)$ and compare with $T^2(x)$ computed in Exercise 15.

**21.** Compute $T \circ D \circ D(x)$ and compare with $T^3(x)$ computed in Exercise 18.

# Chapter 4

# Graphical Analysis

## Exercises

**1.** Use graphical analysis to describe the fate of all orbits for each of the following functions. Use different colors for orbits which behave differently.

1a) $F(x) = 2x$

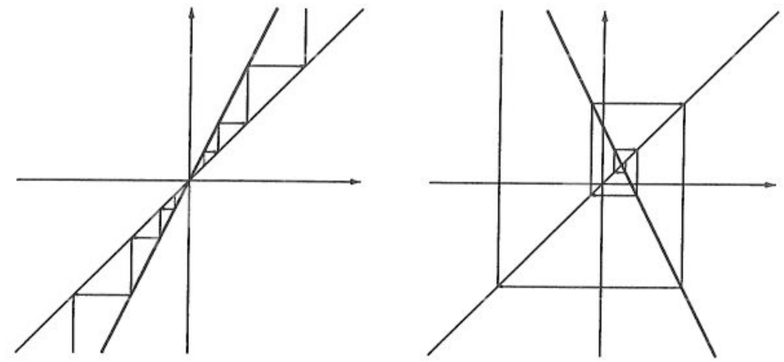
Figure 4.1a clearly shows that $F$ is an odd function,[1] that $\operatorname{fix} F = \{0\}$, and that all orbits tend to $\pm\infty$ (except the orbit of 0 which is fixed).

1b) $F(x) = \frac{1}{3x}$

First of all, note that $x > 0$ implies that $F^n(x) > 0$ for all $n$. Likewise, $x < 0$ implies that $F^n(x) < 0$ for all $n$. Thus orbits are destined to remain in their quadrant of origin for all time (see Figure 4.2). Secondly, not only is $F$ an odd function, but it's also its own inverse. Finally, an easy calculation shows that $\operatorname{fix} F = \{\pm\sqrt{3}/3\}$, and that $F(F(x)) = x$ for all $x$, that is, $\operatorname{per}_2 F = \mathbb{R}$. The function $F$ is an example of what we call a **totally periodic** function.

These results may be generalized somewhat. Let $F(x) = k/x$ for any $k \neq 0$. It can be shown that $\operatorname{fix} F = \{\pm\sqrt{k}\}$ and that $F^2(x) = x$ for all $x$.

[1] An odd function is one for which $F(-x) = -F(x)$ for all $x$. Geometrically speaking, imagine folding the graph along the $x$-axis and then along the $y$-axis, so that the two parts of the curve lie one on top the other.



(a) The graph of $F(x) = 2x$. (b) The graph of $F(x) = -2x + 1$.

Figure 4.1: Two cases of linear maps with repelling fixed points.

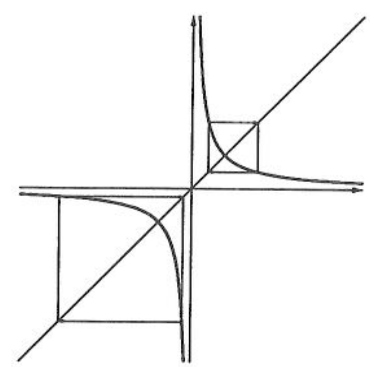
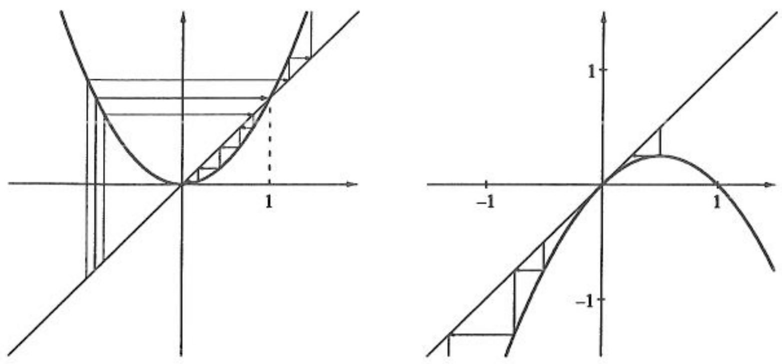


Figure 4.2: The graph of $F(x) = \frac{1}{3x}$, a typical reciprocal function.




(a) The basic quadratic $F(x) = x^2$. (b) The graph of $F(x) = x - x^2$.

Figure 4.3: A pair of quadratic functions.

1c) $F(x) = -2x + 1$ (see Figure 4.1b)

We have that $\operatorname{fix} F = \{1/3\}$ since

$$-2x + 1 = x \Rightarrow 1 = 3x \Rightarrow x = 1/3.$$

It's also true that $|F^n(x)| \to \infty$ for all $x$ except the fixed point. Note the qualitative difference between this linear dynamical system and its cousin in Exercise 1a; that is, as the orbit moves away from the fixed point at $x = 1/3$, it begins to alternate sign and continues to do so for all time.

1d) $F(x) = x^2$

This map was briefly discussed in Section 4.3 of the text where it was pointed out that $\operatorname{fix} F = \{0, 1\}$ and that $-1$ is eventually fixed. In fact,

$$\overline{\operatorname{fix} F} = \{0, \pm 1\}.$$

Now, suppose $x$ is nonnegative. Then the following statements are true:

i. If $x > 1$, then $F^n(x) \to \infty$ as $n \to \infty$.

ii. If $0 \le x < 1$, then $F^n(x) \to 0$ as $n \to \infty$.



iii. If $x = 1$, then $F^n(1) = 1$ for all $n$ (that is, $x = 1$ is a fixed point for $F$, as we've already seen).

But what about negative $x$? Once again there are a number of cases to consider:

iv. If $x < -1$, then $F(x) > 1$. Hence, by (i) above, $F^n(x) \to \infty$ as $n \to \infty$.[2]

v. Now, if $-1 < x \le 0$, then $-x > x^2 \ge 0$. (This is obtained by multiplying through by $x$, which is negative.) But $-x < 1$, and so we have that $0 \le x^2 < 1$. Hence, $F^n(x) \to 0$ as $n \to \infty$ by (ii) above.

vi. When $x = -1$, we see that $F(x) = 1$. But $1 \in \operatorname{fix} F$ and so $x = -1$ is eventually fixed.

Summarizing these results, we write

$$\lim_{n\to\infty} F^n(x) = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } |x| = 1 \\ \infty & \text{if } |x| > 1 \end{cases} . \tag{4.1}$$

We remark that the behavior of $F^n(x)$ for negative $x$ is analogous to that for positive $x$ since one iteration of $F$ maps its argument directly into the *positive* real numbers (i.e. $F(x) = x^2 > 0$ for all $x$). See Figure 4.3a. The importance of this exercise can not be overemphasized.

1e) $F(x) = -x^3$

Here's another odd function (see Figure 4.4), this time with $\operatorname{fix} F = \{0\}$. We also see that $F^2(x) = x^9$, and since

$$\begin{aligned} & x^9 = x \\ \Rightarrow\ & x^9 - x = 0 \\ \Rightarrow\ & x(x^8 - 1) = 0, \end{aligned}$$

we have that $\pm 1 \in \operatorname{per}_2 F$. There are other period 2 points—the remaining 8th roots of unity—but these are complex. Anyway, it turns out that the fixed point is attracting while the 2-cycle is repelling (see

[2] To be precise, it's the orbit of $F(x)$ that tends to $\infty$. But if $F^n(F(x)) \to \infty$, then so does $F^n(x)$ since it's the tail of the sequence that determines convergence.

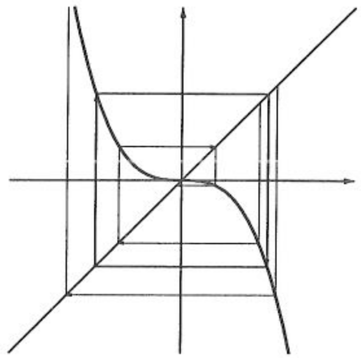


Figure 4.4: The graph of the cubic $F(x) = -x^3$.

Chapter 5), and so

$$\lim_{n\to\infty} |F^n(x)| = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } |x| = 1 \\ \infty & \text{if } |x| > 1 \end{cases} . \qquad (4.2)$$

Compare (4.2) with (4.1): the absolute value signs are required in the former since points along the orbits alternate in sign.

1f) $F(x) = x - x^2$

As seen in Figure 4.3b, this function has a single fixed point at the origin. If $0 < x < 1$, then $F^n(x)$ is a positive decreasing sequence of points converging to 0. On the other hand, if $x < 0$, then $F^n(x) \to -\infty$ as $n \to \infty$. And if $x > 1$, it follows that $F(x) < 0$, and consequently, $F^n(F(x)) \to -\infty$ as $n \to \infty$. Finally, note that $x = 1$ is eventually fixed after one iteration.

Experiments indicate that for $0 < x < 1$, the orbit of $x$ converges to 0 rather slowly. Similarly, for negative $x$ very close to 0, $F^n(x)$ *slowly* moves away from the origin. The explanation of this behavior is that the fixed point is **neutral**, an important characteristic discussed in more detail in Chapter 5.

1g) $S(x) = \sin x$

Looking at Figure 4.5 it appears that all orbits tend to 0, which is true, but experiments suggest that the convergence is very slow. (See Section 3.2 of the text for an illustration of this fact.)



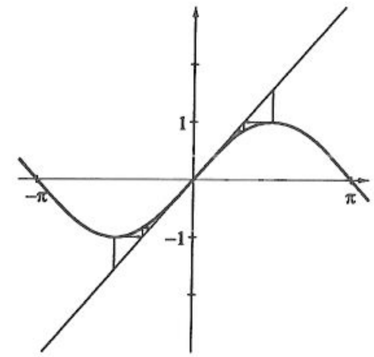



Figure 4.5: The graph of the sine function.

**2.** Use graphical analysis to find $\{ x \mid F^n(x) \to \pm\infty \}$ for each of the following functions.

2a) $F(x) = 2x(1-x)$ (see Figure 4.6a)

For $0 < x < 1$, $F^n(x) \to 1/2$. But when $x < 0$, $F^n(x) \to -\infty$. Now, if $x > 1$, it follows that $F(x) < 0$, and so $F^n(F(x)) \to -\infty$ as well. Thus, $\{ x \mid F^n(x) \to \pm\infty \} = \{ x \mid x < 0 \text{ or } x > 1 \}$. Note that $x = 1$ is eventually fixed after one iteration.

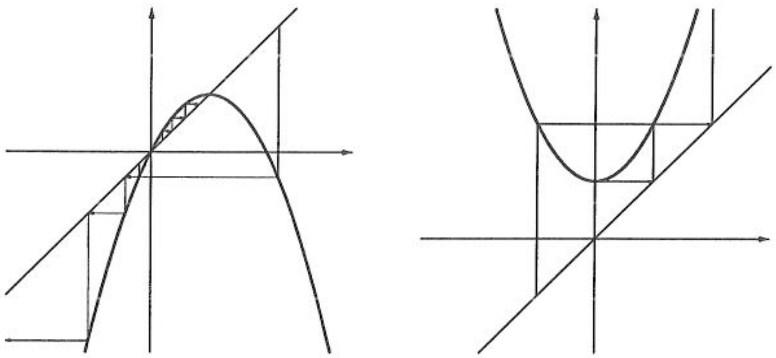
2b) $F(x) = x^2 + 1$ (see Figure 4.6b)

In this case, we have that $F^n(x) \to \infty$ for *all* $x$.

2c) $T(x) = \begin{cases} 2x & \text{if } x \le 1/2 \\ 2 - 2x & \text{if } x > 1/2 \end{cases}$

The map depicted in Figure 4.7 is called the **tent map**. When $x < 0$, $T^n(x) \to -\infty$. When $x > 1$, $T(x) < 0$, and so $T^n(T(x)) \to -\infty$ as $n \to \infty$ as well. But when $0 < x < 1$, $0 < T^n(x) < 1$ for *all* $n$. Thus, $\{ x \mid F^n(x) \to \pm\infty \} = \{ x \mid x < 0 \text{ or } x > 1 \}$. We remark that $x = 1$ is eventually fixed after one iteration since $T(1) = 0$, and 0 is fixed.

**3.** Sketch the phase portraits for each of the functions in Exercise 1.



(a) The graph of $F(x) = 2x(1-x)$. (b) The graph of $F(x) = x^2 + 1$.

Figure 4.6: The dynamics of two more quadratic functions.

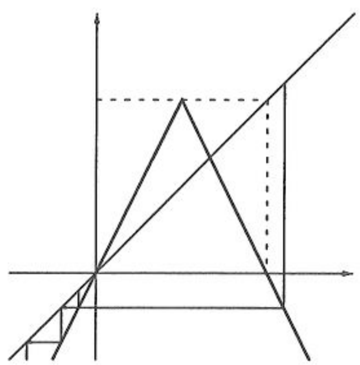


Figure 4.7: The dynamics of the tent map $T(x) = 1 - |2x - 1|$.



See Figure 4.8.

**4.** Perform a complete orbit analysis for each of the following functions.

4a) $F(x) = \frac{1}{2}x - 2$

First, let's find a fixed point for $F$:

$$\begin{aligned} \frac{1}{2}x - 2 &= x \\ \Rightarrow\ -2 &= \frac{1}{2}x \\ \Rightarrow\ -4 &= x. \end{aligned}$$

Thus, $\text{fix}\, F = \{-4\}$, and it so happens that $F^n(x) \to -4$ as $n \to \infty$ for *all* $x$. See Figure 4.9.

4b) $A(x) = |x|$

This exercise was essentially solved at the end of Chapter 3 (see Exercises 6, 7e, and 8) where it was shown that

$$\text{fix}\, A = \{\, x \mid x \geq 0 \,\},$$

and that all other points are eventually fixed after just one iteration.

4c) $F(x) = -x^2$

From the graph of $F$ in Figure 4.10, we see that $\text{fix}\, F = \{-1, 0\}$. Since $F(1) = -1$, we also see that $x = 1$ is eventually fixed. For $-1 < x < 1$, $F^n(x) \to 0$ as $n \to \infty$, and for $|x| > 1$, $F^n(x) \to -\infty$ as $n \to \infty$. That is, 0 is attracting and $-1$ is repelling.

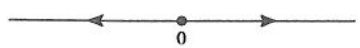
4d) $F(x) = -x^5$

First, let's compute the fixed points of $F$:

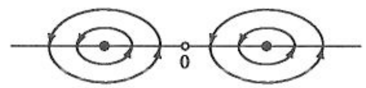
$$\begin{aligned} -x^5 &= x \\ \Rightarrow\ 0 &= x + x^5 \\ \Rightarrow\ 0 &= x(1 + x^4). \end{aligned}$$

Therefore, $0 \in \text{fix}\, F$ and the other four fixed points are complex. Now,

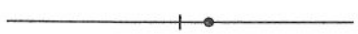
$$F'(x) = -5x^4$$



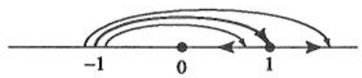

(a) $F(x) = 2x$ has a globally repelling fixed point at the origin.



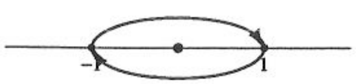

(b) $F(x) = 1/(3x)$ has fixed points at $\pm\sqrt{3}/3$, and all nonzero real numbers are of period 2.



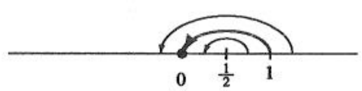
(c) $F(x) = -2x + 1$ also has a globally repelling fixed point, but orbits oscillate in this case.



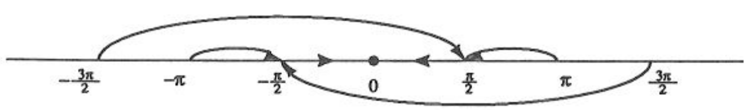

(d) $F(x) = x^2$ has two types of orbits: those that approach 0 from the right, and those that escape to $+\infty$.




(e) $F(x) = -x^3$ has a repelling 2-cycle. (Compare with Figure 4.7 in the text.)




(f) $F(x) = x - x^2$ has a neutral fixed point which is weakly repelling on the left and weakly attracting on the right.




(g) $S(x) = \sin x$ has a globally, but weakly attracting fixed point at the origin.

Figure 4.8: Phase portraits.



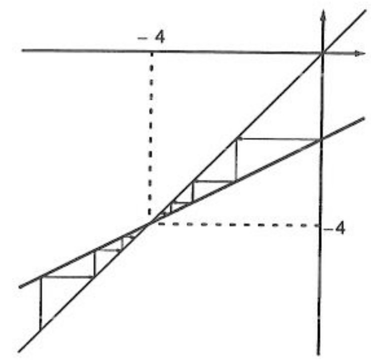



Figure 4.9: The straight line $F(x) = x/2 - 2$.

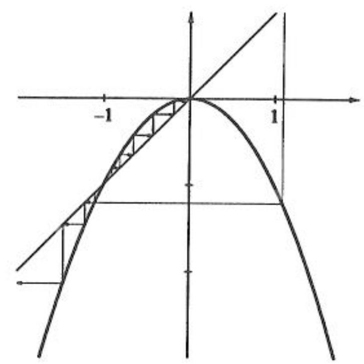



Figure 4.10: The graph of $F(x) = -x^2$.

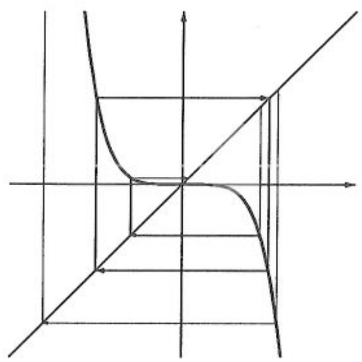


Figure 4.11: The graph of $F(x) = -x^5$.

and so 0 is superattracting since $F'(0) = 0$. Note also that 1 and $-1$ constitute a repelling 2-cycle for $F$. This suggests that $|F^n(x)| \to 0$ as $n \to \infty$ whenever $|x| < 1$. All other orbits escape to $\infty$. See Figure 4.11.

4e) $F(x) = 1/x$

Reciprocal functions of the form $F(x) = k/x$ were solved in general in Exercise 1b where it was shown that all real numbers are of period 2.

4f) $E(x) = e^x$

As shown in Figure 4.12 all orbits quickly escape to infinity under iteration of this exponential map.

**5.** Let $F(x) = |x-2|$. Use graphical analysis to display a variety of orbits of $F$. Use red to display cycles of period 2, blue for eventually fixed orbits, and green for orbits which are eventually periodic.

Recall from Exercise 10 at the end of Chapter 3 that $\operatorname{fix} F = \{1\}$ and that all odd integers are eventually fixed. In fact, these are the only eventually fixed points for $F$.

It was also shown earlier that $\{0, 2\} \subset \operatorname{per}_2 F$ and that all even integers are eventually periodic with prime period 2. In fact,

$$\operatorname{per}_2 F = \{\, x \mid 0 \le x \le 2 \,\},$$



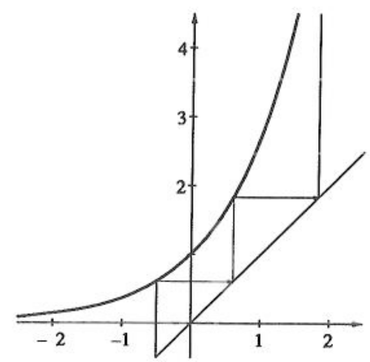



Figure 4.12: The exponential function $E(x) = e^x$.

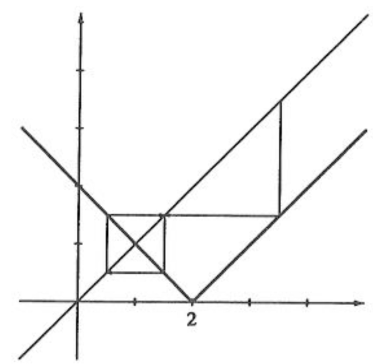



Figure 4.13: Orbits of the slide-and-fold dynamical system $F(x) = |x-2|$.

and moreover, every real number is eventually periodic with period 2. To see this, first suppose that $x > 2$. Then there exists an $m$ such that $0 < F^m(x) \le 2$ (see the detailed argument in Exercise 3.10), and therefore, $F^m(x) \in \mathrm{per}_2 F$. Consequently, $x \in \mathrm{per}_2^m F$. In the event that $x < 0$, we have that $F(x) > 2$, and so $F(x)$ is eventually periodic via a similar argument. See Figure 4.13 for some typical orbits.

**6.** Consider $F(x) = x^2 - 1.1$. First find the fixed points of $F$. Then use the fact that these points are also solutions of $F^2(x) = x$ to find the cycle of prime period 2 for $F$.

We begin by computing

$$\begin{aligned} & x^2 - 1.1 = x \\ \Rightarrow\ & x^2 - x - 1.1 = 0 \\ \Rightarrow\ & x = \frac{1 \pm \sqrt{1-(4)(1)(-1.1)}}{2} \end{aligned}$$

which implies the fixed points of $F$ are

$$\frac{1 \pm \sqrt{5.4}}{2}.$$

Just to be sure, we had better check our work:

$$\begin{aligned} F\left(\frac{1 \pm \sqrt{5.4}}{2}\right) &= \left(\frac{1 \pm \sqrt{5.4}}{2}\right)^2 - 1.1 \\ &= \frac{1 \pm 2\sqrt{5.4} + 5.4}{4} - 1.1 \\ &= \frac{6.4 \pm 2\sqrt{5.4} - 4.4}{4} \\ &= \frac{2 \pm 2\sqrt{5.4}}{4} \\ &= \frac{1 \pm \sqrt{5.4}}{2} \qquad \checkmark \end{aligned}$$

Next, let's compute the second iterate of $F$,

$$\begin{aligned} F^2(x) &= (x^2 - 1.1)^2 - 1.1 \\ &= x^4 - 2.2x^2 + 1.21 - 1.1 \\ &= x^4 - 2.2x^2 + 0.11, \end{aligned}$$



and *its* fixed points (which are also the period 2 points of $F$):

$$\begin{aligned} & x^4 - 2.2x^2 + 0.11 = x \\ \Rightarrow\ & x^4 - 2.2x^2 - x + 0.11 = 0 \\ \Rightarrow\ & (x^2 - x - 1.1)(x^2 + x - 0.1) = 0 \\ \Rightarrow\ & x^2 - x - 1.1 = 0 \quad \text{or} \quad x^2 + x - 0.1 = 0 \\ \Rightarrow\ & x = \frac{1 \pm \sqrt{5.4}}{2} \quad \text{or} \quad x = \frac{-1 \pm \sqrt{1.4}}{2}. \end{aligned}$$

But, you ask, how could *anyone* know how to factor a messy fourth degree polynomial such as that? Well, we know that any fixed point is also a period 2 point, right?[3] This means that the solutions to $x^2 - x - 1.1 = 0$ must also be solutions to $x^4 - 2.2x^2 - x + 0.11 = 0$, and suggests we compute

$$\frac{x^4 - 2.2x^2 - x + 0.11}{x^2 - x - 1.1} = x^2 + x - 0.1$$

by polynomial long division, say. (Thought you'd never use your college algebra, eh?) Study this technique carefully—it will prove invaluable in the sequel.

**7.** All of the following exercises deal with the dynamics of linear functions of the form $F(x) = ax + b$ where $a$ and $b$ are constants.

7a) Find the fixed points of $F(x) = ax + b$.

We have that

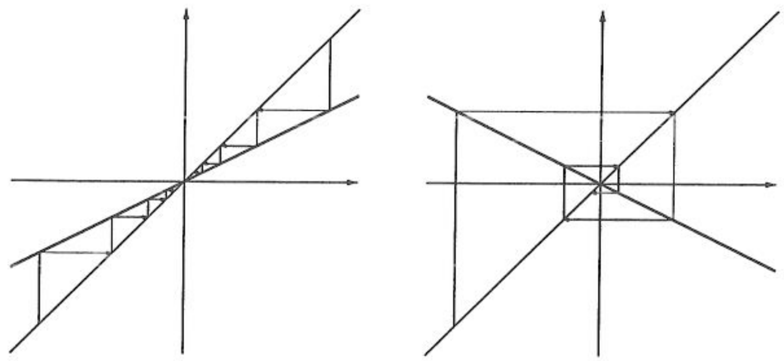
$$\begin{aligned} & ax + b = x \\ \Rightarrow\ & b = x - ax \\ \Rightarrow\ & b = x(1-a) \\ \Rightarrow\ & \frac{b}{1-a} = x \end{aligned}$$

provided $a \ne 1$. In other words, $\mathrm{fix}\, F = \{b/(1-a)\}$.

7b) For which values of $a$ and $b$ does $F$ have no fixed points?

$F$ has no fixed point when its graph is distinct from and parallel to the diagonal line $y = x$, that is, when $a = 1$ and $b \ne 0$.

[3] Actually, a fixed point is a period $n$ point for *any* $n$.



(a) $0 < a < 1$ (b) $-1 < a < 0$

Figure 4.14: Linear maps of the form $F(x) = ax + b$ with attracting fixed points.

7c) For which values of $a$ and $b$ does $F$ have infinitely many fixed points?

$F$ is the identity map when $a = 1$ and $b = 0$; in this case, $\text{fix}\, F = \mathbb{R}$.

7d) For which values of $a$ and $b$ does $F$ have exactly one fixed point?

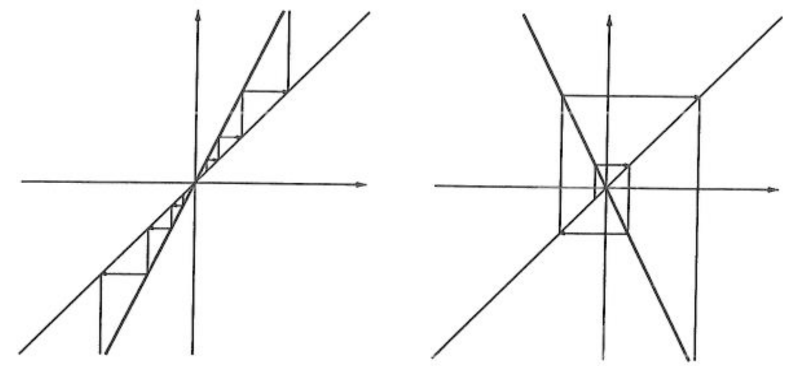
Regardless of $b$, $F$ has exactly one fixed point if and only if $a \neq 1$.

7e) Suppose $F$ has just one fixed point and $0 < |a| < 1$. Using graphical analysis, what can you say about the behavior of all other orbits of $F$? We will call these fixed points **attracting** fixed points later. Why do we use this terminology?

While it's true that all orbits tend toward the fixed point, there are two very important subcases to consider and these are illustrated in Figure 4.14. For $0 < a < 1$, and for all $x$, $F^n(x)$ is a monotonic sequence of points[4] converging to $b/(1-a)$. When $-1 < a < 0$ however, the orbit oscillates about the fixed point, first less than this value, then greater, until finally it squeezes in on the fixed point from either side. In both cases, we say that the map has an **attracting**

[4] A sequence of points is **monotone** if the sequence is nonincreasing or nondecreasing. It is **strictly monotone** if it is strictly increasing or strictly decreasing.





(a) $a > 1$ (b) $a < -1$

Figure 4.15: Linear maps with repelling fixed points.

fixed point since nearby orbits are attracted to it. In the linear case, *all* attracting orbits happen to be "nearby."

7f) What is the behavior of all orbits when $a = 0$?

When $a = 0$, every point is eventually fixed after at most one iteration. (Why "at most?")

7g) Suppose $F$ has just one fixed point and $|a| > 1$. Using graphical analysis, what can you say about the behavior of all other orbits of $F$ in this case? We call such fixed points **repelling**. Can you explain why?

Again, there are two cases to consider (see Figure 4.15): either $a > 1$ or $a < -1$. In either case, $|F^n(x)| \to \infty$ as $n \to \infty$. Note that the orbit oscillates about the fixed point in the latter case, growing arbitrarily large in absolute value.

7h) Perform a complete orbit analysis for $F(x) = x + b$ in case $b > 0$, $b = 0$, and $b < 0$.

i. $b = 0$

Here $F$ is the identity map and $\text{fix}\, F = \mathbb{R}$.

ii. $b > 0$

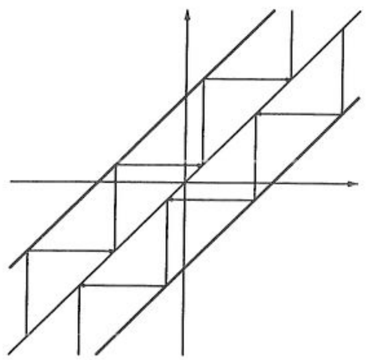


Figure 4.16: Straight lines parallel to the diagonal.

In this case, $\text{per}_n F = \emptyset$ for $n \geq 1$, and $F^n(x) \to \infty$ as $n \to \infty$ for all $x$.

iii. $b < 0$

Again we have $\text{per}_n F = \emptyset$ for $n \geq 1$, but in this case, and for all $x$, $F^n(x) \to -\infty$ as $n \to \infty$.

See the sketches in Figure 4.16 for typical examples.

7i) Perform a complete orbit analysis for $F(x) = -x + b$.

Regardless of $b$, we find that $\text{fix}\, F = \{b/2\}$ and $\text{per}_2 F = \mathbb{R}$. See Figure 4.17.

Here are some additional exercises involving linear functions:

7j) Compute $F^2$ and discuss its dynamics. What is $\text{fix}\, F^2$ (and hence $\text{per}_2 F$)? Under what conditions is this fixed point attracting? Repelling? Under what conditions does $F^2$ admit an oscillating orbit?

7k) Compute $F^3$ and $F^4$. From these deduce a formula for $F^n$. Prove by induction that your formula is correct. What is its limiting value as $n \to \infty$?



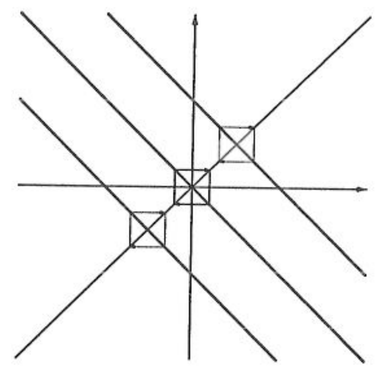


Figure 4.17: Straight lines perpendicular to the diagonal.

# Chapter 5

# Fixed and Periodic Points

## Exercises

**1.** For each of the following functions, find all fixed points and classify them as attracting, repelling, or neutral.

1a) $F(x) = x^2 - x/2$

$$\begin{aligned} x^2 - x/2 = x &\Rightarrow x^2 - 3x/2 = 0 \\ &\Rightarrow x(x - 3/2) = 0 \\ &\Rightarrow x = 0 \quad \text{or} \quad x = 3/2. \end{aligned}$$

Therefore, $\text{fix}\, F = \{0, 3/2\}$.

$F'(x) = 2x - 1/2$.

$F'(0) = -1/2 \Rightarrow 0$ is attracting and oscillating.

$F'(3/2) = 5/2 > 1 \Rightarrow 3/2$ is repelling.

1b) $F(x) = x(1 - x)$

$\text{fix}\, F = \{0\}$.

$F'(x) = 1 - 2x$.

$F'(0) = 1 \Rightarrow 0$ is neutral.

1c) $F(x) = 3x(1 - x)$

$$\begin{aligned} 3x(1 - x) = x &\Rightarrow 2x - 3x^2 = 0 \\ &\Rightarrow x(2 - 3x) = 0 \\ &\Rightarrow x = 0 \quad \text{or} \quad x = 2/3 \end{aligned}$$



Therefore, $\text{fix}\, F = \{0, 2/3\}$.

$F'(x) = 3 - 6x$.

$F'(0) = 3 > 1 \Rightarrow 0$ is repelling.

$F'(2/3) = 3 - 6(2/3) = -1 \Rightarrow 2/3$ is neutral.

1d) $F(x) = (2 - x)/10$

$$\begin{aligned} (2 - x)/10 = x &\Rightarrow 2 - x = 10x \\ &\Rightarrow 2/11 = x \end{aligned}$$

Therefore, $\text{fix}\, F = \{2/11\}$.

$F'(x) = -1/10$.

$F'(2/11) = -1/10 \Rightarrow 2/11$ is attracting and oscillating.

1e) $F(x) = x^4 - 4x^2 + 2$

If $Q(x) = x^2 - 2$, then $Q^2(x) = x^4 - 4x^2 + 2$ by Exercise 3.3, and so $F = Q^2$. Now a fixed point for $Q$ is also a fixed point for $F$ since $F(x) = Q^2(x) = Q(Q(x)) = Q(x) = x$. But the fixed points of $Q$ satisfy the equation $x^2 - x - 2 = 0$. This suggests we factor and solve the resulting 4th-degree polynomial equation as follows:

$$\begin{aligned} & x^4 - 4x^2 + 2 = x \\ \Rightarrow\ & x^4 - 4x^2 - x + 2 = 0 \\ \Rightarrow\ & (x^2 - x - 2)(x^2 + x - 1) = 0 \\ \Rightarrow\ & x^2 - x - 2 = 0 \quad \text{or} \quad x^2 + x - 1 = 0 \\ \Rightarrow\ & (x - 2)(x + 1) = 0 \quad \text{or} \quad x = \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2} \\ \Rightarrow\ & x = 2 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = \frac{-1 \pm \sqrt{5}}{2}. \end{aligned}$$

If we let

$$\phi = \frac{1 + \sqrt{5}}{2},$$

then the reader may check that

$$\text{fix}\, F = \{-\phi, -1, 2, 1/\phi\}.$$

The number $\phi$ is called the **golden ratio**[1] and arises naturally in connection with the ubiquitous Fibonacci sequence.[2] Now, since

$$F'(x) = 4x^3 - 8x = 4x \cdot Q(x),$$

we have that $F'(-1) = 4$ and $F'(2) = 16$, and hence, both integral fixed points are repelling. But what about $-\phi$ and $1/\phi$? Recall that a fixed point of $Q$ is also a fixed point for $F$, and since $Q$ is an even function, we may compute $Q(\phi)$ instead of $Q(-\phi)$. That is,

$$\begin{aligned}\phi^2 - 2 &= \left(\frac{1+\sqrt{5}}{2}\right)^2 - 2\\ &= \frac{6+2\sqrt{5}}{4} - \frac{8}{4}\\ &= \frac{-2+2\sqrt{5}}{4}\\ &= \frac{-1+\sqrt{5}}{2}\\ &= \frac{1}{\phi}.\end{aligned}$$

Similarly, we find that

$$\frac{1}{\phi^2} - 2 = -\phi,$$

and hence, $-\phi$ and $1/\phi$ constitute a 2-cycle for $Q$ (see Exercise 3.3). And since

$$F'(-\phi) = -4\phi(\phi^2 - 2) = -4\phi\frac{1}{\phi} = -4$$

along with

$$F'\left(\frac{1}{\phi}\right) = \frac{4}{\phi}\left(\frac{1}{\phi^2} - 2\right) = \frac{4}{\phi}(-\phi) = -4,$$

$-\phi$ and $1/\phi$ are repelling fixed points for $F$.

[1] Some authors call $\phi$ the golden section while still others define it as $(\sqrt{5}-1)/2$. The latter and $(\sqrt{5}+1)/2$ are reciprocals of one another, and a single unit apart on the real line.

[2] For a particularly lucid introduction to the Fibonacci numbers, see chapter 11 in: Ogilvy, C. Stanley and John T. Anderson. *Excursions in number theory*. New York: Oxford University Press, 1966.



1f) $S(x) = \frac{\pi}{2}\sin x$

$\operatorname{fix} S = \{0, \pm\pi/2\}$.

$S'(x) = \frac{\pi}{2}\cos x$.

$S'(0) = \pi/2 > 1 \Rightarrow 0$ is repelling.

$S'(\pm\pi/2) = (\pi/2)\cos(\pm\pi/2) = 0 \Rightarrow \pm\pi/2$ are superattracting.

1g) $S(x) = -\sin x$

$\operatorname{fix} S = \{0\}$.

$S'(x) = -\cos x$.

$S'(0) = -1$, and so $-1$ is neutral.

1h) $F(x) = x^3 - x$

$$\begin{aligned}x^3 - x = x &\Rightarrow x^3 - 2x = 0\\ &\Rightarrow x(x^2-2) = 0\\ &\Rightarrow x = 0 \quad\text{or}\quad x = \pm\sqrt{2}\end{aligned}$$

Therefore, $\operatorname{fix} F = \{0, \pm\sqrt{2}\}$.

$F'(x) = 3x^2 - 1$.

$F'(0) = -1 \Rightarrow 0$ is neutral.

$F'(\pm\sqrt{2}) = 3(\pm\sqrt{2})^2 - 1 = 5 \Rightarrow \pm\sqrt{2}$ are repelling.

1i) $A(x) = \arctan x$

$\operatorname{fix} A = \{0\}$.

$A'(x) = 1/(1+x^2)$.

$A'(0) = 1 \Rightarrow 0$ is neutral.

1j) $T(x) = \begin{cases} 2x & \text{if } x \le 1/2 \\ 2 - 2x & \text{if } x > 1/2 \end{cases}$

Setting each piece of this two part function equal to $x$ yields $\operatorname{fix} T = \{0, 2/3\}$. Also,

$$T'(x) = \begin{cases} 2 & \text{if } x < 1/2 \\ -2 & \text{if } x > 1/2 \end{cases}$$

since $T$ is piecewise linear, but the derivative of $T$ is not defined at $x = 2$. Therefore, $T'(0) = 2$ and so 0 is repelling. And since $T'(2/3) = -2$, 2/3 is also repelling. In fact, no periodic point for $T$ can be attracting! See Exercise 3 for a related result.

1k) $F(x) = 1/x^2$

$\text{fix}\, F = \{1\}$.

$F'(x) = -2/x^3$.

$F'(1) = -2$. Therefore, $x = 1$ is repelling.

**2.** For each of the following functions, zero lies on a periodic orbit. Classify this orbit as attracting, repelling, or neutral.

2a) $F(x) = 1 - x^2$

Since $F(0) = 1$ and $F(1) = 0$, $\{0,1\} \subseteq \text{per}_2 F$.

$F'(x) = -2x$.

$(F^2)'(0) = F'(0) \cdot F'(1) = 0 \cdot (-2) = 0$, and so this period 2 orbit is superattracting.

2b) $C(x) = \frac{\pi}{2} \cos x$

Since $C(0) = \pi/2$ and $C(\pi/2) = 0$, $\{0, \pi/2\} \subseteq \text{per}_2 C$.

$C'(x) = -(\pi/2) \sin x$.

$(C^2)'(0) = C'(0) \cdot C'(\pi/2) = 0 \cdot (-\pi/2) = 0$, and once again the orbit is superattracting.

2c) $F(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1$

Since $F(0) = 1$, $F(1) = -1$, and $F(-1) = 0$, we have that $\{0, \pm 1\} \subseteq \text{per}_3 F$.

$F'(x) = -\frac{3}{2}x^2 - 3x$.

$(F^3)'(0) = F'(0) \cdot F'(1) \cdot F'(-1) = 0 \cdot (-\frac{9}{2}) \cdot (\frac{3}{2}) = 0$, and so this period 3 orbit is superattracting.

2d) $F(x) = |x-2| - 1$

Note that $F(0) = 1$ and $F(1) = 0$. Thus $0 \in \text{per}_2 F$. In fact, every point is eventually periodic with period 2.

Since

$$|x-2| - 1 = \begin{cases} x - 3 & \text{if } x \geq 2 \\ 1 - x & \text{if } x < 2 \end{cases},$$



it follows that

$$F'(x) = \begin{cases} 1 & \text{if } x > 2 \\ -1 & \text{if } x < 2 \end{cases}$$

and therefore, $(F^2)'(0) = F'(0) \cdot F'(1) = (-1) \cdot (-1) = 1$. This implies that the orbit of 0 is neutral.

2e) $A(x) = -\frac{4}{\pi} \arctan(x+1)$

Since $A(0) = -1$ and $A(-1) = 0$, we see that $\{-1, 0\} \subseteq \text{per}_2 A$.

The reader may verify that

$$A'(x) = \frac{-4}{\pi(1 + (x+1)^2)},$$

and $(A^2)'(0) = A'(0) \cdot A'(-1) = (-2/\pi) \cdot (-4/\pi) = 8/\pi^2 < 1$. This implies that 0 is an attracting periodic point of period 2.

2f) $F(x) = \begin{cases} x + 1 & \text{if } x \leq 3.5 \\ 2x - 8 & \text{if } x > 3.5 \end{cases}$

In this case, $0 \mapsto 1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 0$, and so $0 \in \text{per}_5 F$. Also,

$$F'(x) = \begin{cases} 1 & \text{if } x < 3.5 \\ 2 & \text{if } x > 3.5 \end{cases}$$

from which it follows that $(F^5)'(0) = F'(0) \cdot F'(1) \cdot F'(2) \cdot F'(3) \cdot F'(4) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 = 2$. Thus, 0 is a repelling periodic point of period 5.

**3.** Suppose $x_0$ lies on a cycle of prime period $n$ for the doubling function $D$. Evaluate $(D^n)'(x_0)$. Is this cycle attracting or repelling?

Recall the definition of the doubling map given at the end of Chapter 3:

$$D(x) = 2x \bmod 1 = \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2x - 1 & \text{if } 1/2 \leq x < 1 \end{cases}.$$

(See Figure 5.1.) The crucial fact employed here is that for all $x \neq 1/2$, $D'(x) = 2$. Now, let $x_0 \in \text{per}_n D$, that is, suppose $D^n(x_0) = x_0$,[3] and let

[3] This periodic point can not be equal to 1/2 since 1/2 is eventually fixed. Moreover, for all $k > 0$, it must be true that $D^k(x_0) \neq 1/2$ since each such point is eventually fixed. Indeed, the reader is encouraged to write down an expression for $\overline{\text{fix}\, D}$.

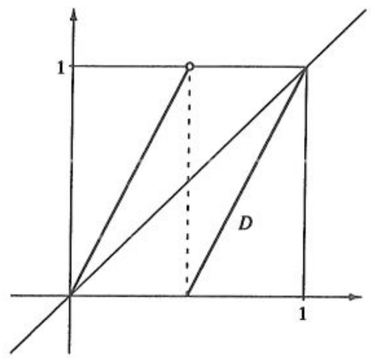



Figure 5.1: The doubling map $D(x) = 2x \bmod 1$.

$x_k = D^k(x_0)$ for $k = 0, 1, 2, \ldots, n-1$. Then

$$\begin{aligned}(D^n)'(x_0) &= \prod_{k=0}^{n-1} D'(x_k)\\ &= D'(x_0)\cdot D'(x_1)\cdot\cdots\cdot D'(x_{n-1})\\ &= \underbrace{2\cdot 2\cdot\cdots\cdot 2}_{n \text{ times}}\\ &= 2^n\end{aligned}$$

which is greater than 1 for all $n > 0$. Hence, *all* periodic points are repelling for the doubling map!

**4.** Each of the following functions has a neutral fixed point. Find this fixed point and, using graphical analysis with an accurate graph, determine if it is weakly attracting, weakly repelling, or neither.

4a) $F(x) = x + x^2$

Either by inspection of the graph of $F$ (see Figure 5.2a), or by solving the equation $F(x) = x$ for $x$, we see that $\operatorname{fix} F = \{0\}$. Since $F'(0) = 1$ (that is, the graph of $F$ is tangent to the diagonal at $x = 0$) this fixed point must be neutral. Moreover, graphical analysis suggests that the fixed point is weakly attracting on the left and weakly repelling on



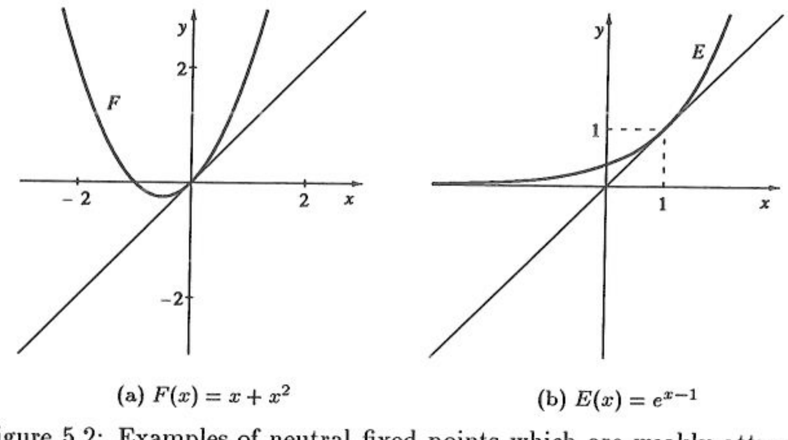



Figure 5.2: Examples of neutral fixed points which are weakly attracting on the left and weakly repelling on the right.

the right. This is a simple consequence of the fact that the graph is concave up at the origin (since $F''(0) = 2 > 0$, by the way).

We remark that $-1$ is eventually fixed after one iteration, and that for all $x$ in the closed interval $[-1, 0]$, $F^n(x) \to 0$ as $n \to \infty$.[4] Also, for $x < -1$, $F(x) > 0$, and therefore, $F^n(F(x)) \to \infty$ as $n \to \infty$ since 0 is weakly repelling on the right.

4b) $F(x) = 1/x$ (see Figure 5.3)

Multiplying both sides of

$$1/x = x$$

by $x$ and solving, we find that $\operatorname{fix} F = \{\pm 1\}$. Also, since

$$F'(x) = \frac{-1}{x^2},$$

we have that $F'(\pm 1) = -1$, and so both fixed points are neutral. But they are neither weakly attracting nor weakly repelling. This is one of those atypical cases where *every* point is periodic (in this case with

[4] But convergence is slow since the fixed point is neutral.

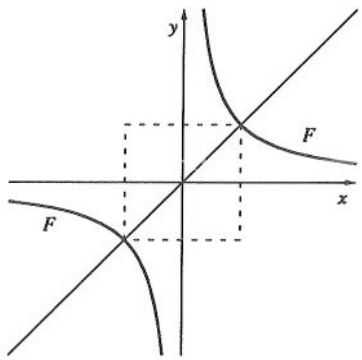



Figure 5.3: The graph of $F(x) = 1/x$, a totally periodic function.

period 2). Such a map is called **totally periodic**, and the reader is encouraged to construct other examples of such functions.

4c) $E(x) = e^{x-1}$ (*Hint:* The fixed point is at $x = 1$.)

Although it's easy to check that 1 is indeed fixed by $E$, it's not so easy to derive this result from first principles, that is, by solving $E(x) = x$ for $x$. But we won't let *that* stop us from analyzing this fixed point! Locally, in a small neighborhood of 1, this problem is reminiscent of Exercise 4a where the neutral point was found to be weakly attracting on the left and weakly repelling on the right. Observe that the graph of $E$ is concave up at the fixed point (see Figure 5.2b) since $E''(1) = 1 > 0$. We remark that being an exponential function, $E$ is its own derivative.

Let's take a detour. The next two problems require detailed knowledge of the sine and tangent functions which is needed to determine whether the origin is weakly attracting or weakly repelling. Specifically, the trigonometric inequalities

$$\sin\theta < \theta < \tan\theta, \qquad 0 < \theta < \pi/2 \tag{5.1}$$

and

$$\tan\theta < \theta < \sin\theta, \qquad -\pi/2 < \theta < 0 \tag{5.2}$$



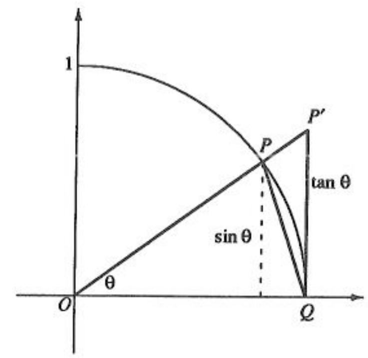



Figure 5.4: The unit circle in the first quadrant.

must be established.[5] We content ourselves here with verifying (5.1) and leave (5.2) as an exercise for the interested reader.

The plan is to show (5.1) geometrically by comparing areas of strategically chosen triangles. We begin by sketching the unit circle in the first quadrant and choosing an angle $\theta$ such that $0 < \theta < \pi/2$. We then construct two triangles $\Delta OPQ$ and $\Delta OP'Q$ as shown in Figure 5.4. Observe that the area of $\Delta OPQ$ is less than the area of the wedge swept out by the angle $\theta$. In fact, since the height of $\Delta OPQ$ is $\sin\theta$, its area is precisely

$$\frac{\sin\theta}{2} \tag{5.3}$$

since the base of the triangle has length 1. Now, the area of the wedge is a certain proportion of the total area of the unit disk, and since a circle sweeps out $2\pi$ radians, this proportion must be $\theta/2\pi$. Therefore, the area of the wedge is

$$\frac{\theta}{2\pi}\pi = \frac{\theta}{2} \tag{5.4}$$

since the area of the unit disk is $\pi$ square units. Comparing areas (5.3) and

[5]These results can be found in most calculus texts. See, for example, Exercises 10–11 on page A24 of: Anton, Howard. *Calculus*. New York: John Wiley and Sons, 1980. Or see the lemma on pages 145–146 of: Loomis, Lynn H. *Calculus*. Reading, MA: Addison-Wesley, 1982.

(5.4), we have

$$\frac{\sin\theta}{2} < \frac{\theta}{2}$$

which implies that

$$\sin\theta < \theta.$$

This result holds for all $\theta$ in the first quadrant, and may even be extended to arbitrary $\theta > 0$. (Exercise: Show $\sin\theta < \theta$ for $\pi/2 < \theta < \pi$, for instance.)

At this point, we've shown half of inequality (5.1). For the rest, observe that the area of $\Delta OP'Q$ is

$$\frac{\tan\theta}{2}$$

since the base of the triangle has length 1. This area clearly exceeds that of the wedge, and so

$$\frac{\tan\theta}{2} > \frac{\theta}{2}$$

which implies that

$$\tan\theta > \theta.$$

We remark that this result can *not* be extended to arbitrary $\theta > 0$. This completes the detour.

4d) $S(x) = \sin x$

From the graph of $S$ (see Figure 5.5a), we see that $\operatorname{fix} S = \{0\}$. But once again it's difficult (if not impossible!) to show this by solving the equation $S(x) = x$ for $x$. At any rate, we see from the graph of $S$ that $\sin x < x$ for all $x > 0$ (see preceding derivation) and that $\sin x > x$ for all $x < 0$. Thus the origin is attracting, but only weakly so since $S'(0) = \cos 0 = 1$.

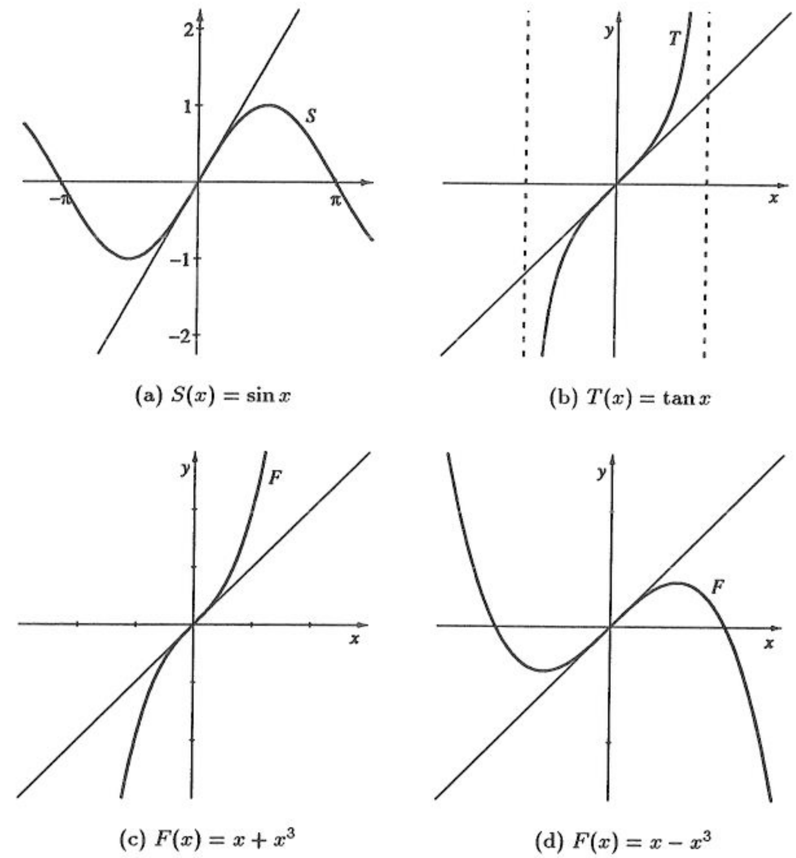
Anticipating Exercise 8, we illustrate an alternative, more mechanical approach to this problem. Observe that $S'(x) = \cos x$, $S''(x) = -S(x)$, and $S'''(x) = -S'(x)$. Thus, $S'(0) = 1$, $S''(0) = 0$, and $S'''(0) = -1 < 0$. By Exercise 8, we conclude that the origin is weakly attracting.

4e) $T(x) = \tan x$

We know that $0 \in \operatorname{fix} T$ since $\tan 0 = 0$,[6] and this fixed point is indeed neutral since $T'(0) = \sec^2 0 = 1$. But is it weakly attracting or weakly

[6] Observe that $T$ has an infinite number of fixed points since it's $\pi$-periodic.






(a) $S(x) = \sin x$

(b) $T(x) = \tan x$

(c) $F(x) = x + x^3$

(d) $F(x) = x - x^3$

Figure 5.5: Examples of neutral fixed points which are also inflection points. Such points are either weakly repelling or weakly attracting.

repelling? From the graph depicted in Figure 5.5b, it appears that $\tan x > x$ for $0 < x < \pi/2$ (see the derivation preceding Exercise 4d) and that $\tan x < x$ for $-\pi/2 < x < 0$, and so the origin must be weakly repelling. In fact, it can be shown that *all* of $T$'s fixed points are repelling, and the reader is invited to experiment with this interesting map on the computer.

The results of Exercise 7 suggest an alternative approach to this problem. The reader may verify that $T'(x) = \sec^2 x$, $T''(x) = 2\tan x \sec^2 x$, and $T'''(x) = 2\sec^2 x(\tan^2 x + \sec^2 x)$. Evaluating these derivatives at the fixed point, we find that $T'(0) = 1$, $T''(0) = 0$, and $T'''(0) = 2 > 0$. The origin is therefore weakly repelling by Exercise 7.

4f) $F(x) = x + x^3$ (see Figure 5.5c)

This is the canonical example of a map with a weakly repelling fixed point. First of all, note that 0 is fixed by $F$. Indeed,

$$\text{fix}\, F = \{0\}.$$

Since $F'(x) = 1 + 3x^2$, we see that $F'(0) = 1$ and so 0 is neutral. The fact that 0 is weakly repelling follows from graphical analysis and the fact that the graph of $F$ lies below the diagonal for negative $x$ and above the diagonal for positive $x$. This observation is verified using the results of Exercise 7; that is, 0 is weakly repelling since $F''(0) = 0$ and $F'''(0) = 6 > 0$.

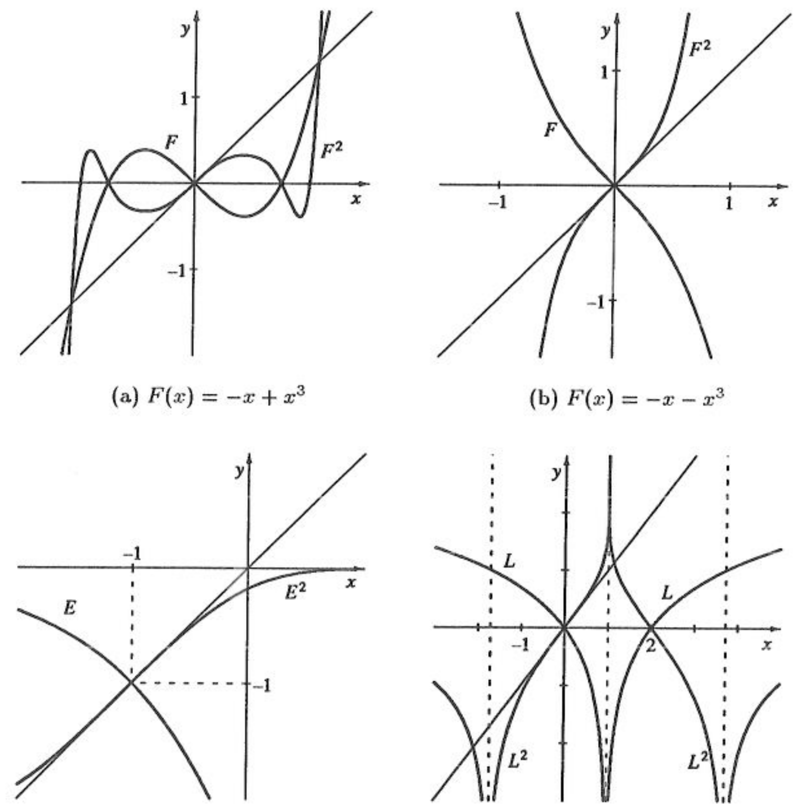
We remark that the graph of $F$ has the same basic shape as the graph in Exercise 4e.

4g) $F(x) = x - x^3$ (see Figure 5.5d)

Likewise, this is the canonical example of a map with a weakly attracting fixed point. Again, 0 is the only fixed point of $F$ with $F'(0) = 1$ and $F''(0) = 0$. But this time the graph of $F$ lies *above* the diagonal to the left of the origin and *below* the diagonal to the right. This is because $F'''(0) = -6 < 0$ (see Exercise 4d for a similar situation) and graphical analysis confirms that the origin is weakly attracting in this case. Note that $F$ has local extrema at $x = \pm 1$ (verify this) which do not effect the local dynamics about the fixed point.

4h) $F(x) = -x + x^3$






(a) $F(x) = -x + x^3$ (b) $F(x) = -x - x^3$

(c) $E(x) = -ee^x$ (d) $L(x) = \ln|x-1|$

Figure 5.6: Neutral fixed points with a derivative of $-1$.

In this case, $F$ has *three* fixed points, only one of which is neutral:

$$\begin{aligned} & x^3 - x = x \\ \Rightarrow\ & x^3 - 2x = 0 \\ \Rightarrow\ & x(x^2 - 2) = 0 \\ \Rightarrow\ & x = 0 \quad \text{or} \quad x = \pm\sqrt{2}. \end{aligned}$$

Since $F'(x) = 3x^2 - 1$, we see that $F'(0) = -1$ while $F'(\pm\sqrt{2}) = 5$. So, 0 is neutral and the other two fixed points are repelling.

For the purpose of comparison, the graphs of $F$ and $F^2$ have been superimposed in Figure 5.6a. Observe that the origin is also a neutral fixed point for $F^2$, but with $(F^2)'(0) = 1$. This is verified by computing

$$\begin{aligned} F^2(x) &= -(-x + x^3) + (-x + x^3)^3 \\ &= (x - x^3) + (x^9 - 3x^7 + 3x^5 - x^3) \\ &= x^9 - 3x^7 + 3x^5 - 2x^3 + x, \end{aligned}$$

from which it follows that

$$(F^2)'(x) = 9x^8 - 21x^6 + 15x^4 - 6x^2 + 1.$$

Using the chain rule, we may check this result as follows:

$$\begin{aligned} (F \circ F)'(x) &= (F' \circ F(x)) \cdot F'(x) \\ &= (3(-x + x^3)^2 - 1)(3x^2 - 1) \\ &= 9x^8 - 21x^6 + 15x^4 - 6x^2 + 1 \qquad \checkmark \end{aligned}$$

Clearly, $F^2(0) = 0$ and $(F^2)'(0) = 1$. Moreover, the origin is an inflection point for $F^2$ since $(F^2)''(0) = 0$ and $(F^2)'''(0) = -12 \neq 0$.

Since the graph of $F^2$ lies above the diagonal to the left of 0 and below the diagonal to the right of 0, the origin must be weakly attracting under iteration of $F^2$. More importantly, $p$ is weakly attracting under iteration of $F$. This is a consequence of

**Proposition 5.1** *Let $p$ be a fixed point for $F$. Then $p$ is a fixed point for $F^2$, and $p$ is neutral for $F$ if and only if $p$ is neutral for $F^2$.*

**Proof**: Suppose $p$ is a fixed point for $F$. Then

$$F^2(p) = F(F(p)) = F(p) = p$$



and so $p \in \operatorname{fix} F^2$. Now,

$$\begin{aligned} (F^2)'(p) &= F'(F(p)) \cdot F'(p) \\ &= [F'(p)]^2, \end{aligned}$$

and so $|F'(p)| = 1$ if and only if $(F^2)'(p) = 1$. Thus $p$ is neutral for $F$ if and only if $p$ is neutral for $F^2$. □

It's important to realize that $(F^2)'(p)$ is always equal to one, and never minus one. We will take advantage of this fact repeatedly in the sequel.

We remark that Proposition 5.1 is easily extended to include the attracting and repelling cases. Also note that the result does *not* say that $p$ is a neutral fixed point for $F$ if and only if it's a neutral fixed point for $F^2$ (can you give a counterexample?).

4i) $F(x) = -x - x^3$ (see Figure 5.6b)

The analysis is exactly the same as in the previous exercise. The trick is to compute and analyze the dynamics of $F^2$ which in this case shows the origin to be weakly repelling.

4j) $E(x) = -ee^x$ (*Hint:* The fixed point is at $x = -1$. Examine in detail the graph of $E^2$ near $x = -1$ using higher derivatives of $E^2$.)

In Figure 5.6c, the graphs of $E$ and $E^2$ have been plotted on the same set of coordinate axes for comparison. Note that the lone fixed point of $E$ is also a fixed point for $E^2$ and that $(E^2)'(-1) = 1$. These properties, readily discernible in the figure, are guaranteed by Proposition 5.1: if $p \in \operatorname{fix} E$, then $p \in \operatorname{fix} E^2$, and $p$ is neutral for $E$ if and only if it's neutral for $E^2$.

Also, it appears that $-1$ is an inflection point for $E^2$, and so we expect $(E^2)'(-1)$ to vanish. Indeed it does, as will be verified below. A similar result will be shown true in general in Exercise 9 which says in effect that if $p$ is a neutral fixed point for $E$ with slope $-1$, then $p$ is likewise a neutral fixed point for $E^2$ but with slope $+1$ *and* $p$ is an inflection point for $E^2$. This is great since we already know how to analyze maps with these properties (see Exercises 4d, 4e, 4f, and 4g).

An analysis of $E^2$ follows. It can be shown that

$$(E^2)'(x) = E(x) \cdot E^2(x)$$

by an application of the chain rule. Similarly,

$$(E^2)''(x) = E(x) \cdot E^2(x) \cdot (1 + E(x))$$

and

$$(E^2)'''(x) = E(x) \cdot E^2(x) \cdot (E(x) + (1 + E(x))^2).$$

Using these formulas, we see that $(E^2)'(-1) = 1$ and $(E^2)''(-1) = 0$ which agrees with earlier observations. Also, $(E^2)'''(-1) = -1 < 0$ which implies that $-1$ is weakly attracting for $E^2$, and hence for $E$. A simple computer experiment will bear this out.

4k) $L(x) = \ln|x - 1|$

We may write

$$L(x) = \begin{cases} \ln(x-1) & \text{if } x > 1 \\ \ln(1-x) & \text{if } x < 1 \end{cases}$$

but note that $L'(x) = 1/(x-1)$ for all $x \neq 1$. The graph of $L$ (see Figure 5.6d) clearly shows that $\operatorname{fix} L = \{0\}$, and this fixed point is neutral since $L'(0) = -1$. In other words, $L$ behaves like $x \mapsto -x$ in the vicinity of the origin. Points whose orbits come close to 0 (that is, points which are nearly eventually fixed) alternate about 0 for some time before crossing the vertical asymptote at $x = 1$.[7] That is, the origin is weakly repelling.

These claims may be verified using the results of Exercise 9. Note that $L''(x) = -(x-1)^{-2}$ and $L'''(x) = 2(x-1)^{-3}$, and so $L''(0) = -1$ while $L'''(0) = -2$. Since $-2L'''(0) - 3[L''(0)]^2 = 1 > 0$, it follows that the origin is weakly repelling.

We remark that the presence of the vertical asymptote at $x = 1$ causes some orbits to fluctuate wildly. Any point nearly on the backward orbit of 1 is such a point. Compute the orbits of $1 \pm \epsilon$ for very small $\epsilon$, for example.

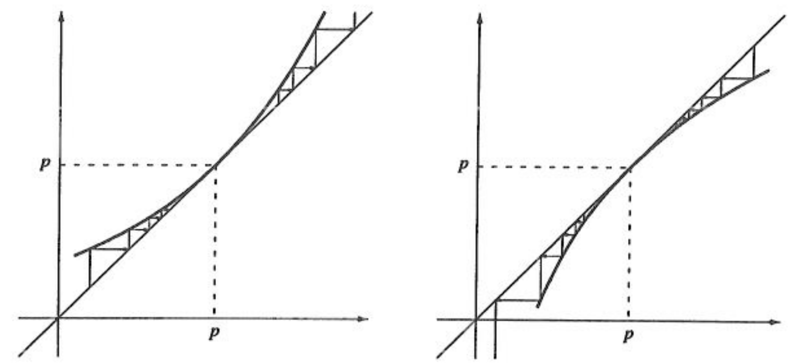
Computer experiments indicate that $L^2$ has fixed points at approximately 1.13022 and $-2.06388$, but this 2-cycle is repelling since

$$L'(1.13022) \cdot L'(-2.06388) \approx -2.5.$$

Note also that $L^2$ has vertical asymptotes at $1 \pm e$.

[7] The reader should compute the orbit of 8.39, for instance.






(a) $F(p) = p, F'(p) = 1, F''(p) > 0$ (b) $F(p) = p, F'(p) = 1, F''(p) < 0$

Figure 5.7: Two cases of a neutral fixed point.

**5.** Suppose that $F$ has a neutral fixed point at $p$ with $F'(p) = 1$. Suppose also that $F''(p) > 0$. What can you say about $p$: is $p$ weakly attracting, weakly repelling, or neither? Use graphical analysis and the concavity of the graph of $F$ near $p$ to support your answer.

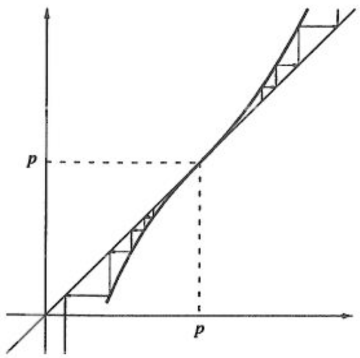
Since $F''(p) > 0$, the graph of $F$ is concave up at the neutral point $p$ (see Figure 5.7a). Graphical analysis clearly shows that $p$ is weakly attracting on the left and weakly repelling on the right.

**6.** Repeat Exercise 5, but this time assume that $F''(p) < 0$.

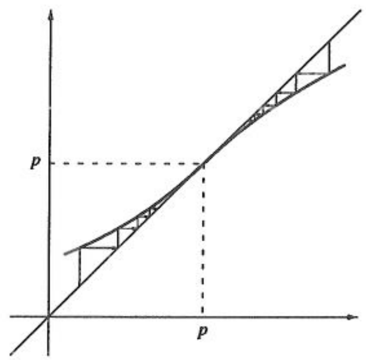
When $F''(p) < 0$, the graph of $F$ is concave down at the neutral point $p$ (see Figure 5.7b). In this case, $p$ is weakly repelling on the left and weakly attracting on the right.

**7.** Suppose that $F$ has a neutral fixed point at $p$ with $F'(p) = 1$ and $F''(p) = 0$. Suppose also that $F'''(p) > 0$. Use graphical analysis and the concavity of the graph of $F$ near $p$ to show that $p$ is weakly repelling.

When $F''(p) = 0$, we know from calculus that $p$ will be an inflection point for $F$ if $F'''(p) \neq 0$. Now suppose $F''(x)$ is negative to the left of $p$, and positive to the right of $p$. Then the graph of $F$ is concave down to the left, and concave up to the right (see Figure 5.8a). Graphical analysis shows that




(a) $F(p) = p, F'(p) = 1, F''(p) = 0$, and $F'''(p) > 0$




(b) $F(p) = p, F'(p) = 1, F''(p) = 0$, and $F'''(p) < 0$

Figure 5.8: Two more cases of a neutral fixed point.

$p$ is weakly repelling in this case. But $F''$ is increasing in a neighborhood of $p$ since it's negative to the left and positive to the right. Therefore, $p$ is weakly repelling provided the derivative of $F''$ is positive at $p$, that is, if $F'''(p) > 0$.

**8.** Repeat Exercise 7, but this time assume that $F'''(p) < 0$. Show that $p$ is weakly attracting.

In this case, we suppose that $F''(x)$ is positive to the left of $p$, and negative to the right of $p$, so that the graph of $F$ is concave up to the left and concave down to the right (see Figure 5.8b). Arguing as above, it follows that $F'''(p) < 0$.

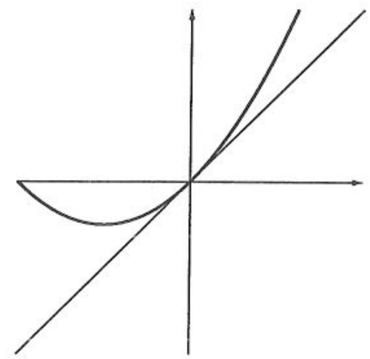
**9.** Combine the results of Exercises 5–8 to state a **Neutral Fixed Point Theorem.**

The four basic cases are illustrated in Figure 5.9 for $p = 0$ and summarized below.

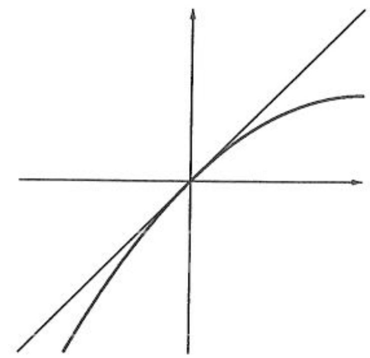
**Theorem 5.2** *Let $p$ be a neutral fixed point for $F$ with $F'(p) = 1$.*

Case 1: *Suppose $F''(p) \neq 0$. If $F''(p) < 0$ (resp. $F''(p) > 0$), then $p$ is*

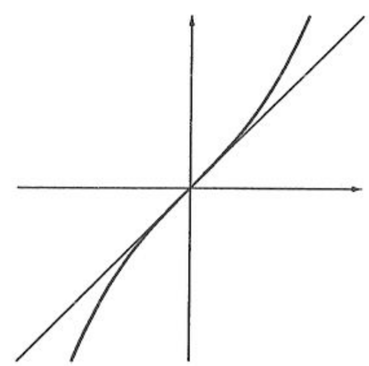




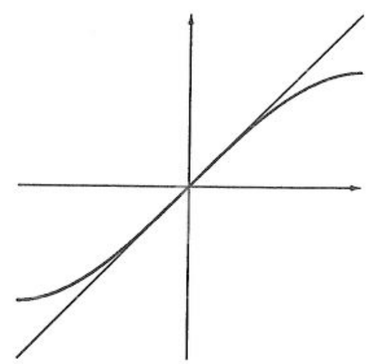
(a) $F(0) = 0, F'(0) = 1, F''(0) > 0$



(b) $F(0) = 0, F'(0) = 1, F''(0) < 0$



(c) $F(0) = 0, F'(0) = 1, F''(0) = 0$, and $F'''(0) > 0$



(d) $F(0) = 0, F'(0) = 1, F''(0) = 0$, and $F'''(0) < 0$

Figure 5.9: The Four Canonical Forms of Neutral Fixed Points.

*weakly attracting (resp. weakly repelling) on the right and weakly repelling (resp. weakly attracting) on the left.*

Case 2: *Suppose* $F''(p) = 0$. *If* $F'''(p) < 0$ *(resp.* $F'''(p) > 0$*), then* $p$ *is weakly attracting (resp. weakly repelling).*

But what if $F'(p) = -1$? As we saw in Exercises 4h–4k, the trick is to apply Theorem 5.2 to $F^2$. First of all, by Proposition 5.1, a neutral fixed point for $F$ is also a neutral fixed point for $F^2$, but with $(F^2)'(p) = 1$. Furthermore, using the chain and product rules for derivatives, we find that

$$(F^2)''(x) = F''(F(x)) \cdot [F'(x)]^2 + F'(F(x)) \cdot F''(x).$$

Evaluating this derivative at $p$, we obtain

$$\begin{aligned}(F^2)''(p) &= F''(F(p)) \cdot [F'(p)]^2 + F'(F(p)) \cdot F''(p) \\ &= F''(p) - F''(p) \\ &= 0\end{aligned}$$

and so case 2 of Theorem 5.2 applies. What remains is the somewhat tedious computation of $(F^2)'''(p)$. The more industrious reader will be inclined to verify that

$$\begin{aligned}&(F^2)'''(x) = \\ &F'''(F(x)) \cdot [F'(x)]^3 + 3F''(F(x)) \cdot F'(x) \cdot F''(x) + F'(F(x)) \cdot F'''(x)\end{aligned}$$

from which it follows that

$$(F^2)'''(p) = -2F'''(p) - 3[F''(p)]^2.$$

When this quantity is negative,[8] the neutral fixed point $p$ is weakly attracting for $F^2$, and hence for $F$.

Let's summarize this result in the following

**Theorem 5.3** *Let* $p$ *be a neutral fixed point for* $F$ *with* $F'(p) = -1$. *If the quantity* $-2F'''(p) - 3[F''(p)]^2$ *is negative (resp. positive), then* $p$ *is weakly attracting (resp. weakly repelling).*

[8] Which is the same as saying $F$ has negative Schwarzian derivative at $p$. (See Chapter 12.)



Unfortunately, there are still cases of neutral fixed points that have yet to be considered. What if $F'(p) = 1$ and $F''(p) = F'''(p) = 0$, for instance? Under these conditions, Theorem 5.2 does not apply and we must continue evaluating higher derivatives at $p$ until one of them is nonzero. Even this will not always work, however, since there are maps having a neutral fixed point satisfying $F'(p) = 1$, and for which $F^{(n)}(p) = 0$ for all $n > 1$. The identity map is one such (trivial) example—can you find others?

# Chapter 6

# Bifurcations

## Exercises

**1.** Each of the following functions undergoes a bifurcation of fixed points at the given parameter value. In each case, use algebraic or graphical methods to identify this bifurcation as either a saddle-node or period-doubling bifurcation, or neither of these. In each case, sketch the phase portrait for typical parameter values below, at, and above the bifurcation value.

1a) $F_\lambda(x) = x + x^2 + \lambda, \quad \lambda = 0$

To find the fixed points of $F$, we must solve

$$x + x^2 + \lambda = x$$

for $x$. This yields

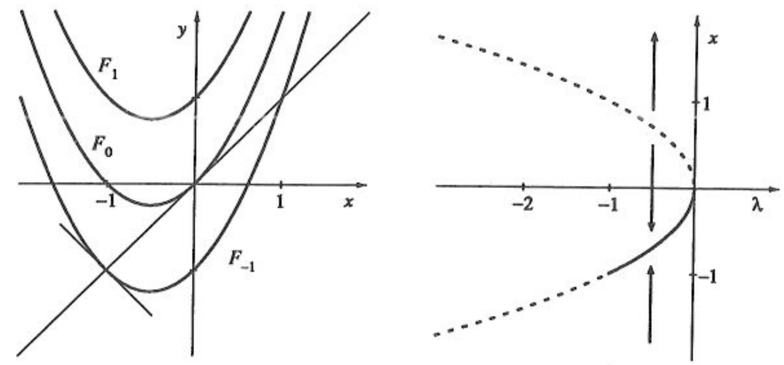
$$x = \pm\sqrt{-\lambda}.$$

Thus, $F_\lambda$ has no fixed points for $\lambda > 0$, one fixed point for $\lambda = 0$, and a pair of fixed points for $\lambda < 0$ (see Figure 6.1a). Clearly, $\lambda = 0$ is a bifurcation point for $F$, and note that $x = 0$ when $\lambda = 0$. Using the fact that $F'_\lambda(x) = 2x + 1$ we see that this point is in fact a saddle-node bifurcation since $F'_0(0) = 1$. Observe that $F'_\lambda$ is independent of $\lambda$.

But we can say more. First of all, it's easily shown that $-(1 + \sqrt{-\lambda})$ is eventually fixed. Secondly, if we evaluate $F'_\lambda$ at the negative fixed point, we get

$$F'_\lambda(-\sqrt{-\lambda}) = 2(-\sqrt{-\lambda}) + 1$$






(a) Some typical family members. (b) Bifurcation diagram.

Figure 6.1: Bifurcation behavior in the quadratic family $F_\lambda(x) = x + x^2 + \lambda$.

$$= 1 - 2\sqrt{-\lambda}$$

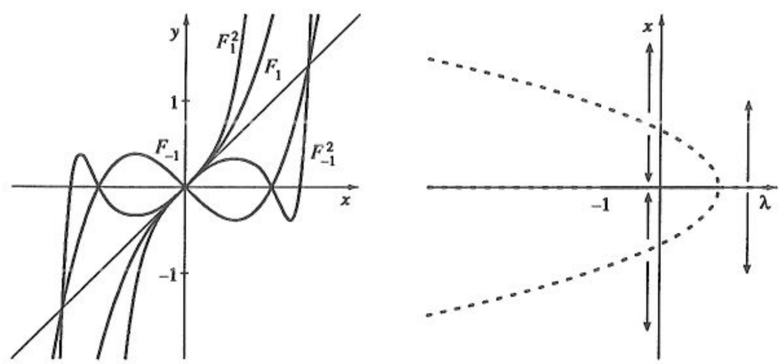
which allows us to determine the $\lambda$ values for which $F_\lambda$ has an attracting fixed point:

$$\begin{aligned} & -1 < F'_\lambda(-\sqrt{-\lambda}) < 1 \\ \Rightarrow\ & -1 < 1 - 2\sqrt{-\lambda} < 1 \\ \Rightarrow\ & -2 < -2\sqrt{-\lambda} < 0 \\ \Rightarrow\ & 1 > \sqrt{-\lambda} > 0 \\ \Rightarrow\ & 1 > -\lambda > 0 \\ \Rightarrow\ & -1 < \lambda < 0. \end{aligned}$$

We conclude from this that $-\sqrt{-\lambda}$ is attracting for $-1 < \lambda < 0$. On the other hand, $+\sqrt{-\lambda}$ is always repelling since $F'_\lambda(+\sqrt{-\lambda}) > 1$ for all $\lambda < 0$. See Figure 6.1b.

1b) $F_\lambda(x) = x + x^2 + \lambda, \quad \lambda = -1$

This problem is a continuation of Exercise 1a (see Figure 6.1). Recall that $-\sqrt{-\lambda}$ is attracting for $-1 < \lambda < 0$ and that $F_\lambda$ experiences a saddle-node bifurcation at $\lambda = 0$. Another bifurcation occurs at the left-hand endpoint of this interval. Indeed, when $\lambda = -1$, we have




(a) Some typical family members and their second iterates. (b) Bifurcation diagram.

Figure 6.2: Bifurcation behavior in the cubic map $F_\lambda(x) = \lambda x + x^3$.

a neutral fixed point at $x = -\sqrt{-(-1)} = -1$ with $F'_{-1}(-1) = -1$. Hence, $F_\lambda$ undergoes a period-doubling bifurcation at $\lambda = -1$.

1c) $F_\lambda(x) = \lambda x + x^3, \quad \lambda = -1$

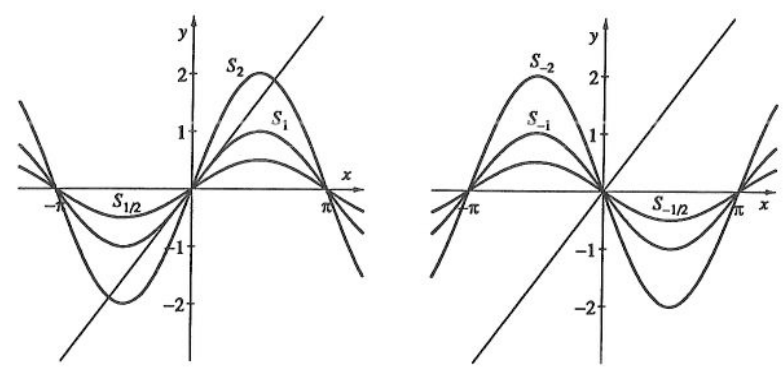
Set $F_\lambda(x)$ equal to $x$ and solve:

$$\begin{aligned} &\lambda x + x^3 = x \\ \Rightarrow\ &(\lambda - 1)x + x^3 = 0 \\ \Rightarrow\ &((\lambda-1) + x^2)x = 0 \\ \Rightarrow\ &x = \pm\sqrt{1-\lambda} \quad \text{or} \quad x = 0. \end{aligned}$$

When $\lambda \geq 1$, there is a single real fixed point at the origin, and for $\lambda < 1$ there are *three* fixed points (see Figure 6.2a). From $F'_\lambda(x) = \lambda + 3x^2$, we compute $F'_\lambda(0) = \lambda$. Hence, the origin is attracting for $-1 < \lambda < 1$. Evaluating $F'_\lambda$ at the remaining fixed points, we get

$$\begin{aligned} F'_\lambda(\pm\sqrt{1-\lambda}) &= \lambda + 3(\pm\sqrt{1-\lambda})^2 \\ &= \lambda + 3(1-\lambda) \\ &= 3 - 2\lambda, \end{aligned}$$






(a) $\mu > 0$. (b) $\mu < 0$.

Figure 6.3: Typical members of the sine family $S_\mu(x) = \mu \sin x$.

and so the fixed points $\pm\sqrt{1-\lambda}$ are attracting when

$$\begin{aligned} &-1 < 3 - 2\lambda < 1 \\ \Rightarrow\ &-4 < -2\lambda < -2 \\ \Rightarrow\ &2 > \lambda > 1, \end{aligned}$$

but note that $\pm\sqrt{1-\lambda}$ is complex in this range. See Figure 6.2b.

These calculations suggest that $F_\lambda$ undergoes bifurcations at $\lambda = -1$, 1, and 2. When $\lambda = -1$, we have already shown that there are three fixed points, and for this parameter value we find $F'_{-1}(0) = -1$ and $F'_{-1}(\pm\sqrt{2}) = 5$. Hence, there is a period-doubling bifurcation at the origin when $\lambda = -1$ and the other two fixed points are repelling.

1d) $F_\lambda(x) = \lambda x + x^3, \quad \lambda = 1$

This problem is a continuation of Exercise 1c. When $\lambda = 1$, $x = 0$ is fixed, and we've already shown that the origin is attracting when $-1 < \lambda < 1$. Since $F'_1(0) = 1$, we see that the origin undergoes a saddle-node bifurcation when $\lambda = 1$. See Figure 6.2.

1e) $S_\mu(x) = \mu \sin x, \quad \mu = 1 \quad$ (see Figure 6.3a)

First of all, note that $S'_\mu(x) = \mu \cos x$. It was suggested earlier in

Exercise 5.4d that fix $S_1 = \{0\}$ and that the origin is weakly attracting; see also Exercise 4.1g. Indeed, the fact that $S_1'(0) = 1$ implies that $x = 0$ undergoes a saddle-node bifurcation at $\mu = 1$. Moreover, since $S_\mu'(0) = \mu$, the origin is attracting for $-1 < \mu < 1$ and repelling for $|\mu| > 1$.

1f) $S_\mu(x) = \mu \sin x, \quad \mu = -1$ (see Figure 6.3b)

Since $S_{-1}'(0) = -1$, it appears that $S_\mu$ undergoes a period-doubling bifurcation at $\mu = -1$. We remark that for $\mu < -1$, $S_\mu$ has an attracting 2-cycle.

The reader may wonder if there other bifurcation points for $S_\mu$, and if so, what are they? We begin to answer this question below.

Since bifurcations occur at neutral fixed points, what we need to do is solve the equations

$$\mu \sin x = x \tag{6.1}$$

and

$$\mu \cos x = \pm 1 \tag{6.2}$$

simultaneously. Now, if we divide (6.1) by (6.2), we get

$$\tan x = \pm x$$

for $\mu \neq 0$. (The tangent function was cursorily examined earlier in Exercise 5.4e.) In other words, the bifurcation points of $S_\mu$ are the fixed points and 2-cycles of $x \mapsto \tan x$.[1]

1h) $E_\lambda(x) = \lambda(e^x - 1), \quad \lambda = -1$

Note that $E_\lambda(0) = 0$ and so $0 \in \text{fix}\, E_\lambda$. Also note that $E_\lambda'(x) = \lambda e^x = E_\lambda(x) + \lambda$. Thus, $E_\lambda'(0) = \lambda$. Therefore, the origin is attracting for $|\lambda| < 1$ and repelling for $|\lambda| > 1$. When $\lambda = -1$, $E_\lambda$ undergoes a period-doubling bifurcation since $E_{-1}(0) = -1$. See Figure 6.4.

1i) $E_\lambda(x) = \lambda(e^x - 1), \quad \lambda = 1$

We have from the previous problem that $E_\lambda'(x) = E_\lambda(x) + \lambda$. It follows that $E_1(0) = 1$ which shows there's a saddle-node bifurcation at $\lambda = 1$. See Figure 6.4 for the graph of $E_1$.

1j) $H_c(x) = x + cx^2, \quad c = 0$

[1] Recall that tangent is an odd function, and that the 2-cycles of an odd function $F$ are solutions to the equation $F(x) = -x$.



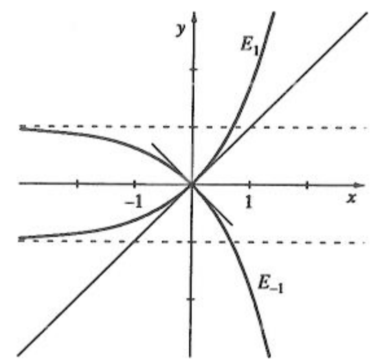



Figure 6.4: Two examples from the exponential family $E_\lambda(x) = \lambda(e^x - 1)$.

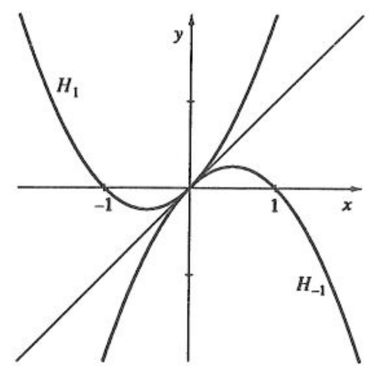



Figure 6.5: Every member of the family $H_c(x) = x + cx^2$ has a neutral fixed point.

Every member of this family is anchored to the origin, with another zero at $x = -1/c$. Figure 6.5 shows that $\operatorname{fix} H_c = \{0\}$, and that $H_c'(0) = 1$ regardless of $c$. Now, let's apply the results of Theorem 5.2. Since $H_c''(0) = 2c$, case 1 of the theorem applies provided $c \neq 0$. For $c < 0$, we see that $H_c''(0) < 0$, and so 0 is weakly attracting on the right and weakly repelling on the left. Similarly, for $c > 0$, $H_c''(0) > 0$, and the origin is weakly repelling on the right and weakly attracting on the left. But when $c = 0$, $H_c''(0) = 0$, and we look to case 2 of Theorem 5.2. Unfortunately, $H_c'''(x)$ is identically zero and so the theorem does not apply. Observe, however, that $H_0$ is the identity map, which is totally periodic.

In summary, this family of maps experiences no bifurcations whatsoever, and provides a good example of why the precise definitions given in Chapter 6 of the text are necessary.

1k) $F_c(x) = x + cx^2 + x^3, \quad c = 0$

First of all, when $c = 0$, $F_0$ is identical to the map in Exercise 1d with $\lambda = 1$. (See $F_1$ and $F_1^2$ in Figure 6.2a.) But a generic member of this family of functions has two fixed points (see Figures 6.6b–c) since

$$\begin{aligned} & x + cx^2 + x^3 = x \\ \Rightarrow\ & cx^2 + x^3 = 0 \\ \Rightarrow\ & (c+x)x^2 = 0 \\ \Rightarrow\ & x = -c \quad \text{or} \quad x = 0. \end{aligned}$$

Now, $F_c'(x) = 1 + 2cx + 3x^2$, and so $F_c'(0) = 1$ regardless of $c$. Hence, this fixed point fails to undergo a bifurcation which can be seen in Figure 6.6d.

Note that $F_c'(-c) = 1 + c^2$ which is strictly greater than one for all $c \neq 0$. Thus, $-c$ is repelling for all $c$ (even $c = 0$ which is weakly repelling). We also remark that $F_{-c}(-x) = -F_c(x)$, a most curious property.

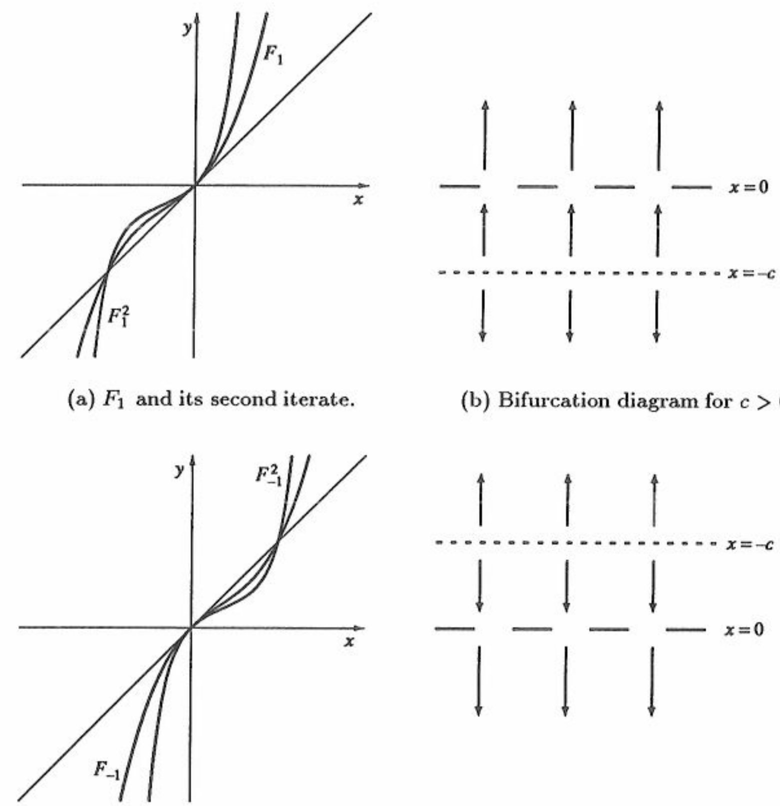
The next four exercises apply to the family $Q_c(x) = x^2 + c$.

**2.** Verify the formulas for the fixed points $p_\pm$ and the 2-cycle $q_\pm$ given in the text.

Recall that

$$p_\pm = \frac{1 \pm \sqrt{1-4c}}{2}$$






(a) $F_1$ and its second iterate. (b) Bifurcation diagram for $c > 0$.

(c) $F_{-1}$ and its second iterate. (d) Bifurcation diagram for $c < 0$.

Figure 6.6: Representatives of the family $F_c(x) = x + cx^2 + x^3$.

and note that these points are real when $1-4c>0$, that is, when $c<1/4$. Now,

$$\begin{aligned}Q_c(p_+) &= Q_c((1+\sqrt{1-4c})/2)\\ &= (1+\sqrt{1-4c})^2/4+c\\ &= (1+2\sqrt{1-4c}+1-4c)/4+c\\ &= (2+2\sqrt{1-4c}-4c+4c)/4\\ &= (1+\sqrt{1-4c})/2\\ &= p_+.\end{aligned}$$

We leave it to the reader to show that $p_-$ is fixed. Now, let's verify that

$$q_\pm = \frac{-1\pm\sqrt{-4c-3}}{2}$$

constitute a 2-cycle for $Q_c$:

$$\begin{aligned}Q_c(q_+) &= Q_c((-1+\sqrt{-4c-3})/2)\\ &= (-1+\sqrt{-4c-3})^2/4+c\\ &= (1-2\sqrt{-4c-3}-4c-3)/4+c\\ &= (-2-2\sqrt{-4c-3}-4c+4c)/4\\ &= (-1-\sqrt{-4c-3})/2\\ &= q_-.\end{aligned}$$

The reader may show that $Q_c(q_-)=q_+$ as well. Thus, $\{q_+,q_-\}\subset \mathrm{per}_2\, Q_c$. We remark that this 2-cycle exists for $-4c-3>0$, that is, for $c<-3/4$.

**3.** Prove that the cycle of period 2 given by $q_\pm$ is attracting for $-5/4 < c < -3/4$.

First of all, note that $Q_c'(x)=2x$. We have that

$$\begin{aligned}(Q_c^2)'(q_+) &= Q_c'(q_+)\cdot Q_c'(q_-)\\ &= (-1+\sqrt{-4c-3})\cdot(-1-\sqrt{-4c-3})\\ &= 1-(-4c-3)\\ &= 4c+4 \qquad (6.3)\end{aligned}$$

which basically solves this and the following two exercises. To find the parameter values for which this 2-cycle is attracting, we must solve

$$-1<4c+4<1$$



$$\begin{aligned}&\Rightarrow -5<4c<-3\\ &\Rightarrow -5/4<c<-3/4.\end{aligned}$$

**4.** Prove that this cycle is neutral for $c=-5/4$.

We apply Equation 6.3. When $c=-5/4$, $4c+4=-1$, and hence, the 2-cycle is neutral for this particular $c$-value.

**5.** Prove that this cycle is repelling for $c<-5/4$.

When $c<-5/4$, we have $4c+4<-1$ from Equation 6.3. Hence, this period 2 orbit is repelling for $c<-5/4$.

Exercises 6–14 deal with the logistic family of functions given by $F_\lambda(x) = \lambda x(1-x)$.

**6.** For which values of $\lambda$ does $F_\lambda$ have an attracting fixed point at $x=0$?

First of all, note that $x=0$ is indeed fixed since $F_\lambda(0)=0$. Using the fact that $F_\lambda'(x)=\lambda(1-2x)$, we see that $F_\lambda'(0)=\lambda$, and so 0 is attracting for $-1<\lambda<1$.

**7.** For which values of $\lambda$ does $F_\lambda$ have a nonzero attracting fixed point?

Let's find the other fixed point of $F_\lambda$:

$$\begin{aligned}&\lambda x(1-x)=x\\ &\Rightarrow \lambda x-\lambda x^2-x=0\\ &\Rightarrow (\lambda-1)x-\lambda x^2=0\\ &\Rightarrow \lambda x^2+(1-\lambda)x=0\\ &\Rightarrow (\lambda x+(1-\lambda))x=0\\ &\Rightarrow x=(\lambda-1)/\lambda \quad\text{or}\quad x=0.\end{aligned} \qquad (6.4)$$

We may conclude from this that

$$\mathrm{fix}\, F_\lambda = \left\{0, \frac{\lambda-1}{\lambda}\right\}.$$

As a check, let's compute

$$F_\lambda\left(\frac{\lambda-1}{\lambda}\right) = \lambda\left(\frac{\lambda-1}{\lambda}\right)\left(1-\frac{\lambda-1}{\lambda}\right)$$

$$= (\lambda - 1)\left(\frac{\lambda - (\lambda - 1)}{\lambda}\right)$$
$$= (\lambda - 1)\left(\frac{1}{\lambda}\right)$$
$$= \frac{\lambda - 1}{\lambda} \quad \checkmark$$

Now,

$$F'_\lambda\left(\frac{\lambda-1}{\lambda}\right) = \lambda\left(1 - 2\frac{\lambda-1}{\lambda}\right) = \lambda\left(\frac{\lambda - 2(\lambda-1)}{\lambda}\right) = \lambda - 2\lambda + 2 = 2 - \lambda.$$

Thus, $(\lambda - 1)/\lambda$ is attracting for

$$-1 < 2 - \lambda < 1$$
$$\Rightarrow -3 < -\lambda < -1$$
$$\Rightarrow 1 < \lambda < 3.$$

**8.** Describe the bifurcation that occurs when $\lambda = 1$.

For $\lambda \neq 0$, $F_\lambda$ has two fixed points, namely 0 and $(\lambda-1)/\lambda$. For $1 < \lambda < 3$, $(\lambda - 1)/\lambda$ is attracting; likewise, for $-1 < \lambda < 1$, 0 is attracting. As $\lambda$ decreases through the bifurcation point at $\lambda = 1$, the fixed point $(\lambda-1)/\lambda$ transfers its attractiveness to 0 which continues to attract orbits until $\lambda = -1$. For $\lambda < -1$, both fixed points are repelling.

**9.** Sketch the phase portrait and bifurcation diagram near $\lambda = 1$.

See Figure 6.7. The horizontal asymptote at $x = 1$ implies that there is no member of the logistic family having 1 as a fixed point. The vertical asymptote at the origin implies that all members of the logistic family have two fixed points (and the two are unique for all $\lambda \neq 1$) except the degenerate $F_0$ which is identically zero.

**10.** Describe the bifurcation that occurs when $\lambda = 3$.

Recall from Exercise 8 that 0 is attracting for $-1 < \lambda < 1$, while $(\lambda-1)/\lambda$ is attracting for $1 < \lambda < 3$. Thus, both fixed points are repelling for $\lambda > 3$.



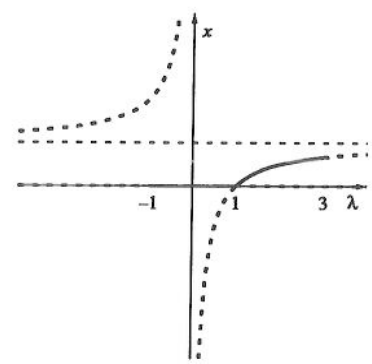



Figure 6.7: Bifurcation diagram for the logistic family.

Note that

$$F'_\lambda\left(\frac{\lambda-1}{\lambda}\right)\Big|_{\lambda=3} = -1$$

which suggests a 2-cycle may be lurking in the shadows.

**14.** Compute an explicit formula for the periodic points of period 2 for $F_\lambda$.

First compute the second iterate of $F_\lambda$:

$$\begin{aligned} F_\lambda^2(x) &= \lambda(\lambda x(1-x))(1 - \lambda x(1-x)) \\ &= \lambda(\lambda x - \lambda x^2)(1 - \lambda x + \lambda x^2) \\ &= \lambda(\lambda x - \lambda^2 x^2 + \lambda^2 x^3 - \lambda x^2 + \lambda^2 x^3 - \lambda^2 x^4) \\ &= \lambda(\lambda x - (\lambda + \lambda^2)x^2 + 2\lambda^2 x^3 - \lambda^2 x^4) \\ &= \lambda^2 x - \lambda^2(1+\lambda)x^2 + 2\lambda^3 x^3 - \lambda^3 x^4. \end{aligned}$$

To find the fixed points of $F_\lambda^2$ (i.e. the period 2 points of $F_\lambda$), set $F_\lambda^2(x)$ equal to $x$, rearrange terms, and get

$$(\lambda^2 - 1)x - \lambda^2(1+\lambda)x^2 + 2\lambda^3 x^3 - \lambda^3 x^4 = 0. \qquad (6.5)$$

This 4th-degree polynomial may be factored into a pair of quadratics. One of these factors *must* be $(\lambda - 1)x - \lambda x^2$ since the fixed points of $F_\lambda$ are also fixed points of $F_\lambda^2$ (see Equation 6.4). Long dividing out this 2nd-degree polynomial from (6.5), we find that

$$\frac{(\lambda^2-1)x-\lambda^2(1+\lambda)x^2+2\lambda^3x^3-\lambda^3x^4}{(\lambda-1)x-\lambda x^2}=\lambda^2x^2-\lambda(\lambda+1)x+(\lambda+1)$$

and so

$$[(\lambda-1)x-\lambda x^2][\lambda^2x^2-\lambda(\lambda+1)x+(\lambda+1)]=0.$$

This new factor may be solved using the quadratic formula:

$$\begin{aligned} x &= \frac{\lambda(\lambda+1)\pm\sqrt{\lambda^2(\lambda+1)^2-4\lambda^2(\lambda+1)}}{2\lambda^2} \\ &= \frac{\lambda(\lambda+1)\pm\lambda\sqrt{(\lambda+1)^2-4(\lambda+1)}}{2\lambda^2} \\ &= \frac{(\lambda+1)\pm\sqrt{(\lambda+1)(\lambda-3)}}{2\lambda}. \end{aligned}$$

Hence, this 2-cycle exists when $\lambda > 3$ or $\lambda < -1$ which agrees with the results of Exercise 8.

# Chapter 7

# The Quadratic Family

## Exercises

**1.** List the intervals which are removed in the third and fourth stages of the construction of the Cantor middle-thirds set.

Step 1: $(1/3, 2/3)$

Step 2: $(1/9, 2/9) \cup (7/9, 8/9)$

Step 3: $(1/27, 2/27) \cup (7/27, 8/27) \cup (19/27, 20/27) \cup (25/27, 26, 27)$

Step 4:

$(1/81, 2/81) \cup (7/81, 8/81) \cup (19/81, 20/81) \cup (25/81, 26/81) \cup (55/81, 56/81) \cup (61/81, 62/81) \cup (73/81, 74/81) \cup (79/81, 80/81)$

**2.** Compute the sum of the lengths of all the intervals which are removed from the interval $[0, 1]$ in the construction of the Cantor middle-thirds set.

From Exercise 1, it should be clear that $2^{n-1}$ open intervals are removed at the $n$th stage of the construction of the Cantor middle-thirds set, each having width $1/3^n$. The combined length of these intervals is

$$1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{27} + \cdots + 2^{n-1} \cdot \frac{1}{3^n} + \cdots,$$

but

$$\sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i} = \sum_{i=1}^{\infty} \frac{2^i}{2 \cdot 3^i}$$



$$= \frac{1}{2} \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^i$$
$$= \frac{1}{2} \left(\frac{2/3}{1 - 2/3}\right)$$
$$= 1.$$

In other words, construction of the Cantor middle-thirds set is a subtractive process which removes an infinite number of open intervals with combined length equal to that of the original unit interval. Apparently, little is left behind at the conclusion of this process... or so it seems.

In the next five exercises,[1] find the rational numbers whose ternary expansion is given by:

**3.**
$$\begin{aligned} 0.\overline{21} &= \left(\tfrac{2}{3} + \tfrac{1}{3^2}\right) + \left(\tfrac{2}{3^3} + \tfrac{1}{3^4}\right) + \cdots \\ &= \tfrac{7}{3^2} + \tfrac{7}{3^4} + \cdots \\ &= 7\left(\tfrac{1}{9} + \tfrac{1}{9^2} + \cdots\right) \\ &= 7\left(\tfrac{1/9}{1-1/9}\right) \\ &= 7\left(\tfrac{1/9}{8/9}\right) \\ &= \tfrac{7}{8} \end{aligned}$$

**4.**
$$\begin{aligned} 0.\overline{022} &= \left(\tfrac{0}{3} + \tfrac{2}{3^2} + \tfrac{2}{3^3}\right) + \left(\tfrac{0}{3^4} + \tfrac{2}{3^5} + \tfrac{2}{3^6}\right) + \cdots \\ &= \tfrac{8}{3^3} + \tfrac{8}{3^6} + \cdots \\ &= 8\left(\tfrac{1}{27} + \tfrac{1}{27^2} + \cdots\right) \\ &= 8\left(\tfrac{1/27}{1-1/27}\right) \\ &= 8\left(\tfrac{1/27}{26/27}\right) \\ &= \tfrac{4}{13} \end{aligned}$$

**5.**
$$\begin{aligned} 0.00\overline{2} &= \tfrac{2}{3^3} + \tfrac{2}{3^4} + \cdots \\ &= \tfrac{2}{3^3}\left(1 + \tfrac{1}{3} + \cdots\right) \\ &= \tfrac{2}{3^3}\left(\tfrac{1}{1-1/3}\right) \\ &= \tfrac{2}{3^3}\left(\tfrac{3}{2}\right) \\ &= \tfrac{1}{9} \end{aligned}$$

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[1] In almost all cases, the technique is to group successive pairs or triplets of terms and simplify. Grouping of terms is permissible since a geometric series is absolutely convergent.

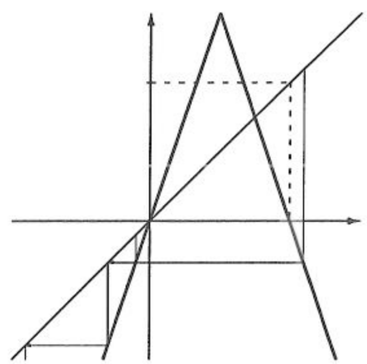

Figure 7.1: The graph of $T(x) = 3/2 - |3x - 3/2|$.

**6.** $$\begin{aligned} 0.212\overline{01} &= \tfrac{2}{3} + \tfrac{1}{3^2} + \tfrac{2}{3^3} + \left(\tfrac{0}{3^4} + \tfrac{1}{3^5}\right) + \left(\tfrac{0}{3^6} + \tfrac{1}{3^7}\right) + \cdots \\ &= \tfrac{18+3+2}{3^3} + \tfrac{1}{3^5}\left(1 + \tfrac{1}{9} + \cdots\right) \\ &= \tfrac{23}{3^3} + \tfrac{1}{3^5}\left(\tfrac{1}{1-1/9}\right) \\ &= \tfrac{23}{3^3} + \tfrac{1}{3^5}\cdot\tfrac{9}{8} \\ &= \tfrac{23}{3^3} + \tfrac{1}{8\cdot 3^3} \\ &= \tfrac{8\cdot 23+1}{8\cdot 3^3} \\ &= \tfrac{185}{216} \end{aligned}$$

**7.** $$\begin{aligned} 0.0\overline{101} &= \left(\tfrac{1}{3^2} + \tfrac{1}{3^4}\right) + \left(\tfrac{1}{3^5} + \tfrac{1}{3^7}\right) + \cdots \\ &= \tfrac{10}{3^4} + \tfrac{10}{3^7} + \cdots \\ &= \tfrac{10}{3^4}\left(1 + \tfrac{1}{27} + \cdots\right) \\ &= \tfrac{10}{3^4}\cdot\tfrac{27}{26} \\ &= \tfrac{5}{39} \end{aligned}$$

**Dynamics on the Cantor middle-thirds set:** The following exercises deal with the function

$$T(x) = \begin{cases} 3x & \text{if } x \le 1/2 \\ 3 - 3x & \text{if } x > 1/2 \end{cases}.$$

**9.** Sketch the graph of $T$ and show by graphical analysis that, if $x > 1$ or $x < 0$, then $T^n(x) \to -\infty$ as $n \to \infty$.

See Figure 7.1.



**10.** Find the fixed points for $T$. What is the ternary expansion of these points?

The origin is obviously a fixed point. The remaining fixed point is obtained by solving

$$\begin{aligned} 3 - 3x = x &\Rightarrow 3 = 4x \\ &\Rightarrow 3/4 = x \end{aligned}$$

Check: $T(3/4) = 3 - 3(3/4) = 3 - 9/4 = 12/4 - 9/4 = 3/4$. ✓

The ternary expansion of 0 is trivial, but what about 3/4? We know from the example on page 77 of the text that

$$0.\overline{02} = 2\sum_{i=1}^{\infty}\left(\frac{1}{9}\right)^i = \frac{1}{4}.$$

So all we need to do is multiply each of these quantities by 3. But multiplying a ternary expansion by 3 shifts the ternary point one place to the right. Thus, $3/4 = 0.\overline{20}$. This may also be seen by writing

$$\begin{aligned} & 2\sum_{i=1}^{\infty}\left(\frac{1}{9}\right)^i = \frac{1}{4} \\ \Rightarrow\ & 2\sum_{i=1}^{\infty}\frac{3}{3^{2i}} = \frac{3}{4} \\ \Rightarrow\ & 2\sum_{i=1}^{\infty}\frac{1}{3^{2i-1}} = \frac{3}{4} \\ \Rightarrow\ & 2\left(\frac{1}{3} + \frac{1}{3^3} + \cdots\right) = \frac{3}{4} \\ \Rightarrow\ & \left(\frac{2}{3} + \frac{2}{3^3} + \cdots\right) = \frac{3}{4} \\ \Rightarrow\ & 0.\overline{20} = \frac{3}{4}. \end{aligned}$$

**11.** Show that 3/13 and 3/28 lie on 3-cycles for $T$.

$$\begin{aligned} T\left(\tfrac{3}{13}\right) &= \tfrac{9}{13} \\ T\left(\tfrac{9}{13}\right) &= 3 - 3\left(\tfrac{9}{13}\right) = \tfrac{12}{13} \\ T\left(\tfrac{12}{13}\right) &= 3 - 3\left(\tfrac{12}{13}\right) = \tfrac{3}{13} \end{aligned} \qquad \begin{aligned} T\left(\tfrac{3}{28}\right) &= 3\left(\tfrac{3}{28}\right) = \tfrac{9}{28} \\ T\left(\tfrac{9}{28}\right) &= 3\left(\tfrac{9}{28}\right) = \tfrac{27}{28} \\ T\left(\tfrac{27}{28}\right) &= 3 - 3\left(\tfrac{27}{28}\right) = \tfrac{3}{28} \end{aligned}$$

**12.** Show that if $x \in (1/3, 2/3)$, then $T^n(x) \to -\infty$ as $n \to \infty$.

Note that $T$ is strictly increasing on $(1/3, 1/2]$ and strictly decreasing on $[1/2, 2/3)$. Therefore,

$$\frac{1}{3} < x \le \frac{1}{2} \Rightarrow T\left(\frac{1}{3}\right) < T(x) \le T\left(\frac{1}{2}\right) \Rightarrow 1 < T(x) \le \frac{3}{2}$$

and

$$\frac{1}{2} < x \le \frac{2}{3} \Rightarrow T\left(\frac{1}{2}\right) < T(x) \le T\left(\frac{2}{3}\right) \Rightarrow \frac{3}{2} \ge T(x) > 1.$$

So, any way you look at it, $1 < T(x) \le 3/2$ when $x \in (1/3, 1/2)$, and in Exercise 9 we showed that if $x > 1$, then $T^n(x) \to -\infty$ as $n \to \infty$.

**13.** Show that if $x \in (1/9, 2/9)$ or $x \in (7/9, 8/9)$, then $T^n(x) \to -\infty$ as $n \to \infty$.

The solution to this problem requires some knowledge of $T^2$. To find an expression for $T^2(x)$ we begin by finding those $x$ for which $T(x) \le 1/2$. If $x \le 1/2$, then $T(x) = 3x \le 1/2$ when $x \le 1/6$. Similarly, if $x \ge 1/2$, then $T(x) = 3 - 3x \le 1/2$ when $x \ge 5/6$. These results suggest we compute the value of $T^2(x)$ on the following partition of the real line:

$$\begin{aligned}
x < 1/6 \quad &\Rightarrow \quad T(x) \le 1/2 \quad \Rightarrow \quad T^2(x) = T(T(x)) = T(3x) = 3(3x) = 9x \\
1/6 \le x < 1/2 \quad &\Rightarrow \quad 1/2 \le T(x) \le 3/2 \quad \Rightarrow \quad T^2(x) = T(T(x)) = T(3x) = 3 - 3(3x) = 3 - 9x \\
1/2 \le x < 5/6 \quad &\Rightarrow \quad 1/2 \le T(x) \le 3/2 \quad \Rightarrow \quad T^2(x) = T(T(x)) = T(3 - 3x) = 3 - 3(3 - 3x) = 9x - 6 \\
x \ge 5/6 \quad &\Rightarrow \quad T(x) \le 1/2 \quad \Rightarrow \quad T^2(x) = T(T(x)) = T(3 - 3x) = 3(3 - 3x) = 9 - 9x
\end{aligned}$$



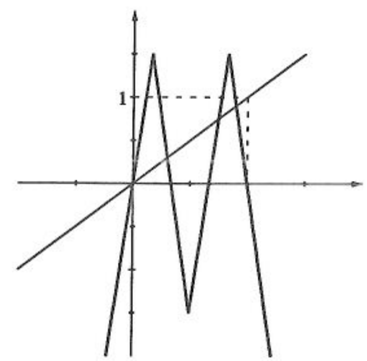


Figure 7.2: The second iterate of $T(x) = 3/2 - |3x - 3/2|$.

Thus, we have shown that

$$T^2(x) = \begin{cases} 9x & \text{if } x \le 1/6 \\ 3 - 9x & \text{if } 1/6 \le x \le 1/2 \\ 9x - 6 & \text{if } 1/2 \le x \le 5/6 \\ 9 - 9x & \text{if } x \ge 5/6 \end{cases}.$$

See Figure 7.2 for the graph of $T^2$. By the way, the reader is invited to show that

$$\operatorname{fix} T^2 = \left\{0, \frac{3}{10}, \frac{3}{4}, \frac{9}{10}\right\}.$$

We're now ready to solve the original exercise. Note that $T^2$ is strictly increasing on $(1/9, 1/6]$ and strictly decreasing on $[1/6, 2/9)$. Therefore,

$$\frac{1}{9} < x \le \frac{1}{6} \Rightarrow T^2\left(\frac{1}{9}\right) < T^2(x) \le T^2\left(\frac{1}{6}\right) \Rightarrow 1 < T^2(x) \le \frac{3}{2}$$

and

$$\frac{1}{6} < x \le \frac{2}{9} \Rightarrow T^2\left(\frac{1}{6}\right) < T^2(x) \le T^2\left(\frac{2}{9}\right) \Rightarrow \frac{3}{2} \ge T^2(x) > 1.$$

If $x \in (1/9, 2/9)$, then $T^2(x) \in (1, 3/2]$, and by Exercise 9 it follows that $T^n(x) \to -\infty$ as $n \to \infty$.

Similar arguments show that if $x \in (7/9, 8/9)$, then $T^n(x) \to -\infty$ as $n \to \infty$.

**14.** Let $\Gamma = \{x \in [0,1] \mid T^n(x) \in [0,1] \text{ for all } n\}$. Prove $\Gamma = K$, the Cantor middle-thirds set.

Let $x = 0.a_1a_2a_3\ldots$ and partition the unit interval so that $[0,1] = [0,1/3] \cup (1/3,2/3) \cup [2/3,1]$. One of the key ideas of the proof is that the value of $a_1$ alone determines which of these three intervals contains $x$. If $a_1 = 0$, then $x \in [0,1/3]$; if $a_1 = 1$, then $x \in (1/3,2/3)$; and if $a_1 = 2$, then $x \in [2/3,1]$. At first glance, the boundary points appear to be exceptions to this general rule since $1/3 = 0.1\overline{0}$ and $2/3 = 0.1\overline{2}$, but even these may be written as $0.0\overline{2}$ and $0.2\overline{0}$, respectively.

Another key idea is the important result of Exercise 12: all $x$ in the interval $(1/3,2/3)$ have orbits which shoot off to $-\infty$. But these are precisely those $x$ whose first ternary digit is equal to 1. Indeed, the claim is that any $x$ having *some* ternary digit equal to 1 has an orbit which escapes to $-\infty$! This is true because

$$T(x) = \begin{cases} 0.a_2a_3a_4\ldots & \text{if } 0 \le x \le 1/2 \\ 0.\hat{a}_2\hat{a}_3\hat{a}_4\ldots & \text{if } 1/2 \le x \le 1 \end{cases}$$

where

$$\hat{a}_i = \begin{cases} 0 & \text{if } a_i = 2 \\ 1 & \text{if } a_i = 1 \\ 2 & \text{if } a_i = 0 \end{cases}.$$

Note that this definition of $T$ is unambiguous at $x = 1/2$ which itself has ternary expansion $0.\overline{1}$.

Now the proof that $\Gamma = K$ goes as follows. Suppose $x \in \Gamma$. Then the ternary expansion of $x$ can not have a 1 in it, for if it did, $T^n(x) \in (1/3,2/3)$ for some $n$ and the orbit would escape to $-\infty$. Thus, $x$ is also in $K$. Conversely, suppose $x \in K$. Then the ternary expansion of $x$ again has no 1s. Thus, $T^n(x) \notin (1/3,2/3)$ for all $n$ and so the orbit of $x$ can not escape. Hence, $x$ is in $\Gamma$.

**15.** Suppose $x \in \Gamma$ has ternary expansion $0.a_1a_2a_3\ldots$. What is the ternary expansion of $T(x)$? Be careful: there are two very different cases!

By the combined results of Exercises 9 & 12, we know that either $x \in [0,1/3]$ or $x \in [2/3,1]$. If $x \in [0,1/3]$, then its leading ternary digit is 0; otherwise, it's 2. These are the two cases that must be dealt with.



If $a_1 = 0$, then $T(x) = 3x$. As we've already seen in Exercise 10, multiplying a ternary expansion by 3 simply shifts the ternary point one place to the right. If $a_1 = 2$, then $T(x) = 3 - 3x = 3(1-x)$. Finally, if $x = 0.a_1a_2a_3\ldots$, then $1 - x = 0.\hat{a}_1\hat{a}_2\hat{a}_3\ldots$ where

$$\hat{a}_i = \begin{cases} 0 & \text{if } a_i = 2 \\ 2 & \text{if } a_i = 0 \end{cases}.$$

Putting all these facts together, we have that

$$T(x) = \begin{cases} 0.a_2a_3a_4\ldots & \text{if } a_1 = 0 \\ 0.\hat{a}_2\hat{a}_3\hat{a}_4\ldots & \text{if } a_1 = 2 \end{cases}$$

for $x \in \Gamma$.