Chapter 9

Symbolic Dynamics

Exercises

1. List all cycles of prime period 4 for the shift map.

There are twelve periodic points of prime period 4:

$$\sigma^4(\overline{0001}) = \sigma^3(\overline{0010}) = \sigma^2(\overline{0100}) = \sigma(\overline{1000}) = (\overline{0001});$$
$$\sigma^4(\overline{0011}) = \sigma^3(\overline{0110}) = \sigma^2(\overline{1100}) = \sigma(\overline{1001}) = (\overline{0011});$$
$$\sigma^4(\overline{1011}) = \sigma^3(\overline{0111}) = \sigma^2(\overline{1110}) = \sigma(\overline{1101}) = (\overline{1011}).$$

Note that all 2-cycles and fixed points are trivially 4-cycles.

Compute $d[\mathbf{s}, \mathbf{t}]$ where:

2. $\mathbf{s} = (\overline{100})$, $\mathbf{t} = (\overline{001})$

$$\begin{aligned} d[(\overline{100}), (\overline{001})] &= \left(1 + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{32}\right) + \left(\frac{1}{64} + \frac{1}{256}\right) + \cdots \\ &= \frac{5}{4} + \frac{5}{32} + \frac{5}{256} + \cdots \\ &= 5\left(\frac{1}{4} + \frac{1}{32} + \frac{1}{256} + \cdots\right) \\ &= 5 \cdot \frac{1/4}{1 - 1/8} \\ &= \frac{10}{7} \end{aligned}$$

3. $\mathbf{s} = (\overline{100})$, $\mathbf{t} = (\overline{010})$

$$\begin{aligned} d[(\overline{100}), (\overline{010})] &= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{8} + \frac{1}{16}\right) + \left(\frac{1}{64} + \frac{1}{128}\right) + \cdots \\ &= \frac{3}{2} + \frac{3}{16} + \frac{3}{128} + \cdots \\ &= 3\left(\frac{1}{2} + \frac{1}{16} + \frac{1}{128} + \cdots\right) \\ &= 3 \cdot \frac{1/2}{1 - 1/8} \\ &= \frac{12}{7} \end{aligned}$$

Note that $(\overline{010})$ is further away from $(\overline{100})$ than $(\overline{001})$ because it differs from $(\overline{100})$ in the second place while $(\overline{001})$ differs in the third. Both $(\overline{010})$ and $(\overline{001})$ differ from $(\overline{100})$ in the first place, and so each must be at least one unit away. But how far are they from each other?

$$\begin{aligned} d[(\overline{001}), (\overline{010})] &= \left(\frac{1}{2} + \frac{1}{4}\right) + \left(\frac{1}{16} + \frac{1}{32}\right) + \left(\frac{1}{128} + \frac{1}{256}\right) + \cdots \\ &= \frac{3}{4} + \frac{3}{32} + \frac{3}{256} + \cdots \\ &= 3 \cdot \frac{1/4}{1 - 1/8} \\ &= \frac{6}{7} \end{aligned}$$

Thus, they are closer to each other than they are to $(\overline{100})$ because they agree in the first position.

4. $\mathbf{s} = (\overline{1011})$, $\mathbf{t} = (\overline{01})$

$$\begin{aligned} d[(\overline{1011}), (\overline{0101})] &= \left(1 + \frac{1}{2} + \frac{1}{4}\right) + \left(\frac{1}{16} + \frac{1}{32} + \frac{1}{64}\right) + \cdots \\ &= \frac{7}{4} + \frac{7}{64} + \cdots \\ &= 7 \cdot \frac{1/4}{1 - 1/16} \\ &= \frac{28}{15} \end{aligned}$$

Considering the fact that the maximum distance between any pair of strings in Σ is two units, the above 4-cycles are surprisingly far apart. (Do you see why?)

5. Find all points in Σ whose distance from $(000\ldots)$ is exactly $1/2$.

First of all, any such string $\mathbf{t}$ must be in M_0, otherwise $d[(000\ldots),\mathbf{t}] \geq 1$. It turns out that there are two strings $\mathbf{t}$ such that $d[(000\ldots),\mathbf{t}] = 1/2$, namely, $(01\overline{0})$ and $(00\overline{1})$.

This result may be generalized. Let $\mathbf{s} = (s_0s_1\ldots)$. There exist two strings $\mathbf{t}$ such that $d[\mathbf{s},\mathbf{t}] = 1/2^n$, namely, $(s_0s_1\ldots\hat{s}_ns_{n+1}s_{n+2}\ldots)$ and $(s_0s_1\ldots s_n\hat{s}_{n+1}\hat{s}_{n+2}\ldots)$.

6. Give an example of a sequence midway between $(000\ldots)$ and $(111\ldots)$. Give a second such example. Are there any other such points? Why or why not?

One of the two points midway between $(\overline{0})$ and $(\overline{1})$ is $(0\overline{1})$ since $d[(0\overline{1}),(\overline{0})] = d[(0\overline{1}),(\overline{1})] = 1$. The other point is $(1\overline{0})$, and these are the only such symbol sequences by virtue of the Proximity Theorem.

7. Let $M_{01} = \{\, \mathbf{s} \in \Sigma \mid s_0 = 0, s_1 = 1 \,\}$ and $M_{101} = \{\, \mathbf{s} \in \Sigma \mid s_0 = 1, s_1 = 0, s_2 = 1 \,\}$. What is the minimum distance between a point in M_{01} and a point in M_{101}? Give an example of two sequences that are this close to each other.

Let $\mathbf{s} \in M_{01}$ and $\mathbf{t} \in M_{101}$. Then $d[\mathbf{s},\mathbf{t}] \geq 3/2$ since $\mathbf{s}$ and $\mathbf{t}$ differ in the first two positions. For example, let $\mathbf{s} = (01\overline{1})$ and $\mathbf{t} = (10\overline{1})$. Then $d[\mathbf{s},\mathbf{t}] = 3/2$. More generally, suppose $\mathbf{t} = (101t_3t_4\ldots)$. Then $\mathbf{s} = (011t_3t_4\ldots)$ is exactly $3/2$ units away from $\mathbf{t}$.

8. What is the maximum distance between a point in M_{01} and a point in M_{101}? Give an example of two sequences that are this far apart.

The maximum distance is 2 units since the strings need not agree at any position. Given any $\mathbf{t} \in M_{101}$, all strings in M_{01} of the form $(010\hat{t}_3\hat{t}_4\ldots)$ are 2 units away from $\mathbf{t}$.

The N-Shift:

The following seven exercises deal with the analogue of the shift map and sequence space for sequences that have more than two possible entries, the space of sequences of N symbols.

10. Let Σ_N denote the space of sequences whose entries are the positive integers $0, 1, \ldots, N-1$, and let σ_N be the shift map on Σ_N. For $\mathbf{s},\mathbf{t} \in \Sigma_N$, let

$$d_N[\mathbf{s},\mathbf{t}] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{N^i}.$$

Prove that d_N is a metric on Σ_N.

The function d_N is nonnegative since $|s_i - t_i| \geq 0$ for all i, and it vanishes if and only if $\mathbf{s} = \mathbf{t}$. So all we really need to show is symmetry and the triangle inequality. Symmetry follows since $|s_i - t_i| = |t_i - s_i|$ for all numbers s_i and t_i. The triangle inequality is a consequence of the triangle inequality for ordinary numbers:

$$\begin{aligned} d_N[\mathbf{s},\mathbf{t}] + d_N[\mathbf{t},\mathbf{u}] &= \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{N^i} + \sum_{i=0}^{\infty} \frac{|t_i - u_i|}{N^i} \\ &= \sum_{i=0}^{\infty} \frac{|s_i - t_i| + |t_i - u_i|}{N^i} \\ &\geq \sum_{i=0}^{\infty} \frac{|s_i - u_i|}{N^i} \\ &= d_N[\mathbf{s},\mathbf{u}]. \end{aligned}$$

Thus d_N is a metric and $\langle d_N, \Sigma_N \rangle$ is a metric space.

11. What is the maximal distance between a pair of sequences in Σ_N?

The maximum value of $|s_i - t_i|$ is $N-1$, and therefore, the maximum distance between two sequences in Σ_N is

$$\begin{aligned} d_N[\mathbf{s},\mathbf{t}] &= \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{N^i} \\ &\leq \sum_{i=0}^{\infty} \frac{N-1}{N^i} \\ &= (N-1)\frac{1}{1-1/N} \\ &= N. \end{aligned}$$

So, in general, the maximum distance can be no more than the size of the alphabet.

12. How many fixed points does σ_N have? How many 2-cycles? How many cycles of prime period 2?

Let σ_N be the shift map on Σ_N. Specifically, let $\sigma_N\colon \Sigma_N \to \Sigma_N$ with

$$\sigma_N(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots).$$

Now σ_N has N fixed points; indeed,

$$\operatorname{fix} \sigma_N = \{(000\ldots), (111\ldots), \ldots, (kkk\ldots)\}$$

where $k = N - 1$. Recall that σ_2 has two points of prime period 2 and we wonder if σ_N has N points of prime period 2. It turns out that this is *not* the case since any sequence of the form $(\overline{s_0 s_1})$ is of period 2, and there are N^2 such points. But N of these are fixed, and so there are $N^2 - N = N(N-1)$ points of prime period 2.

13. How many points in Σ_N are fixed by σ_N^n?

For starters, how many periodic points of period 3 are there? We have that

$$\operatorname{per}_3 \sigma_N = \{ (\overline{s_0 s_1 s_2}) \mid s_0, s_1, s_2 \in \Sigma_N \}$$

and so $|\operatorname{per}_3 \sigma_N| = N^3$. Of these, $N^3 - N = N(N^2-1)$ are of prime period 3 since N of them are fixed. It's not very difficult to see that

$$|\operatorname{per}_n \sigma_N| = N^n,$$

in general. For example, σ_N has N^4 4-cycles and $N^4 - N^2 = N^2(N^2-1)$ sequences of prime period 4, but it's not clear how many points of *prime* period n there are in general.

14. Prove that $\sigma_N\colon \Sigma_N \to \Sigma_N$ is continuous.

First we give a careful proof that $\sigma\colon \Sigma \to \Sigma$ is continuous, a result which boils down to the following important fact:

Lemma 9.1 *If* $d[(s_0 s_1 s_2 \ldots), (t_0 t_1 t_2 \ldots)] < 1/2^{n+1}$, *then*

$$d[(s_1 s_2 s_3 \ldots), (t_1 t_2 t_3 \ldots)] \le 1/2^n.$$

Proof: If $d[(s_0 s_1 s_2 \ldots), (t_0 t_1 t_2 \ldots)] < 1/2^{n+1}$, then s_i agrees with t_i for $i \in \{0, 1, \ldots, n+1\}$ (by the Proximity Theorem). And if $s_i = t_i$ for $i \in \{0, 1, \ldots, n+1\}$ (and hence, for $i \in \{1, 2, \ldots, n+1\}$), then

$$d[(s_1 s_2 s_3 \ldots), (t_1 t_2 t_3 \ldots)] \le 1/2^n,$$

also by the Proximity Theorem. □

Using this result, the proof that σ is continuous proceeds as follows:

1. let $\epsilon > 0$ be given, and
2. choose n such that $1/2^n < \epsilon$;
3. let $\delta = 1/2^{n+1}$, and
4. suppose $d[(s_0 s_1 s_2 \ldots), (t_0 t_1 t_2 \ldots)] < \delta$;
5. then $d[(s_1 s_2 s_3 \ldots), (t_1 t_2 t_3 \ldots)] \le 1/2^n < \epsilon$.

Next we need a generalization of the Proximity Theorem. Let $\mathbf{s}, \mathbf{t} \in \Sigma_N$ and suppose $s_i = t_i$ for $i \in \{0, 1, \ldots, n\}$, then

$$\begin{aligned} d[\mathbf{s}, \mathbf{t}] &= \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{N^i} \\ &\le \sum_{i=n+1}^{\infty} \frac{N-1}{N^i} \\ &= (N-1)\frac{1/N^{n+1}}{1 - 1/N} \\ &= \frac{1}{N^n}. \end{aligned}$$

The converse is also true: if $d[\mathbf{s}, \mathbf{t}] < 1/N^n$, then $s_i = t_i$ for $i \in \{0, 1, \ldots, n\}$. We remark, however, that it is *not* the case that $s_i = t_i$ for $i \in \{0, 1, \ldots, n\}$ if and only if $d[\mathbf{s}, \mathbf{t}] \le 1/N^n$, and the reader is encouraged to find an appropriate counterexample (see Exercise 5 for a hint).

To show that σ_N is continuous, let $\epsilon > 0$ be given and choose n so that $1/N^n < \epsilon$. Now suppose $\delta = 1/N^{n+1}$ and $d[\mathbf{s}, \mathbf{t}] < \delta$. Then $s_i = t_i$ for $i \in \{0, 1, \ldots, n+1\}$, and

$$d[\sigma_N(\mathbf{s}), \sigma_N(\mathbf{t})] \le 1/N^n < \epsilon,$$

both by the above analogue to the Proximity Theorem. Thus σ_N is continuous.

15. Now define

$$d_\delta[\mathbf{s}, \mathbf{t}] = \sum_{k=0}^{\infty} \frac{\delta_k(\mathbf{s}, \mathbf{t})}{N^k}$$

where

$$\delta_k(\mathbf{s},\mathbf{t}) = \begin{cases} 0 & \text{if } s_k = t_k \\ 1 & \text{if } s_k \neq t_k \end{cases} . \tag{9.1}$$

Prove that d_δ is also a metric on Σ_N.

Is d_δ a metric? And is the particular choice of δ_k important? What if $\delta_k(\mathbf{s},\mathbf{t}) = s_k + t_k \bmod N$, or

$$\delta_k(\mathbf{s},\mathbf{t}) = \begin{cases} 0 & \text{if } s_k = t_k \\ N-1 & \text{if } s_k \neq t_k \end{cases} , \tag{9.2}$$

for instance? Well, certainly $d_\delta[\mathbf{s},\mathbf{t}] \geq 0$ since $\delta_k(\mathbf{s},\mathbf{t}) \geq 0$ by definition in all three cases. But is $d_\delta[\mathbf{s},\mathbf{t}] = 0$ if and only if $\mathbf{s} = \mathbf{t}$? The answer is "yes" in the case of (9.1) and (9.2), and "no" when $\delta_k(\mathbf{s},\mathbf{t}) = s_k + t_k \bmod N$ (since $\delta_k((\overline{0}),(\overline{N})) = 0$, for instance).

Both (9.1) and (9.2) are symmetric, that is, $d_\delta[\mathbf{s},\mathbf{t}] = d_\delta[\mathbf{t},\mathbf{s}]$, but what of the elusive triangle inequality? Consider the situation in (9.1) where we must show that $\delta_k(\mathbf{s},\mathbf{t}) + \delta_k(\mathbf{t},\mathbf{u}) \geq \delta_k(\mathbf{s},\mathbf{u})$. There are four cases to check:

$\delta_k(\mathbf{s},\mathbf{t})$	$\delta_k(\mathbf{t},\mathbf{u})$	$\delta_k(\mathbf{s},\mathbf{u})$
0	0	0
0	1	1
1	0	1
1	1	0 or 1

In all cases, $\delta_k(\mathbf{s},\mathbf{t}) + \delta_k(\mathbf{t},\mathbf{u}) \geq \delta_k(\mathbf{s},\mathbf{u})$ and so d_δ satisfies the triangle inequality. Likewise, enumeration will also show that (9.2) gives rise to a metric space.

16. What is the maximum distance between two points in Σ_N when the previous metric d_δ is used?

Since $\delta_k(\mathbf{s},\mathbf{t})$ given in (9.1) is at most 1, $d_\delta[\mathbf{s},\mathbf{t}]$ can be at most $N/(N-1)$ because

$$\begin{aligned} d_\delta[\mathbf{s},\mathbf{t}] &= \sum_{k=0}^{\infty} \frac{\delta_k(\mathbf{s},\mathbf{t})}{N^k} \\ &\leq \sum_{k=0}^{\infty} \frac{1}{N^k} \\ &= \frac{N}{N-1}. \end{aligned}$$

We leave it to the reader to determine the maximum distance between two points in Σ_N for $\delta_k(\mathbf{s},\mathbf{t})$ in (9.2).

18. Each of the following defines a function on the space of sequences Σ. In each case, decide if the given function is continuous. If so, prove it. If not, explain why.

18a) $F(s_0s_1s_2\ldots) = (0s_0s_1s_2\ldots)$.

Let $\mathbf{s} = (s_0s_1s_2\ldots)$ and $\mathbf{t} = (t_0t_1t_2\ldots)$. Then $F(\mathbf{s}) = (0s_0s_1s_2\ldots)$, $F(\mathbf{t}) = (0t_0t_1t_2\ldots)$, and

$$\begin{aligned} d[F(\mathbf{s}),F(\mathbf{t})] &= \sum_{i=1}^{\infty} \frac{|s_{i-1}-t_{i-1}|}{2^i} \\ &= \frac{1}{2}\sum_{i=1}^{\infty} \frac{|s_{i-1}-t_{i-1}|}{2^{i-1}} \\ &= \frac{1}{2}\sum_{i=0}^{\infty} \frac{|s_i-t_i|}{2^i} \\ &= \frac{1}{2}d[\mathbf{s},\mathbf{t}]. \end{aligned}$$

This suggests we prove F continuous as follows: Let $\epsilon > 0$ be given, and choose n such that $1/2^n < \epsilon$. Now let $\delta = 1/2^{n-1}$ and suppose $d[\mathbf{s},\mathbf{t}] < \delta$. Then

$$d[F(\mathbf{s}),F(\mathbf{t})] = \frac{1}{2}d[\mathbf{s},\mathbf{t}] < \frac{1}{2}\delta = \frac{1}{2^n} < \epsilon,$$

and so F is continuous.

18b) $G(s_0s_1s_2\ldots) = (0s_00s_10s_2\ldots)$.

We proceed as in Exercise 18a. In the present case, $d[G(\mathbf{s}),G(\mathbf{t})]$ is strictly less than $d[\mathbf{s},\mathbf{t}]/2$ however, since

$$\begin{aligned} d[G(\mathbf{s}),G(\mathbf{t})] &= \sum_{i=0}^{\infty} \frac{|s_i-t_i|}{2^{2i+i}} \\ &= \frac{1}{2}\sum_{i=0}^{\infty} \frac{|s_i-t_i|}{4^i} \end{aligned}$$

$$< \frac{1}{2}\sum_{i=0}^{\infty}\frac{|s_i - t_i|}{2^i}$$
$$= \frac{1}{2}d[\mathbf{s},\mathbf{t}].$$

Therefore,

$$d[G(\mathbf{s}),G(\mathbf{t})] < \frac{1}{2}d[\mathbf{s},\mathbf{t}] < \frac{1}{2}\delta = \frac{1}{2^n} < \epsilon$$

upon choosing appropriate values for n and δ.

18c) $H(s_0s_1s_2\ldots) = (s_1s_0s_3s_2s_5s_4\ldots)$.

Suppose $\mathbf{s}$ and $\mathbf{t}$ are such that $s_{2k+1} \neq t_{2k+1}$ and $s_{2k} = t_{2k}$ for all k. In other words, $\mathbf{s}$ and $\mathbf{t}$ differ with respect to every other entry starting with s_1 and t_1. Then

$$d[H(\mathbf{s}),H(\mathbf{t})] = \sum_{i=0}^{\infty}\frac{|s_{2i+1} - t_{2i+1}|}{2^{2i}}$$
$$= 2\sum_{i=0}^{\infty}\frac{|s_{2i+1} - t_{2i+1}|}{2^{2i+1}}$$
$$= 2d[\mathbf{s},\mathbf{t}].$$

For example, let $\mathbf{s} = (000\ldots)$ and $\mathbf{t} = (\overline{01})$. Then $H(\mathbf{s}) = \mathbf{s}$ and $H(\mathbf{t}) = (\overline{10})$, and so we have

$$\frac{4}{3} = d[H(\mathbf{s}),H(\mathbf{t})] = 2d[\mathbf{s},\mathbf{t}] = 2\cdot\frac{2}{3} = \frac{4}{3}.$$

Now suppose $s_{2k} \neq t_{2k}$ and $s_{2k+1} = t_{2k+1}$ for all k. These symbol sequences still differ with respect to every other entry, but offset by one position. In this case,

$$d[H(\mathbf{s}),H(\mathbf{t})] = \sum_{i=0}^{\infty}\frac{|s_{2i} - t_{2i}|}{2^{2i+1}}$$
$$= \frac{1}{2}\sum_{i=0}^{\infty}\frac{|s_{2i} - t_{2i}|}{2^{2i}}$$
$$= \frac{1}{2}d[\mathbf{s},\mathbf{t}].$$

Now, the claim is that for $\mathbf{s},\mathbf{t} \in \Sigma$,

$$\frac{1}{2}d[\mathbf{s},\mathbf{t}] \le d[H(\mathbf{s}),H(\mathbf{t})] \le 2d[\mathbf{s},\mathbf{t}]. \tag{9.3}$$

If you believe this, then the continuity of H follows almost immediately. As usual, let $\epsilon > 0$ be given, and choose n so that $1/2^n < \epsilon$. Then let $\delta = 1/2^{n+1}$ and suppose $d[\mathbf{s},\mathbf{t}] < \delta$. But $d[H(\mathbf{s}),H(\mathbf{t})] \le 2d[\mathbf{s},\mathbf{t}]$ by (9.3), and moreover,

$$2d[\mathbf{s},\mathbf{t}] < 2\delta = 1/2^n < \epsilon.$$

Thus $d[H(\mathbf{s}),H(\mathbf{t})] < \epsilon$ and so H is continuous.

18d) $J(s_0s_1s_2\ldots) = (\hat{s}_0\hat{s}_1\hat{s}_2\ldots)$ where $\hat{s}_j = 1$ if $s_j = 0$, and $\hat{s}_j = 0$ if $s_j = 1$.

We claim that $d[J(\mathbf{s}),J(\mathbf{t})] = d[\mathbf{s},\mathbf{t}]$. This is because $|s_j - t_j| = |\hat{s}_j - \hat{t}_j|$ which is easily seen by enumerating all possible cases (of which there are only four). The proof that J is continuous is immediate. Just choose $\delta = \epsilon$.

18e) $K(s_0s_1s_2\ldots) = ((1-s_0)(1-s_1)(1-s_2)\ldots)$.

We remark that the mapping K makes sense only if we think of the s_i as binary digits (which they aren't!). In this context, $K = J$ of Exercise 18d and so K is continuous since J is.

18f) $L(s_0s_1s_2\ldots) = (s_0s_2s_4s_6\ldots)$.

The trick here is to pick δ small enough so that $\mathbf{s}$ and $\mathbf{t}$ agree on the first $2n+1$ entries.

As always, let $\epsilon > 0$ be given and choose n so that $1/2^n < \epsilon$. Now let $\delta = 1/2^{2n}$ and suppose $d[\mathbf{s},\mathbf{t}] < \delta$. Then by the Proximity Theorem, $s_i = t_i$ for $i = 0,1,\ldots,2n$, and so,

$$d[L(\mathbf{s}),L(\mathbf{t})] = \sum_{i=n+1}^{\infty}\frac{|s_{2i} - t_{2i}|}{2^i}$$
$$\le \sum_{i=n+1}^{\infty}\frac{1}{2^i}$$
$$= \frac{1/2^{n+1}}{1-1/2}$$
$$= \frac{1}{2^n} < \epsilon.$$

Thus L is continuous.

18g) $M(s_0s_1s_2\ldots) = (s_0s_{10}s_{100}s_{1000}\ldots)$.

This exercise parallels the previous one, except this time we must choose δ to be very, very small. Specifically, we may choose $\delta = 1/2^{10^n}$ so that $\mathbf{s}$ and $\mathbf{t}$ agree on the first $10^n + 1$ entries.

19. Define a different distance function d' on Σ by $d'[\mathbf{s},\mathbf{t}] = 1/(k+1)$ where k is the least index for which $s_k \neq t_k$ and $d'[\mathbf{s},\mathbf{s}] = 0$. Is d' a metric?

Yes, d' is a metric. Observe that $d'[\mathbf{s},\mathbf{t}] \geq 0$ for all $\mathbf{s}$ and $\mathbf{t}$, and that $d'[\mathbf{s},\mathbf{t}] = 0$ if and only if $\mathbf{s} = \mathbf{t}$. That $d'[\mathbf{s},\mathbf{t}] = d'[\mathbf{t},\mathbf{s}]$ is also obvious from the definition. But does d' satisfy the triangle inequality? Let $\mathbf{s}$, $\mathbf{t}$, and $\mathbf{u}$ be symbol sequences in Σ, and let $k_1 = \min\{\, i \in \mathbb{N} \mid s_i \neq t_i \,\}$ and $k_2 = \min\{\, i \in \mathbb{N} \mid t_i \neq u_i \,\}$. A key result is

$$d'[\mathbf{s},\mathbf{u}] = \frac{1}{\min\{k_1, k_2\}},$$

but clearly,

$$\begin{aligned}\frac{1}{\min\{k_1,k_2\}} &< \frac{1}{\min\{k_1,k_2\}} + \frac{1}{\min\{k_1,k_2\}} \\ &\leq \frac{1}{k_1} + \frac{1}{k_2},\end{aligned}$$

and so the triangle inequality is satisfied.

Chapter 10

Chaos

Exercises

For each of the following sets, decide whether or not the set is dense in $[0,1]$.

1. S_1 is the set of all real numbers in $[0,1]$ except those of the form $1/2^n$ for $n = 1, 2, 3, \ldots$.

Yes, S_1 is dense in $[0,1]$. One way to show this is to first let $w = 1/2^n$ for some fixed positive integer n, and then produce a sequence x_k in S_1 such that $x_k \to w$ as $k \to \infty$.

We use a bisection technique on the interval $[w, 2w]$. Begin by letting $x_0 = 2w = 1/2^{n-1}$ and forming the sequence

$$x_k = \frac{w + x_{k-1}}{2}$$

for $k > 0$. For example,

$$x_1 = \frac{w + x_0}{2} = \frac{1/2^n + 1/2^{n-1}}{2} = \frac{3}{2^{n+1}},$$

$$x_2 = \frac{w + x_1}{2} = \frac{1/2^n + 3/2^{n+1}}{2} = \frac{5}{2^{n+2}},$$

$$x_3 = \frac{w + x_2}{2} = \frac{1/2^n + 5/2^{n+2}}{2} = \frac{9}{2^{n+3}},$$

and in general, it appears that

$$x_k = \frac{2^k + 1}{2^{n+k}}$$

which the reader is asked to show by induction. Note carefully that $x_k \in S_1$ for $k > 0$. We claim that $x_k \to w$ as $k \to \infty$.

To see this, consider the function

$$F_w(x) = \frac{w + x}{2}$$

where w is any real number. Note that F_w is a linear map with attracting fixed point w. Moreover, its basin of attraction is the whole real line.[1]

In summary, let $w = 1/2^n \notin S_1$, $x_0 = 2w$, and $F_w(x) = (w + x)/2$. Then $F_w^k(x_0) \to w$ as $k \to \infty$, and $F_w^k(x_0) \in S_1$ for all positive integers k. This proves that S_1 is dense in $[0,1]$.

2. S_2 is the set of all rationals in $[0,1]$ of the form $p/2^n$ where p and n are natural numbers.

Note that S_2 contains all multiples of $1/2$, all multiples of $1/4$, etc. In fact, $x \in S_2$ if and only if x has a terminating binary expansion. Assuming this to be true for the moment, we may easily show that S_2 is dense in $[0,1]$ as follows:

Let $w = 0.b_1b_2b_3\ldots$ be an arbitrary point in $[0,1]$. If w terminates, then we are done, so suppose it does not. Then the sequence

$$0.b_1,\ 0.b_1b_2,\ 0.b_1b_2b_3, \ldots$$

obviously converges to w, and each element of this sequence is in S_2. q.e.d.

Using a bisection technique, we now exhibit such a sequence in S_2 converging to w. Take the unit interval, divide it in half, and determine which half contains w. Discard the half which does not. Halve the remaining interval, and again ask which half contains w. Continue this halving process, each time throwing away the half interval which does not contain w. This **binary search** technique, as it's called, captures w to any degree of precision one cares to specify.

A corresponding algorithm is given in Figure 10.1. The variables l_k and r_k are the left-hand and right-hand endpoints of the subintervals, respectively,

[1] **We say that w is a globally attracting fixed point for F_w.**

```
k := 1
l_k := 0; r_k := 1
loop
   m_k := (l_k + r_k)/2
   if w < m_k then
      b_k := 0; r_k := m_k
   else if w > m_k then
      b_k := 1; l_k := m_k
   end if
   k := k + 1
end loop
```

Figure 10.1: A binary search algorithm for w.

and m_k is the computed midpoint. We'll describe the b_k in a moment, but first observe the following facts:

1. both l_1 and r_1 are in S_2, and hence, $m_1 \in S_2$ (in fact, $m_1 = 1/2$); arguing inductively, if l_k and r_k are contained in S_2, then so is m_k, and hence, all l_k and r_k are contained in S_2;
2. since $w \notin S_2$, $w \neq m_k$ for all k;
3. $m_k \to w$ as $k \to \infty$.

The binary search algorithm in Figure 10.1 can also be thought of as a binary tree with the elements of S_2 at the nodes, and 0s and 1s decorating the edges. See Figure 10.2.

In this tree, a movement to the left traverses an edge marked with a 0, while a motion to the right picks up a 1. These bits correspond to the binary expansion of w, and are precisely the b_k given in the algorithm. In fact, the concatenation of all these b_k corresponds to one of the terminating binary expansions mentioned earlier.

3. S_3 is the Cantor middle-thirds set.

The Cantor set K can not be dense. There can't possibly be a sequence in K converging to $1/2$, for example, since the interval $(1/3, 2/3)$ was removed at the first stage of the Cantor set construction.

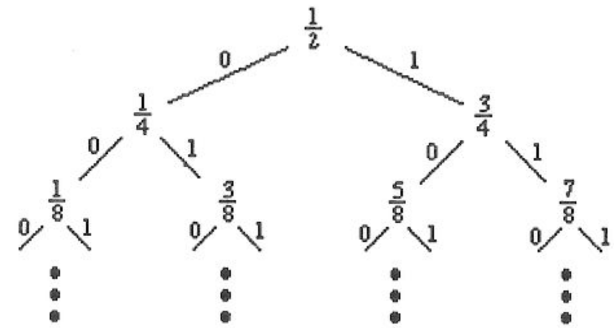


Figure 10.2: A binary tree of the elements in S_2.

4. S_4 is the complement of the Cantor middle-thirds set.

Recall that the ternary expansion of $x \in K$ has no 1s. Any point which *does* have a 1 in its ternary expansion is in K's complement. Moreover, any finite string can be prepended to such a point, and the result is *still* in K's complement. Thus the complement of K is dense in $[0, 1]$ since there are (uncountably many) points in K's complement arbitrarily close to any point x in the Cantor set. Just prepend the first $n+1$ bits of x to your favorite point in the complement of K.

5. S_5 is the complement of any subset of $[0, 1]$ which has countably many elements.

Note that the set S_1 in Exercise 1 is a special case of S_5. Yes, the unit interval remains dense even after removing a countably infinite number of points.

The following argument depends on the fact that open intervals are uncountable sets.[2] Let x be any point in the complement of S_5, and let $\epsilon > 0$ be given. Now let $N = N(x)$ be any ϵ-neighborhood of x. Then $N \cap S_5$ is nonempty since N is an open interval and hence, uncountable. Hence, S_5 is dense in $[0, 1]$.

For each of the following sets, decide whether or not the set is dense in Σ. Give reasons.

6. $T_1 = \{ (s_0 s_1 s_2 \ldots) \mid s_4 = 0 \}$.

[2] Thanks to John Thoo for reminding me of this.

Any point $\mathbf{s} \notin T_1$ is of the form $(s_0s_1s_2s_31s_5s_6\ldots)$. The point in T_1 closest to $\mathbf{s}$ is $(s_0s_1s_2s_30s_5s_6\ldots)$, and in fact,

$$d[(s_0s_1s_2s_31s_5s_6\ldots),(s_0s_1s_2s_30s_5s_6\ldots)] = \frac{1}{2^4}.$$

Therefore, T_1 can not possibly be dense in Σ.

7. T_2 is the complement of T_1.

Observe that $T_2 = \{\,(s_0s_1s_2\ldots) \mid s_4 = 1\,\}$. By an argument virtually identical to the one in Exercise 6, T_2 can not possibly be dense in Σ.

8. $T_3 = \{\,(s_0s_1s_2\ldots) \mid$ the sequence ends in all 0s$\,\}$.

We will show that T_3 is dense. Take an arbitrary $\mathbf{s} = (s_0s_1s_2\ldots)$ in Σ and construct a sequence of points $\mathbf{s}_n$ such that

$$\mathbf{s}_n = (s_0s_1\ldots s_{n-1}\overline{0}).$$

Note that each $\mathbf{s}_n \in T_3$ and that $\mathbf{s}_n \to \mathbf{s}$ as $n \to \infty$. Thus T_3 is dense in Σ. (See Exercise 2 for a related problem.)

9. $T_4 = \{\,(s_0s_1s_2\ldots) \mid$ at most one of the $s_j = 0\,\}$.

This set can not possibly be dense in Σ. Consider the point $(00\overline{1})$ in T_4's complement. There is no sequence of points in T_4 converging to $(00\overline{1})$. Indeed, the closest point in T_4 to $(00\overline{1})$ is $(0\overline{1})$, and

$$d[(00\overline{1}),(0\overline{1})] = 1/2.$$

In fact, there is no sequence in T_4 converging to *any* point in its complement. Let $\mathbf{s} \notin T_4$. Then $\mathbf{s}$ has at least two 0s. Now suppose s_k is the second of these two 0s. Then each point in T_4 is at least $1/2^k$ units away from $\mathbf{s}$, and so there can be no sequence in T_4 converging to it.

10. $T_5 = \{\,(s_0s_1s_2\ldots) \mid$ infinitely many of the $s_j = 0\,\}$.

Take a point in T_5's complement which has but finitely many 0s. Then it has infinitely many 1s, and in fact, the tail of the sequence must be all 1s and hence of the form $(s_0s_1\ldots s_n\overline{1})$ for some n. But the sequence of points

$$(s_0s_1\ldots s_n\overline{0}),\ (s_0s_1\ldots s_n1\overline{0}),\ (s_0s_1\ldots s_n11\overline{0}),\ldots$$

converges to $(s_0s_1\ldots s_n\overline{1})$, and each such point is an element of T_5. Thus T_5 is dense in Σ. (See Exercise 8 for a related problem.)

11. T_6 is the complement of T_5.

As argued in Exercise 10, a point in T_6 must end in all 1s. Consequently, we proceed as in Exercise 8 by constructing a sequence of points, each ending in all 1s, converging to any point in Σ.

We remark that T_6 is *not* the same as

$$\{\,(s_0s_1s_2\ldots) \mid \text{infinitely many of the } s_j = 1\,\}$$

since there are strings in the latter which are not in T_6. The string $(\overline{01})$ is one such example.

12. $T_7 = \{\,(s_0s_1s_2\ldots) \mid$ no two consecutive $s_j = 0\,\}$.

In words, T_7 consists of those strings in which every 0 is followed by a 1. The complement of this set consists of all strings with a consecutive pair of 0s. The string $(00\overline{1})$ is one such example. Unfortunately, there is no string in T_7 close to this string—in fact, the element in T_7 closest to $(00\overline{1})$ is $(0\overline{1})$, and $d[(00\overline{1}),(0\overline{1})] = 1/2$. Therefore, T_7 is not dense.

13. T_8 is the complement of T_7.

The complement of T_7 is dense in Σ. As mentioned in Exercise 12, T_8 is the set of strings containing a consecutive pair of 0s. Now take any point in Σ and construct a sequence in T_8 converging to it. (The sequence in Exercise 8 will serve this purpose just fine.)

15. Is the orbit of the point (01 001 0001 00001 ...) under σ dense in Σ?

No—in fact, the orbit of (01 001 0001 00001 ...) stays away from M_{11} altogether. And there's really nothing special about the systematically increasing number of 0s in this string. No element of

$$\{\text{strings of 0s and 1s not having 11 as a substring}\}$$

has an orbit which is dense in Σ.

16. Is it possible to give an example of an orbit under σ that accumulates on (i.e., comes arbitrarily close to but never equals) the two fixed points of σ, but which is not dense?

Consider the orbit of

$$(01\ 0011\ 000111\ 00001111\ldots)$$

under σ. This orbit comes arbitrarily close to either of the fixed points, but it is not dense since it stays away from all other periodic points, for example.

17. Prove that, if $\mathrm{s} \in \Sigma$, there are sequences t arbitrarily close to s for which $d[\sigma^n(\mathrm{s}), \sigma^n(\mathrm{t})] = 2$ for all sufficiently large n.

Let $\mathrm{s} = (s_0 s_1 s_2 \ldots) \in \Sigma$ and consider the point

$$\mathrm{t} = (s_0 s_1 \ldots s_n \hat{s}_{n+1} \hat{s}_{n+2} \ldots).$$

By the Proximity Theorem, we know that

$$d[\mathrm{s}, \mathrm{t}] \leq \frac{1}{2^n},$$

but

$$d[\sigma^k(\mathrm{s}), \sigma^k(\mathrm{t})] = 2$$

for all $k > n$.

18. Prove that the set of endpoints of removed intervals in the Cantor middle-thirds set is a dense subset of the Cantor set.

Points in the Cantor set have no 1s in their ternary expansion, while endpoints of removed intervals correspond to terminating ternary expansions. The rest of the proof is straightforward (see Exercise 2).

19. Let $V(x) = 2|x| - 2$. Find the fixed points of V and V^2. Compute an expression for V^3.

The function $V: [-2, 2] \to [-2, 2]$ is a piecewise linear approximation to our old friend $Q_{-2}: [-2, 2] \to [-2, 2]$. See Figure 10.3. Note that both V and Q_{-2} are 2-to-1 and onto the closed interval $[-2, 2]$.

Now, by definition of absolute value, we have that

$$V(x) = \begin{cases} 2x - 2 & \text{if } 0 \leq x \leq 2 \\ -2x - 2 & \text{if } -2 \leq x \leq 0 \end{cases}.$$

Setting each part of this piecewise linear map equal to x and solving, we get

$$\text{fix}\, V = \{-2/3, 2\}.$$

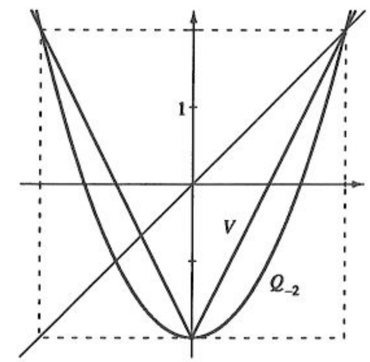


Figure 10.3: The quadratic map $Q_{-2}(x) = x^2 - 2$ and its piecewise linear approximation $V(x) = 2|x| - 2$.

Now recall that a formula for $V^2(x)$ was derived in Section 10.2 of the text. It is

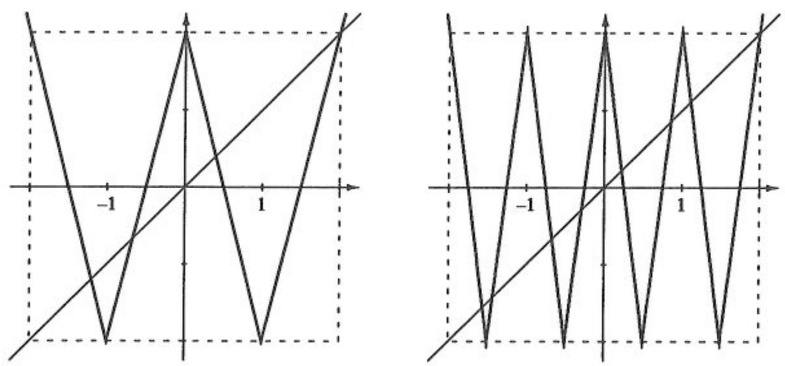
$$V^2(x) = \begin{cases} 4x - 6 & \text{if } 1 \leq x \leq 2 \\ -4x + 2 & \text{if } 0 \leq x \leq 1 \\ 4x + 2 & \text{if } -1 \leq x \leq 0 \\ -4x - 6 & \text{if } -2 \leq x \leq -1 \end{cases}$$

and *its* fixed points may be found similarly. (See Figure 10.4a.) The reader should verify that

$$\text{fix}\, V^2 = \left\{ -\frac{6}{5}, -\frac{2}{3}, \frac{2}{5}, 2 \right\}.$$

We now derive an expression for $V^3(x)$. First observe that V partitions the closed interval $[-2, 2]$ into two subintervals, $[-2, 0]$ and $[0, -2]$. Let's denote this partition by $[-2, 0, 2]$. Now, the second iterate V^2 further partitions $[-2, 2]$ into the four subintervals given by $[-2, -1, 0, 1, 2]$. But where did the numbers 1 and -1 come from? Well, they happen to be the zeroes of V. (This is no accident. Do you see why this must be so? If not, take a moment to review the derivation of $V^2(x)$ in the text.)

So let's iterate this procedure! What are the zeroes of V^2? Setting each part of $V^2(x)$ equal to zero and solving, we get $\{3/2, 1/2, -1/2, -3/2\}$, which is not all that surprising considering the symmetry properties of V



(a) The graph of V^2. (b) The graph of V^3.

Figure 10.4: The second and third iterates of $V(x) = 2|x| - 2$.

and its iterates. These calculations suggest the partition

$$\left[-2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\right]$$

for V^3. (Exercise: What partition might we try for V^4?)

Back to the derivation of $V^3(x)$. There are eight cases to consider, four of which are detailed below:

$$\begin{aligned}
-2 \le x \le -3/2 \quad &\Rightarrow \quad 1 \le V(x) \le 2 \text{ and } 0 \le V^2(x) \le 2 \\
&\Rightarrow \quad V^3(x) = V(V^2(x)) \\
&\qquad = V(-4x-6) \\
&\qquad = 2(-4x-6) - 2 \\
&\qquad = -8x - 14 \\
-3/2 \le x \le -1 \quad &\Rightarrow \quad 0 \le V(x) \le 1 \text{ and } -2 \le V^2(x) \le 0 \\
&\Rightarrow \quad V^3(x) = V(V^2(x)) \\
&\qquad = V(-4x-6) \\
&\qquad = -2(-4x-6) - 2 \\
&\qquad = 8x + 10
\end{aligned}$$

$$\begin{aligned}
-1 \le x \le -1/2 \quad &\Rightarrow \quad -1 \le V(x) \le 0 \text{ and } -2 \le V^2(x) \le 0 \\
&\Rightarrow \quad V^3(x) = V(V^2(x)) \\
&\qquad = V(4x+2) \\
&\qquad = -2(4x+2) - 2 \\
&\qquad = -8x - 6 \\
-1/2 \le x \le 0 \quad &\Rightarrow \quad -2 \le V(x) \le -1 \text{ and } 0 \le V^2(x) \le 2 \\
&\Rightarrow \quad V^3(x) = V(V^2(x)) \\
&\qquad = V(4x+2) \\
&\qquad = 2(4x+2) - 2 \\
&\qquad = 8x + 2
\end{aligned}$$

Completing the remaining four cases we get the following expression for $V^3(x)$:

$$V^3(x) = \begin{cases} -8x - 14 & \text{if } -2 \le x \le -3/2 \\ 8x + 10 & \text{if } -3/2 \le x \le -1 \\ -8x - 6 & \text{if } -1 \le x \le -1/2 \\ 8x + 2 & \text{if } -1/2 \le x \le 0 \\ -8x + 2 & \text{if } 0 \le x \le 1/2 \\ 8x - 6 & \text{if } 1/2 \le x \le 1 \\ -8x + 10 & \text{if } 1 \le x \le 3/2 \\ 8x - 14 & \text{if } 3/2 \le x \le 2 \end{cases}.$$

The graph of V^3 is given in Figure 10.4b.

20. Prove that the doubling function given by

$$D(x) = \begin{cases} 2x & \text{if } x < 1/2 \\ 2x - 1 & \text{if } x \ge 1/2 \end{cases}$$

is chaotic on $[0, 1)$. Compare this result with your observations in Experiment 3.6.

Choose $x \in [0, 1)$ and write it in binary; that is, let

$$x = 0.b_1b_2b_3\ldots$$

where $b_i \in \{0, 1\}$. Now, compute $D(x)$. If $b_1 = 0$, then $0 \le x \le 1/2$ and

$$D(x) = 0.b_2b_3b_4\ldots$$

since doubling shifts the binary point one place to the right. If $b_1 = 1$, then $1/2 \le x < 1$ and

$$D(x) = (1.b_2b_3b_4\ldots) - 1 = 0.b_2b_3b_4\ldots.$$

In either case, $D(x) = 0.b_2b_3b_4\ldots$ and so we see that doubling is equivalent to the shift. Consequently, anything true (in the dynamical sense) of the shift is also true of doubling, and in particular, D is chaotic because σ is.

21. Prove that the function

$$T(x) = \begin{cases} 2x & \text{if } x \le 1/2 \\ 2-2x & \text{if } x > 1/2 \end{cases}$$

is chaotic on $[0,1]$.

We will show that the doubling map is semi-conjugate to the tent map T via T itself! That is, we will show that

$$\begin{array}{ccc} [0,1] & \xrightarrow{D} & [0,1] \\ T\downarrow & & \downarrow T \\ [0,1] & \xrightarrow{T} & [0,1] \end{array}$$

commutes. (Note that the doubling map D needs to be defined on the *closed* unit interval for this to work.) Thus T will be chaotic by virtue of Exercise 20. This is because orbits under iteration of D map to dynamically equivalent orbits under T. In fact, we now prove by induction that

$$T \circ D^{n-1} = T^n \qquad (10.1)$$

for all $n > 0$. Suppose Equation 10.1 is true for $n := k$. Then

$$\begin{aligned} &T \circ D^{k-1} = T^k \\ \Rightarrow\ &T \circ T \circ D^{k-1} = T \circ T^k, \end{aligned}$$

and since $T \circ D = T \circ T$ (this will be verified in a moment) we have

$$T \circ D^k = T^{k+1}$$

which completes the inductive proof. We remark that (10.1) gives an explicit formula for $T^n(x)$ since we already know that $D^{n-1}(x) = 2^{n-1}x \bmod 1$.

We now show that D is semi-conjugate to T via T, or in other words, that $T \circ D = T \circ T$. There are four cases to consider for $T \circ T$:

$$\begin{aligned}
0 \le x \le 1/4 \quad &\Rightarrow \quad 0 \le T(x) \le 1/2 \quad \Rightarrow \quad T\circ T(x) = T(2x) = 2(2x) = 4x \\
1/4 \le x \le 1/2 \quad &\Rightarrow \quad 1/2 \le T(x) \le 1 \quad \Rightarrow \quad T\circ T(x) = T(2x) = 2-2(2x) = 2-4x \\
1/2 \le x \le 3/4 \quad &\Rightarrow \quad 1/2 \le T(x) \le 1 \quad \Rightarrow \quad T\circ T(x) = T(2-2x) = 2-2(2-2x) = 4x-2 \\
3/4 \le x \le 1 \quad &\Rightarrow \quad 0 \le T(x) \le 1/2 \quad \Rightarrow \quad T\circ T(x) = T(2-2x) = 2(2-2x) = 4-4x
\end{aligned}$$

Similarly, there are four cases for $T \circ D$:

$$\begin{aligned}
0 \le x \le 1/4 \quad &\Rightarrow \quad 0 \le D(x) \le 1/2 \quad \Rightarrow \quad T\circ D(x) = T(2x) = 2(2x) = 4x \\
1/4 \le x < 1/2 \quad &\Rightarrow \quad 1/2 \le D(x) < 1 \quad \Rightarrow \quad T\circ D(x) = T(2x) = 2-2(2x) = 2-4x \\
1/2 \le x \le 3/4 \quad &\Rightarrow \quad 0 \le D(x) \le 1/2 \quad \Rightarrow \quad T\circ D(x) = T(2x-1) = 2(2x-1) = 4x-2 \\
3/4 \le x < 1 \quad &\Rightarrow \quad 1/2 \le D(x) < 1 \quad \Rightarrow \quad T\circ D(x) = T(2x-1) = 2-2(2x-1) = 4-4x
\end{aligned}$$

We have to be a little bit careful at $x = 1/2$ since D is not continuous there, and also at $x = 1$ since we haven't yet defined $D(1)$. But the reader may check that $T \circ D(1/2) = T \circ T(1/2) = 0$, and that $T \circ D(1) = T \circ T(1) = 0$ provided we define $D(1)$ to be either 0 or 1. It is also straightforward to check that both $T \circ T$ and $T \circ D$ are continuous on $[0,1]$. So what we have shown is that

$$T \circ D(x) = T \circ T(x) = \begin{cases} 4x & \text{if } 0 \le x \le 1/4 \\ 2-4x & \text{if } 1/4 \le x \le 1/2 \\ 4x-2 & \text{if } 1/2 \le x \le 3/4 \\ 4-4x & \text{if } 3/4 \le x \le 1 \end{cases},$$

and so D is conjugate to T via T. See Figure 3.9a for the graph of $T \circ T = T \circ D$.

22. Use the results of the previous exercise to construct a conjugacy between T on the interval $[0,1]$ and $G(x) = 2x^2 - 1$ on the interval $[-1,1]$.

We will show that

$$\begin{array}{ccc} [0,1] & \xrightarrow{T} & [0,1] \\ \downarrow U & & \downarrow U \\ [-1,1] & \xrightarrow{G} & [-1,1] \end{array}$$

commutes, where $U(x) = \cos(\pi x)$. In other words, we will show that

$$U \circ T = G \circ U. \tag{10.2}$$

Note that $U\colon [0,1] \to [-1,1]$ is a homeomorphism. Now, the left-hand side of (10.2) is given by

$$U \circ T(x) = \begin{cases} \cos(2\pi x) & \text{if } 0 \le x \le 1/2 \\ \cos(2\pi x - 2\pi) & \text{if } 1/2 \le x \le 1 \end{cases}.$$

But $\cos(2\pi x - 2\pi) = \cos(2\pi x)$, and so

$$U \circ T(x) = \cos(2\pi x).$$

Applying the double-angle formula for cosine, the right-hand side of (10.2) is $G \circ U(x) = 2\cos^2(\pi x) - 1 = \cos(2\pi x)$, and so

$$U \circ T(x) = G \circ U(x).$$

Hence, G is chaotic since T is chaotic (see Exercise 21).

23. Construct a conjugacy which is valid on all of $\mathbb{R}$ between G in the previous exercise and Q_{-2}. (*Hint:* Use a linear function of the form $x \mapsto ax + b$.)

We seek a linear function L such that

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{G} & \mathbb{R} \\ \downarrow L & & \downarrow L \\ \mathbb{R} & \xrightarrow{Q_{-2}} & \mathbb{R} \end{array}$$

commutes. That is, we seek a homeomorphism L such that

$$L \circ G = Q_{-2} \circ L.$$

Suppose $L\colon \mathbb{R} \to \mathbb{R}$ is of the form $L(x) = ax + b$. Then $L \circ G(x) = a(2x^2 - 1) + b = 2ax^2 - a + b$, whereas $Q_{-2} \circ L(x) = (ax+b)^2 - 2 = a^2x^2 + 2abx + b^2 - 2$. Equating coefficients, we obtain the system of equations

$$\begin{cases} 2a &= a^2 \\ 0 &= 2ab \\ b - a &= b^2 - 2 \end{cases}$$

which has solution

$$\begin{cases} a &= 2 \\ b &= 0 \end{cases}.$$

Let's check this result: $L \circ G(x) = 2(2x^2 - 1) = 4x^2 - 2$, and $Q_{-2} \circ L(x) = (2x)^2 - 2 = 4x^2 - 2$. ✓

24. Prove that $F_4(x) = 4x(1-x)$ is chaotic on $[0,1]$.

Our goal is to find a linear map $W\colon [-2,2] \to [0,1]$ such that

$$F_4 \circ W = W \circ Q_{-2}.$$

Suppose $W(x) = ax + b$. Then

$$\begin{aligned} F_4 \circ W(x) &= 4(ax+b)(1 - ax - b) \\ &= 4(ax - a^2x^2 - abx + b - abx - b^2) \\ &= 4((a - 2ab)x - a^2x^2 + b - b^2) \end{aligned} \tag{10.3}$$

whereas

$$\begin{aligned} W \circ Q_{-2}(x) &= a(x^2 - 2) + b \\ &= ax^2 - 2a + b. \end{aligned} \tag{10.4}$$

Equating coefficients in (10.3) and (10.4), we arrive at the following system of equations:

$$-4a^2 = a \tag{10.5}$$
$$4(a - 2ab) = 0 \tag{10.6}$$
$$4(b - b^2) = -2a + b. \tag{10.7}$$

From (10.5), we see that $a = 0$ or $a = -1/4$. Plugging the latter into (10.6), we find that $-1 + 2b = 0$ which says that $b = 1/2$. Checking these results against (10.7), we have

$$\begin{aligned} 4\left(\frac{1}{2} - \frac{1}{4}\right) &= -2\left(-\frac{1}{4}\right) + \frac{1}{2} \\ \Rightarrow\ 2 - 1 &= \frac{1}{2} + \frac{1}{2} \\ \Rightarrow\ 1 &= 1. \quad \checkmark \end{aligned}$$

Thus,

$$\begin{aligned} W(x) &= -\frac{1}{4}x + \frac{1}{2} \\ &= \frac{2 - x}{4}, \end{aligned}$$

and the reader may check that $F_4 \circ W(x) = W \circ Q_{-2}(x)$. Hence, F_4 is chaotic by virtue of Exercise 23 where it was shown that Q_{-2} was chaotic. Indeed, combining the results of Exercises 20–24, we have that

$$\begin{array}{ccccccccc} [0,1] & \xrightarrow{T} & [0,1] & \xrightarrow{U} & [-1,1] & \xrightarrow{L} & [-2,2] & \xrightarrow{W} & [0,1] \\ \downarrow D & & \downarrow T & & \downarrow G & & \downarrow Q_{-2} & & \downarrow F_4 \\ [0,1] & \xrightarrow{T} & [0,1] & \xrightarrow{U} & [-1,1] & \xrightarrow{L} & [-2,2] & \xrightarrow{W} & [0,1] \end{array}$$

commutes. Note that all conjugacies but the first are homeomorphisms.

Chapter 11

Sarkovskii's Theorem

Exercises

1. Can a continuous function on $\mathbb{R}$ have a periodic point of period 48 and not one of period 56? Why?

Yes. Observe that $56 = 2^3 \cdot 7$ precedes $48 = 2^4 \cdot 3$ in the Sarkovskii ordering. Thus, if a continuous function F has a cycle of period 56, then it also has a cycle of period 48. Now there is a continuous function F with a cycle of period 48 that does not have a cycle of period 56 (in fact, by the theorem on p. 138 of the text, there exists a continuous function with period 48 having no cycles of any period preceding 48 in the Sarkovskii ordering) but this is not *necessarily* so. In other words, a continuous function F with a periodic point of period 48 may or may not have a periodic point of period 56.

2. Can a continuous function on $\mathbb{R}$ have a periodic point of period 176 and not one of period 96? Why?

No. In this case, $176 = 2^4 \cdot 11$ precedes $96 = 2^5 \cdot 3$ in the Sarkovskii ordering, and so, by Sarkovskii's theorem,[1] a continuous function with a periodic point of period 176 must also have a periodic point of period 96.

3. Give an example of a function $F\colon [0,1] \to [0,1]$ that has a periodic point of period 3 and *no* other periods. Can this happen?

[1] For an interesting proof of Sarkovskii's theorem, see: Harvey Kaplan (1987). A cartoon-assisted proof of Sarkovskii's theorem. *Amer. J. Physics* 55(11), 1023–1032.

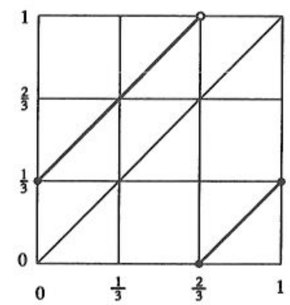


Figure 11.1: A discontinuous piecewise linear map having only period 3 points.

Well, Sarkovskii's theorem applies only to continuous functions and so it follows that such a function must be discontinuous. Consider the function

$$F(x) = \begin{cases} x + 1/3 & \text{if } 0 \le x < 2/3 \\ x - 2/3 & \text{if } 2/3 \le x \le 1 \end{cases}$$

which has *lots* of 3-cycles:

$$0 \mapsto 1/3 \mapsto 2/3 \mapsto 0$$

$$1/6 \mapsto 1/2 \mapsto 5/6 \mapsto 1/6$$

$$1/12 \mapsto 5/12 \mapsto 9/12 \mapsto 1/12$$

$$1/5 \mapsto 8/15 \mapsto 13/15 \mapsto 1/5$$

Indeed, the graph in Figure 11.1 clearly shows that

$$F[0,1/3] = [1/3,2/3]$$

$$F[1/3,2/3] = [2/3,1)$$

$$F[2/3,1] = [0,1/3]$$

and so *every* point is of period 3 (except the endpoint 1 which is eventually period 3).

Here are some related exercises:

Exercise. Assign values to each of the endpoints in (11.1) and (11.2). Justify your choices as best you can.

$$S_2(x) = \begin{cases} x + 1/2 & \text{if } 0 < x < 1/2 \\ x - 1/2 & \text{if } 1/2 < x < 1 \end{cases} \tag{11.1}$$

$$S_4(x) = \begin{cases} x + 1/4 & \text{if } 0 < x < 3/4 \\ x - 3/4 & \text{if } 3/4 < x < 1 \end{cases} \tag{11.2}$$

Exercise. In general, show that for any integer $m > 1$,

$$S_m(x) = \begin{cases} x + 1/m & \text{if } 0 < x < (m-1)/m \\ x - (m-1)/m & \text{if } (m-1)/m < x < 1 \end{cases}$$

has nothing but periodic points of period m. Also show that

$$S_m = S_{m/(m-1)}^{-1}.$$

(In Exercise 3, for example, it is easily checked that $F = S_3 = S_{3/2}^{-1}$.)

One final note: Babbage[2] gives many examples of functions F such that $F^3(x) = x$. None is continuous of course, and all are conjugate to the linear fractional transformation

$$M(x) = \frac{ax + b}{d - \frac{a^2+ad+d^2}{b}x}$$

which the reader may check satisfies $M^3(x) = x$.

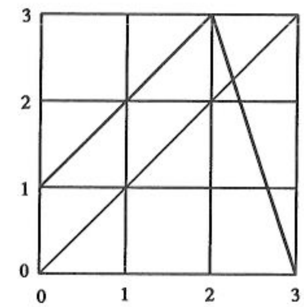
4. The graphs in Figure 11.2 each have a 4-cycle given by $\{0, 1, 2, 3\}$. One of these functions has cycles of all other periods, and one has only periods 1, 2, and 4. Identify which function has each of these properties.

Note that each of these functions is continuous and so we need only find a 3-cycle to identify which has cycles of all periods. On the other hand, it may be easier to prove the nonexistence of the 3-cycle, which we now do.

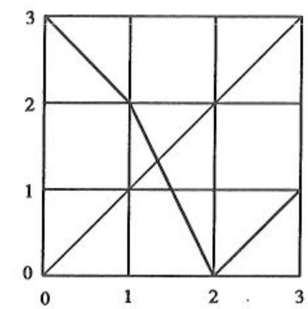
First consider the graph in Figure 11.2b. By inspection, we find the cycles

$$1/2 \mapsto 5/2 \mapsto 1/2$$

[2]Charles Babbage (1816). An essay towards the calculus of functions, part II. *Philosophical Transactions of the Royal Society* **106**, 179–256.



(a) Graph of F.



(b) Graph of G.

Figure 11.2: Two graphs with period 4.

$$1/4 \mapsto 11/4 \mapsto 3/4 \mapsto 9/4 \mapsto 1/4$$

$$1/8 \mapsto 23/8 \mapsto 7/8 \mapsto 17/8 \mapsto 1/8$$

and other 4-cycles are readily found. Indeed, we will show in what follows that G has nothing but 4-cycles.

The equation of the piecewise linear graph in Figure 11.2b is

$$G(x) = \begin{cases} 3 - x & \text{if } 0 \le x \le 1 \\ 4 - 2x & \text{if } 1 \le x \le 2 \\ x - 2 & \text{if } 2 \le x \le 3 \end{cases}.$$

Now let $0 \le x \le 1$. Then $-1 \le -x \le 0$ and so $2 \le 3 - x \le 3$. In other words, $G[0,1] = [2,3]$, and similarly, $G[2,3] = [0,1]$. (These facts are easily verified by inspection of the graph of G.) This means that no point in $[0,1] \cup [2,3]$ can have odd period. Now let's see what happens when we take an arbitrary $x \in [0,1]$ and iterate G.

$$\begin{aligned} G(x) &= 3 - x && \text{since } x \in [0,1] \\ G^2(x) &= G(3-x) = (3-x) - 2 = 1 - x && \text{since } 3 - x \in [2,3] \\ G^3(x) &= G(1-x) = 3 - (1-x) = 2 + x && \text{since } 1 - x \in [0,1] \\ G^4(x) &= G(2+x) = (2+x) - 2 = x && \text{since } 2 + x \in [2,3] \end{aligned}$$

So every point in $[0,1] \cup [2,3]$ is of period 4. Now $G[1,2] = [0,2]$ and $G[0,2] = [0,3]$. But as soon as a point leaves the closed interval $[1,2]$, it's locked into a 4-cycle. So the question is: are there points in $[1,2]$ that remain in $[1,2]$ for all time? The answer is **no**! The orbits of points close to the fixed point $x = 4/3$ oscillate away from the fixed point (since it's repelling) and eventually enter $[0,1]$, an interval of 4-cycles.

In summary, G has a repelling fixed point, a neutral 2-cycle, and a whole bunch of neutral period 4 points. Everything else is eventually periodic with period 4. We remark that G is the **double** of $x \mapsto 3 - x$ on $[0,3]$ which is easily seen to have nothing but period 2 points (see Devaney's *An Introduction to Chaotic Dynamical Systems*, Second Edition, pp. 67–68).

Now what about the dynamics of F? From the graph in Figure 11.2a, it's clear that

$$F[0,1] = [1,2] \tag{11.3}$$

$$F[1,2] = [2,3] \tag{11.4}$$

$$F[2,3] = [0,3] \tag{11.5}$$

and so we have $[0,1] \subset [0,3] = F^3[0,1]$. Therefore, F has a period 3 point in $[0,1]$ and this 3-cycle must be repelling. (How do we know it's repelling? Because the mappings (11.3–11.5) have constant slope 1, 1, and -3, respectively, and hence, the derivative of the period 3 point is $1 \cdot 1 \cdot (-3) = -3$.)

6. Consider the piecewise linear graph in Fig. 11.3. Prove that this function has a cycle of period 7 but not period 5.

Assuming $F\colon [1,7] \to [1,7]$, the 7-cycle

$$1 \mapsto 4 \mapsto 5 \mapsto 3 \mapsto 6 \mapsto 2 \mapsto 7 \mapsto 1$$

is repelling since each lap of F in Fig. 11.3 has constant slope greater than or equal to one in absolute value. Now, let's apply F five times to each of the unit intervals in $[1,7]$ and see what happens:

- $[1,2] \mapsto [4,7] \mapsto [1,5] \mapsto [3,7] \mapsto [1,6] \mapsto [2,7]$ and $[1,2] \cap [2,7] = \{3\} \in \mathrm{per}_7 F$.
- $[2,3] \mapsto [6,7] \mapsto [1,2] \mapsto [4,7] \mapsto [1,5] \mapsto [3,7]$ and $[2,3] \cap [3,7] = \{3\} \in \mathrm{per}_7 F$.

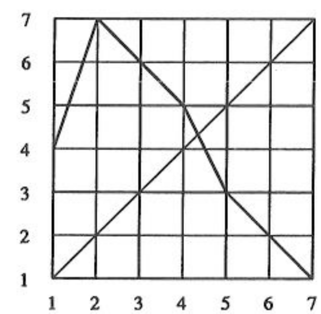


Figure 11.3: This function has a 7-cycle but no 5-cycle.

- $[3,4] \mapsto [5,6] \mapsto [2,3] \mapsto [6,7] \mapsto [1,2] \mapsto [4,7]$ and $[3,4] \cap [4,7] = \{4\} \in \mathrm{per}_7 F$.
- $[5,6] \mapsto [2,3] \mapsto [6,7] \mapsto [1,2] \mapsto [4,7] \mapsto [1,5]$ and $[5,6] \cap [1,5] = \{5\} \in \mathrm{per}_7 F$.
- $[6,7] \mapsto [1,2] \mapsto [4,7] \mapsto [1,5] \mapsto [3,7] \mapsto [1,6]$ and $[6,7] \cap [1,6] = \{6\} \in \mathrm{per}_7 F$.

Thus, there are no period 5 points in these intervals. Also notice that

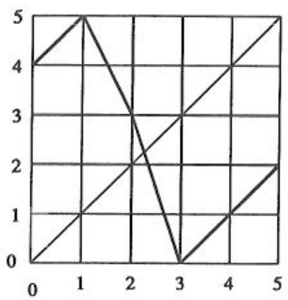
$$[3,4] \mapsto [5,6] \mapsto [2,3] \mapsto [6,7] \mapsto [1,2] \mapsto \cdots \mapsto [1,7].$$

But what about $[4,5]$? Since

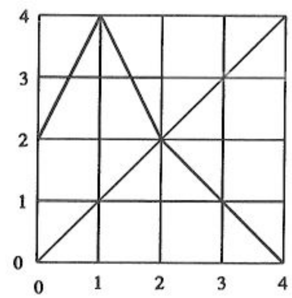
$$[4,5] \mapsto [3,5] \mapsto [3,6] \mapsto [2,6] \mapsto [2,7] \mapsto [1,7] \tag{11.6}$$

there is indeed a period 5 point in $[4,5]$. But we claim that there's exactly *one* such point in $[4,5]$, and in fact, it's the fixed point $x = 13/3$. This is because each of the mappings in (11.6) is strictly decreasing and therefore $F^5\colon [4,5] \to [1,7]$ is strictly decreasing (since the composition of an odd number of decreasing functions is decreasing). Thus F^5 has a unique fixed point in $[4,5]$, and moreover, this point must be the fixed point of F.

Does F have a 3-cycle? No, for if it did, it would also have a 5-cycle which we've already shown does not exist.



(a)



(b)

Figure 11.4: Two graphs with no odd periods.

By the way, there's a nice pattern here that deserves mention.

Period 5, but not period 3:

$$1 \mapsto 3 \mapsto 4 \mapsto 2 \mapsto 5 \mapsto 1. \tag{11.7}$$

Period 7, but not period 5:

$$1 \mapsto 4 \mapsto 5 \mapsto 3 \mapsto 6 \mapsto 2 \mapsto 7 \mapsto 1. \tag{11.8}$$

See how (11.8) is obtained from (11.7)? Just add one to each point in the period 5 orbit and then add the iteration $7 \mapsto 1$ at the end. Similarly, we may construct a map with a period 9 orbit, but not period 7. The necessary 9-cycle would be

$$1 \mapsto 5 \mapsto 6 \mapsto 4 \mapsto 7 \mapsto 3 \mapsto 8 \mapsto 2 \mapsto 9 \mapsto 1. \tag{11.9}$$

The map corresponding to (11.9) is not hard to construct, and is left as an exercise.

7. Consider the graph in Figure 11.4a. Prove that this function has a cycle of period 6 but no cycles of any odd period.

This function has the 6-cycle

$$0 \mapsto 4 \mapsto 1 \mapsto 5 \mapsto 2 \mapsto 3 \mapsto 0$$

which may be checked easily from the graph, and so F has periodic points of all even periods. The mappings

$$[0,1] \mapsto [4,5] \mapsto [1,2] \mapsto [3,5] \mapsto [0,2] \mapsto [3,5] \mapsto \cdots$$

$$[2,3] \mapsto [0,2] \mapsto [0,5] \mapsto \cdots$$

clearly show the absence of odd periodic points, however. Take period 3 points, for instance. Since

$$F^3[0,1] = [3,5] \tag{11.10}$$

$$F^3[1,2] = [3,5] \tag{11.11}$$

$$F^3[3,5] = [0,2], \tag{11.12}$$

there are no period 3 points in these intervals since $[0,2]$ maps to $[3,5]$, and vice versa. The only remaining interval is $[2,3]$, but it contains absolutely *no* periodic points save the lone fixed point of F. So F has no period 3 points.

Let's also check period 5 points. Again, there aren't any since (11.10–11.12) hold with F^3 replaced with F^5. In fact, there are no periodic points with *any* odd period since

$$F^{2n+1}[0,1] = [3,5]$$

$$F^{2n+1}[1,2] = [3,5]$$

$$F^{2n+1}[3,5] = [0,2]$$

for all positive integers n.

Here's a totally different approach to this problem: First, verify that the piecewise linear function in Figure 11.5 has the period 3 orbit

$$0 \mapsto 1/2 \mapsto 1 \mapsto 0.$$

Since this function is continuous, Sarkovskii's theorem guarantees that it has periodic points of *all* periods.

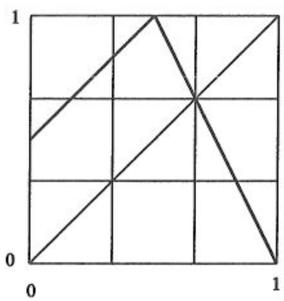


Figure 11.5: A continuous function with a 3-cycle.

Now construct the double of this function[3] to produce the graph in Figure 11.4a.[4] Since the period of every orbit has doubled, this function has only even periodic points.

8. Consider the function whose graph is displayed in Figure 11.4b. Prove that this function has cycles of all even periods but no odd periods (except 1).

Observe, first of all, that 2 is fixed by F.[5] Inspection of the graph in Figure 11.4b shows that $[3,4] \mapsto [0,1]$ and $[2,3] \mapsto [1,2]$, both of which map to $[2,4]$. But $[2,4] \mapsto [0,2]$, and vice versa. In general,

$$F^{2n+1}[0,1] = F^{2n+1}[1,2] = [2,4]$$

and

$$F^{2n+1}[2,3] = F^{2n+1}[3,4] = [0,2],$$

and hence, there can be no odd periodic points. On the other hand, all even periodic orbits exist since

$$F^{2n}[0,1] = F^{2n}[1,2] = [0,2]$$

[3]See pp. 67–68 of R. L. Devaney's *An Introduction to Chaotic Dynamical Systems*, Second Edition, Addison-Wesley, 1989.

[4]This is not quite true, but the discrepancy is not important.

[5]Indeed, any multiple of a power of two in $[0,4]$ is eventually fixed.

and

$$F^{2n}[2,3] = F^{2n}[3,4] = [2,4].$$

Thus we have that

$$[0,1] \subseteq [0,2] \supseteq [1,2]$$

and

$$[2,3] \subseteq [2,4] \supseteq [3,4]$$

and so even periodic orbits are guaranteed to exist.

9. Consider the subshift of finite type $\Sigma' \subset \Sigma$ determined by the rules 1 may follow 0 and both 0 and 1 may follow 1, as discussed in Section 11.4.

9a) Prove that periodic points for σ are dense in Σ'.

Let $\mathbf{s} = (s_0 s_1 s_2 \ldots) \in \Sigma'$. Given $\epsilon > 0$, choose an integer n such that $1/2^n < \epsilon$. Now consider

$$\mathbf{t} = (\overline{s_0 s_1 \ldots s_n 1})$$

and observe that $\mathbf{s}$ and $\mathbf{t}$ agree up through their respective $(n+1)$st entries.[6] By the Proximity Theorem,

$$d[\mathbf{s}, \mathbf{t}] \leq 1/2^n < \epsilon$$

and so we've found a periodic point arbitrarily close to $\mathbf{s}$. Hence, these periodic points are dense in Σ'.

9b) Prove that there is a dense orbit for σ in Σ'.

Like the full shift, there is a point in Σ' with a dense orbit, but it is slightly more difficult to write down. It's constructed just like $\hat{\mathbf{s}}$ on p. 116 of the text, but of course any sub-block containing a pair of adjacent zeros, or any pair of adjacent sub-blocks ending and starting with zeros, must be omitted, for such a point is not in Σ'.

If you believe that every transitive dynamical system has a dense orbit (see p. 117 of the text), then the proof is easy. Let $\mathbf{s} = (s_0 s_1 s_2 \ldots)$ and $\mathbf{t} = (t_0 t_1 t_2 \ldots)$ be any two points in Σ', and let $\epsilon > 0$ be given. Then there is a third point, namely,

$$(s_0 s_1 \ldots s_n 1 t_0 t_1 \ldots t_n \ldots)$$

[6]Note that $\mathbf{t} = (\overline{s_0 s_1 \ldots s_n})$ will not work since both s_0 and s_n may be zero, and hence $\mathbf{t} \notin \Sigma'$.

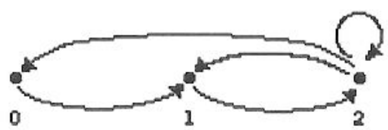


Figure 11.6: A subshift of finite type on three symbols.

which is within ϵ of $\mathbf{s}$, and whose orbit comes within ϵ of $\mathbf{t}$. Note that a 1 has been inserted between the s_i and the t_i in the event that $s_n = t_0 = 0$. The tail of the sequence is arbitrary.

Incidentally, the previous two facts taken together show that the subshift is sensitive to initial conditions[7] from which it follows that the subshift is chaotic.

The following four problems deal with the subshift of Σ_3, the space of sequences of 0's, 1's, and 2's, determined by the rules that 1 may follow 0, 2 may follow 1, and 0, 1, or 2 may follow 2.

Let $\Sigma_3 = \left\{ (s_0 s_1 s_2 \ldots) \,\middle|\, s_i \in \{0,1,2\} \right\}$ and let

$$\Sigma_3 \supset \Sigma_3' = \{ (s_0 s_1 s_2 \ldots) \mid 2 \text{ must follow } 1 \text{ must follow } 0 \}.$$

See Figure 11.6.

10. Is this subset of Σ_3 closed?

Yes. Choose $\mathbf{s} \notin \Sigma_3'$. Then there exists a k such that both s_k and s_{k-1} are both zero or one. Choose $\epsilon < 1/2^k$ and let $\mathbf{t}$ be any point in Σ_3 such that $d[\mathbf{s}, \mathbf{t}] < \epsilon$. By the Proximity theorem, $\mathbf{s}$ and $\mathbf{t}$ agree up through the kth position. In particular, $s_k = t_k$ and $s_{k-1} = t_{k-1}$, and so t_k and t_{k-1} are both zero or one. Hence, $\mathbf{t} \notin \Sigma_3'$ and so the complement of Σ_3' is open. Therefore, Σ_3' itself is closed.

11. Are periodic points dense for this subshift?

Yes. The proof mirrors the one given in Exercise 9a except here we let

$$\mathbf{t} = (\overline{s_0 s_1 \ldots s_n 2})$$

[7] See J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey (1992). On Devaney's definition of chaos. *Amer. Math. Monthly* **99**(4), 332–334.

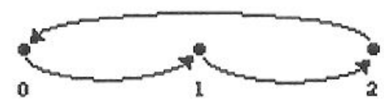


Figure 11.7: A subshift of finite type with a 3-cycle but no fixed or period 2 points.

where n is chosen in such a way that $1/2^n < \epsilon$ and $s_n = 2$ (which can always be done since $\mathbf{s}$ contains an infinite number of 2s). This ensures that $\mathbf{t} \in \Sigma_3'$.

12. Is there a dense orbit for this subshift?

We will show that this subshift is transitive. Let $\mathbf{s} = (s_0 s_1 s_2 \ldots)$ and $\mathbf{t} = (t_0 t_1 t_2 \ldots)$ be any two points in Σ_3', and let $\epsilon > 0$ be given. Choose n such that $1/2^n < \epsilon$ and $s_n = 2$. Then any point beginning with the sequence $s_0 s_1 \ldots s_n t_0 t_1 \ldots t_n$ is within ϵ of $\mathbf{s}$ and has an orbit which comes within ϵ of $\mathbf{t}$.

13. How many periodic points of periods 2, 3, and 4 satisfy these rules?

Period 1:	$(\overline{2})$	1
Period 2:	$(\overline{2}), (\overline{12}), (\overline{21})$	3
Period 3:	$(\overline{2}), (\overline{012}), (\overline{120}), (\overline{201}), (\overline{122}), (\overline{221}), (\overline{212})$	7
Period 4:	$(\overline{2}), (\overline{12}), (\overline{21}), (\overline{0122}), (\overline{1220}), (\overline{2201}), (\overline{2012}),$ $(\overline{2122}), (\overline{1222}), (\overline{2221}), (\overline{2212})$	11

True or false: There are $3 + 7 + 11 = 21$ periodic points of period 5?

14. Construct a subshift of finite type in Σ_3 which has a period 3 point but no fixed or period 2 points.

Consider the subshift depicted in Figure 11.7. It has but three points, all of which are 3-cycles.

15. Discuss the dynamics of the subshift given by the directed graph in Figure 11.8a. Are periodic points dense for this subshift? Is there a dense orbit? How many periodic points of period n does this subshift have?

There's not much difference between this subshift on three symbols and the full shift on two symbols. The only difference is that strings may start with

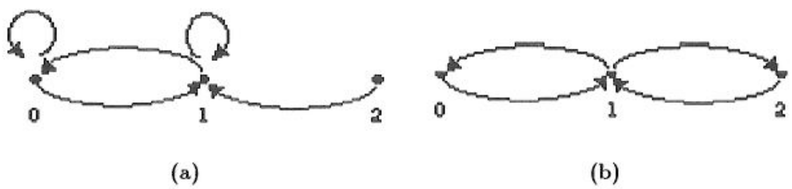


Figure 11.8: More subshifts of finite type.

the symbol 2 in the case of the subshift. Indeed, for $s \in \Sigma_3'$, we have that $\sigma(\mathrm{s}) \in \Sigma_2$, and in fact, $\mathrm{per}_n\, \sigma$ is the same as before (the size of $\overline{\mathrm{per}_n\, \sigma}$ has doubled, however).

16. Discuss the dynamics of the subshift given by the directed graph in Figure 11.8b. Are periodic points dense for this subshift? Is there a dense orbit?

Except for those strings beginning with 1, there's a 1-1 correspondence between elements of Σ_3' and Σ_2: just substitute 0 for every 01, and 1 for each 21. It seems that this subshift is equivalent to the one in Exercise 15.

Chapter 12

The Role of the Critical Orbit

Exercises

1. Compute the Schwarzian derivative for the following functions and decide if $SF(x) < 0$ for all x.

1a) $F(x) = x^2$; $F'(x) = 2x$; $F''(x) = 2$; and $F'''(x) = 0$.

$$\begin{aligned} SF(x) &= \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left[\frac{F''(x)}{F'(x)}\right]^2 \\ &= 0 - \frac{3}{2}\left[\frac{2}{2x}\right]^2 \\ &= -\frac{3}{2x^2} \\ &< 0 \qquad \text{for all } x. \end{aligned}$$

In fact, for the quadratic family, $Q_c(x) = x^2 + c$,

$$SQ_c(x) = -\frac{3}{2x^2}$$

for all c.

1b) $F(x) = x^3$; $F'(x) = 3x^2$; $F''(x) = 6x$; and $F'''(x) = 6$.

$$\begin{aligned} SF(x) &= \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left[\frac{F''(x)}{F'(x)}\right]^2 \\ &= \frac{6}{3x^2} - \frac{3}{2}\left[\frac{6x}{3x^2}\right]^2 \\ &= -\frac{4}{x^2} \\ &< 0 \qquad \text{for all } x. \end{aligned}$$

1c) $F(x) = e^{3x}$; $F'(x) = 3e^{3x}$; $F''(x) = 9e^{3x}$; and $F'''(x) = 27e^{3x}$.

$$\begin{aligned} SF(x) &= \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left[\frac{F''(x)}{F'(x)}\right]^2 \\ &= \frac{27e^{3x}}{3e^{3x}} - \frac{3}{2}\left[\frac{9e^{3x}}{3e^{3x}}\right]^2 \\ &= -\frac{9}{2} \\ &< 0. \end{aligned}$$

In general, for $F(x) = e^{kx}$,

$$SF(x) = -\frac{k^2}{2}.$$

1d) $F(x) = \cos(x^2+1)$; $F'(x) = -2x\sin(x^2+1)$; and $F''(x) = -4x^2\cos(x^2+1) - 2\sin(x^2+1)$. Also,
$F'''(x) = 8x^3\sin(x^2+1) - 8x\cos(x^2+1) - 4x\cos(x^2+1)$
$8x^3\sin(x^2+1) - 12x\cos(x^2+1)$.

$$\begin{aligned} &SF(x) \\ &= \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left[\frac{F''(x)}{F'(x)}\right]^2 \\ &= \frac{8x^3\sin(x^2+1) - 12x\cos(x^2+1)}{-2x\sin(x^2+1)} \\ &\qquad - \frac{3}{2}\left[\frac{-4x^2\cos(x^2+1) - 2\sin(x^2+1)}{-2x\sin(x^2+1)}\right]^2 \end{aligned}$$

$$\begin{aligned}
&= 6\cot(x^2+1) - 4x^2 - \frac{3}{2}\left(2x\cot(x^2+1) + \frac{1}{x}\right)^2 \\
&= 6\cot(x^2+1) - 4x^2 - \frac{3}{2}\left(4x^2\cot^2(x^2+1) + 4\cot(x^2+1) + \frac{1}{x^2}\right) \\
&= -\left(4x^2 + 6x^2\cot^2(x^2+1) + \frac{3}{2x^2}\right) \\
&< 0 \qquad \text{for all } x.
\end{aligned}$$

We better check our work. Let $C(x) = \cos x$ and $G(x) = x^2 + 1$ so that $C \circ G(x) = \cos(x^2+1)$. An easy computation yields

$$\begin{aligned}
SC(x) &= \frac{\sin x}{-\sin x} - \frac{3}{2}\left[\frac{-\cos x}{-\sin x}\right]^2 \\
&= -1 - \frac{3}{2}\cot^2 x
\end{aligned}$$

and

$$\begin{aligned}
SG(x) &= \frac{0}{2x} - \frac{3}{2}\left[\frac{2}{2x}\right]^2 \\
&= -\frac{3}{2x^2}.
\end{aligned}$$

So, by the chain rule for Schwarzian derivatives,

$$\begin{aligned}
S(C \circ G)(x) &= SC(G(x)) \cdot [G'(x)]^2 + SG(x) \\
&= \left(-1 - \frac{3}{2}\cot^2(x^2+1)\right) \cdot 4x^2 - \frac{3}{2x^2} \\
&= -4x^2 - 6x^2\cot^2(x^2+1) - \frac{3}{2x^2},
\end{aligned}$$

which checks.

1e) $F(x) = \arctan x$; $F'(x) = (1+x^2)^{-1}$; and $F''(x) = -2x(1+x^2)^{-2}$.
$F'''(x) = 8x^2(1+x^2)^{-3} - 2(1+x^2)^{-2}$
$2(1+x^2)^{-3}(4x^2 - (1+x^2))$
$2(1+x^2)^{-3}(3x^2 - 1)$.

$$\begin{aligned}
SF(x) &= \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left[\frac{F''(x)}{F'(x)}\right]^2 \\
&= \frac{2(1+x^2)^{-3}(3x^2-1)}{(1+x^2)^{-1}} - \frac{3}{2}\left[\frac{-2x(1+x^2)^{-2}}{(1+x^2)^{-1}}\right]^2 \\
&= \frac{2(3x^2-1)}{(1+x^2)^2} - \frac{3}{2}\left[\frac{-2x}{1+x^2}\right]^2 \\
&= \frac{12x^2 - 4 - 12x^2}{2(1+x^2)^2} \\
&= -\frac{2}{(1+x^2)^2} \\
&< 0 \qquad \text{for all } x.
\end{aligned}$$

2. Is it true that $S(F+G)(x) = SF(x) + SG(x)$? If so, prove it. If not, give a counterexample.

False. Unlike ordinary differentiation, the Schwarzian derivative does not distribute over addition. Let $F(x) = G(x) = e^x$. Then

$$\begin{aligned}
SF(x) + SG(x) &= 2\left(\frac{e^x}{e^x} - \frac{3}{2}\left[\frac{e^x}{e^x}\right]^2\right) \\
&= 2\left(1 - \frac{3}{2}\right) \\
&= -1,
\end{aligned}$$

but

$$\begin{aligned}
S(F+G)(x) &= \frac{e^x + e^x}{e^x + e^x} - \frac{3}{2}\left[\frac{e^x + e^x}{e^x + e^x}\right]^2 \\
&= 1 - \frac{3}{2} \\
&= -\frac{1}{2}.
\end{aligned}$$

3. Is it true that $S(F \cdot G)(x) = SF(x) \cdot G(x) + F(x) \cdot SG(x)$? If so, prove it. If not, give a counterexample.

Unfortunately, there is no product-like rule for Schwarzian derivatives. Let $F(x) = e^{2x}$ and $G(x) = e^{3x}$. By Exercise 1c, $SF(x) = -2$ and $SG(x) = -9/2$. But

$$SF(x) \cdot G(x) + F(x) \cdot SG(x) = -2e^{3x} - (9/2)e^{2x}$$

$$\begin{aligned} &\neq -25/2 \\ &= S(F \cdot G)(x) \end{aligned}$$

since $F \cdot G(x) = e^{5x}$.

4. Is it true that $S(cF)(x) = cSF(x)$ where c is a constant? If so, prove it. If not, give a counterexample.

Neither multiplicative nor additive constants have any effect on the Schwarzian derivative. Since $(cF)^{(n)} = c \cdot F^{(n)}$ for all n, we have that

$$\begin{aligned} S(cF)(x) &= \frac{cF'''(x)}{cF'(x)} - \frac{3}{2}\left[\frac{cF''(x)}{cF'(x)}\right]^2 \\ &= \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left[\frac{F''(x)}{F'(x)}\right]^2 \\ &= SF(x). \end{aligned}$$

Also, since $(F + c)^{(n)} = F^{(n)}$ for all n,

$$S(F + c)(x) = SF(x).$$

5. Give an example of a function that has $SF(x) > 0$ for at least some x-values.

Exercise 4i at the end of Chapter 5 provides such an example. There we found the origin to be a weakly repelling fixed point for $F(x) = -x - x^3$ since

$$-2F'''(0) - 3[F''(0)]^2 = 12.$$

(This is the same as computing $SF(0) = 6$, by the way.) In fact,

$$\begin{aligned} SF(x) &= \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left[\frac{F''(x)}{F'(x)}\right]^2 \\ &= \frac{-6}{-1 - 3x^2} - \frac{3}{2}\left[\frac{-6x}{-1 - 3x^2}\right]^2 \\ &= \frac{6 - 36x^2}{(1 + 3x^2)^2} \end{aligned}$$

which is positive when

$$6 - 36x^2 > 0$$

or when

$$|x| < \frac{\sqrt{6}}{6}.$$

This example illustrates but one of the two cases mentioned in the proof of the Schwarzian Min-Max Principle given in the text.

6. Prove that $S(1/x) = 0$ and $S(ax + b) = 0$. Conclude that $SF(x) = 0$ where

$$F(x) = \frac{1}{ax + b}.$$

Let $R(x) = 1/x$ and $L(x) = ax + b$. Then $R'(x) = -1/x^2$, $R''(x) = 2/x^3$, and $R'''(x) = -6/x^4$. Also, $L'(x) = a$ and $L^{(n)}(x) = 0$ for all $n > 1$. Therefore,

$$\begin{aligned} SR(x) &= \frac{-6x^{-4}}{-x^{-2}} - \frac{3}{2}\left[\frac{2x^{-3}}{-x^{-2}}\right]^2 \\ &= \frac{6}{x^2} - \frac{6}{x^2} \\ &= 0 \end{aligned}$$

and

$$SL(x) = \frac{0}{a} - \frac{3}{2}\left[\frac{0}{a}\right]^2 = 0.$$

Applying the chain rule for Schwarzian derivatives,

$$\begin{aligned} S(R \circ L)(x) &= SR(L(x)) \cdot [L'(x)]^2 + SL(x) \\ &= 0 \cdot a^2 + 0 \\ &= 0. \end{aligned}$$

Question: Do all totally periodic functions have zero Schwarzian derivative?

7. Compute $SM(x)$ where

$$M(x) = \frac{ax + b}{cx + d}.$$

We will show that a linear fractional transformation, or Möbius transformation, has zero Schwarzian derivative. Our goal is to find constants k_1,

k_2, and k_3 such that

$$\frac{ax+b}{cx+d} = k_1 + \frac{k_2}{x+k_3}.$$

Then the Möbius transformation is the composition of functions that we have already shown to have zero Schwarzian derivative (see Exercise 6). Hence, the Möbius transformation itself has zero Schwarzian derivative by repeated applications of the chain rule. If $c \neq 0$, then we may write

$$\frac{ax+b}{cx+d} = \frac{\frac{a}{c}x + \frac{b}{c}}{x + \frac{d}{c}}. \tag{12.1}$$

Now,

$$\begin{aligned} k_1 + \frac{k_2}{x+k_3} &= \frac{k_1(x+k_3)+k_2}{x+k_3} \\ &= \frac{k_1 x + (k_1 k_3 + k_2)}{x+k_3}. \end{aligned} \tag{12.2}$$

Equating coefficients in (12.1) and (12.2), we see immediately that $k_1 = a/c$ and $k_3 = d/c$. Also,

$$\begin{aligned} \frac{b}{c} &= k_1 k_3 + k_2 \\ &= \frac{a}{c}\frac{d}{c} + k_2 \end{aligned}$$

which implies that

$$\begin{aligned} k_2 &= \frac{b}{c} - \frac{a}{c}\frac{d}{c} \\ &= \frac{b-ad}{c}. \end{aligned}$$

Thus,

$$\frac{ax+b}{cx+d} = \frac{a}{c} + \frac{\frac{b-ad}{c}}{x+\frac{d}{c}}$$

and we are done since we have shown that the Möbius transformation is the composition of both linear and inverse transformations.

9. Give a formula for $S(F \circ G \circ H)(x)$ in terms of SF, SG, SH, and the derivatives of these functions?

Two applications of the chain rule for Schwarzian derivatives yields

$$\begin{aligned} &S((F \circ G) \circ H)(x) \\ &= S(F \circ G)(H(x)) \cdot [H'(x)]^2 + S(H)(x) \\ &= \left[SF(G \circ H(x)) \cdot [(G' \circ H)(x)]^2 + SG(H(x))\right] \\ &\qquad \cdot [H'(x)]^2 + SH(x) \qquad (12.3) \\ &= SF(G \circ H(x)) \cdot [(G' \circ H)(x) \cdot H'(x)]^2 \\ &\qquad + SG(H(x)) \cdot [H'(x)]^2 + SH(x) \qquad (12.4) \\ &= SF(G \circ H(x)) \cdot [(G \circ H)'(x)]^2 + S(G \circ H)(x) \\ &= S(F \circ (G \circ H))(x). \end{aligned}$$

Either of (12.3) or (12.4) solves the exercise, but we have shown more, that is, composition of functions is associative with respect to the Schwarzian derivative.

10. Compute the Schwarzian derivatives for each of the following functions:

10c) $F(x) = \sin\left(e^{x^2+2}\right)$

We would like to apply the results of Exercise 9. To this end, we begin by letting $H(x) = \sin x$. Then $H'(x) = \cos x$, $H''(x) = -\sin x$, $H'''(x) = -\cos x$, and

$$\begin{aligned} SH(x) &= \frac{-\cos x}{\cos x} - \frac{3}{2}\left[\frac{-\sin x}{\cos x}\right]^2 \\ &= -1 - \frac{3}{2}\tan^2 x. \end{aligned}$$

Now let $E(x) = e^x$. But E is its own derivative, and so

$$\begin{aligned} SE(x) &= \frac{e^x}{e^x} - \frac{3}{2}\left[\frac{e^x}{e^x}\right]^2 \\ &= 1 - \frac{3}{2} \\ &= -\frac{1}{2}. \end{aligned}$$

For $Q(x) = x^2 + 2$, we have $Q'(x) = 2x$, $Q''(x) = 2$, $Q'''(x) = 0$, and

$$SQ(x) = 0 - \frac{3}{2}\left[\frac{2}{2x}\right]^2$$

$$= -\frac{3}{2x^2}.$$

Note that $E \circ Q(x) = e^{x^2+2}$, and so

$$H \circ E \circ Q(x) = \sin\left(e^{x^2+2}\right).$$

To apply Exercise 9, we must compute

$$\begin{aligned} S(E \circ Q)(x) &= SE(Q(x)) \cdot [Q'(x)]^2 + SQ(x) \\ &= -\frac{1}{2}[2x]^2 - \frac{3}{2x^2} \\ &= -2x^2 - \frac{3}{2x^2}. \end{aligned}$$

Finally, since $(E \circ Q)'(x) = 2xe^{x^2+2}$, we see that

$$\begin{aligned} &S(H \circ E \circ Q)(x) \\ &= SH(E \circ Q(x)) \cdot [(E \circ Q)'(x)]^2 + S(E \circ Q)(x) \\ &= -\left(1 + \frac{3}{2}\tan^2\left(e^{x^2+2}\right)\right)\left[4x^2e^{2x^2+4}\right] - \left(2x^2 + \frac{3}{2x^2}\right) \\ &< 0 \qquad \text{for all } x. \end{aligned}$$

10d) $F(x) = \frac{1}{\cos(x^2-2)}$

Let $C(x) = \cos x$. By Exercise 1d,

$$SC(x) = -1 - \frac{3}{2}\cot^2(x).$$

Now let $Q(x) = x^2-2$. By Exercise 1a and the fact that $S(F+c) = SF$ (which was shown in Exercise 4), we see that

$$SQ(x) = -\frac{3}{2x^2}.$$

In fact, $SQ_c(x) = -3/(2x^2)$ for all c. Now consider $C \circ Q$. By the chain rule for Schwarzian derivatives,

$$\begin{aligned} S(C \circ Q)(x) &= SC(Q(x)) \cdot [Q'(x)]^2 + SQ(x) \\ &= \left(-1 - \frac{3}{2}\cot^2(x^2-2)\right)4x^2 - \frac{3}{2x^2} \\ &= -\left(4x^2 + 6x^2\cot^2(x^2-2) + \frac{3}{2x^2}\right) \end{aligned}$$

which is practically the same answer obtained in Exercise 1d. At any rate, the last step is to let $R(x) = 1/x$. By Exercise 6,

$$\begin{aligned} S(R \circ C \circ Q)(x) &= SR(C \circ Q(x)) \cdot [(C \circ Q)'(x)]^2 + S(C \circ Q)(x) \\ &= S(C \circ Q)(x) \end{aligned}$$

since $SR = 0$.

Chapter 13

Newton's Method

Exercises

1. Use graphical analysis to completely describe all orbits of the associated Newton iteration function for F.

1a) $F(x) = 4 - 2x \Rightarrow F'(x) = -2 \Rightarrow F''(x) = 0$.

$NF(x) = x - F(x)/F'(x) = x - (4 - 2x)/(-2) = x - (x - 2) = 2$.

All orbits are eventually fixed on the fixed point 2 after one iteration. See Figure 13.1.

1b) $F(x) = x^2 - 2x \Rightarrow F'(x) = 2x - 2 \Rightarrow F''(x) = 2$.

$$\begin{aligned} NF(x) &= x - \frac{F(x)}{F'(x)} \\ &= x - \frac{x^2 - 2x}{2x - 2} \\ &= \frac{2x^2 - 2x - (x^2 - 2x)}{2x - 2} \\ &= \frac{x^2}{2x - 2}. \end{aligned}$$

All orbits, except that of 1, converge on a root. The basins are apparent from the graph (see Figure 13.2).

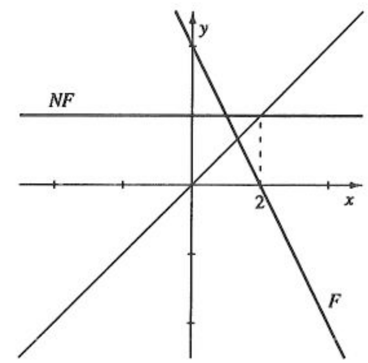


Figure 13.1: The Newton I.F. for $F(x) = 4 - 2x$ is constant.

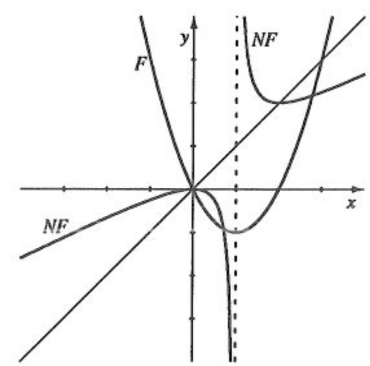


Figure 13.2: The Newton I.F. for $F(x) = x^2 - 2x$ is a typical example.

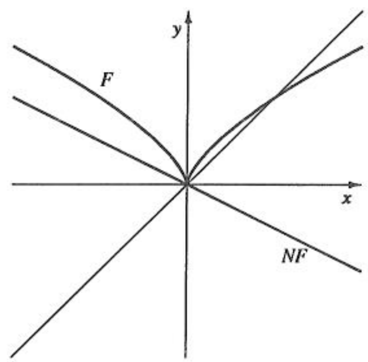


Figure 13.3: The Newton I.F. for $F(x) = x^{2/3}$ is linear.

1c) $F(x) = x^{2/3} \Rightarrow F'(x) = \frac{2}{3}x^{-1/3} \Rightarrow F''(x) = \frac{-2}{9}x^{-4/3}$.

$$\begin{aligned} NF(x) &= x - \frac{x^{2/3}}{\frac{2}{3}x^{-1/3}} \\ &= x - 3x/2 \\ &= -x/2. \end{aligned}$$

Observe in Figure 13.3 that F is not differentiable at the origin, yet NF has an attracting fixed point there. In fact, all orbits are attracted to 0, but *not* quadratically.

This example is typical of an entire class of functions, namely, $F_d(x) = x^d$ with $1/2 < d < 1$. In each case, F_d is not differentiable at the origin, yet Newton's method converges linearly with slope $(d-1)/d$. For $d > 1$, the derivative exists (but vanishes) and the Newton iteration function still converges.

1d) $F(x) = x^4 + x^2 = x^2(x^2+1)$.

$F'(x) = 4x^3 + 2x = 2x(2x^2+1)$.

$F''(x) = 12x^2 + 2$.

$$NF(x) = x - \frac{x^4 + x^2}{4x^3 + 2x}$$

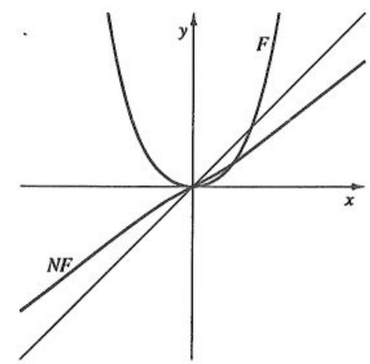


Figure 13.4: The Newton I.F. for $F(x) = x^4 + x^2$.

$$\begin{aligned} &= \frac{4x^4 + 2x^2 - (x^4 + x^2)}{4x^3 + 2x} \\ &= \frac{3x^3 + x}{4x^2 + 2}. \end{aligned}$$

Curiously, the graph of NF depicted in Figure 13.4 is almost linear. Indeed,

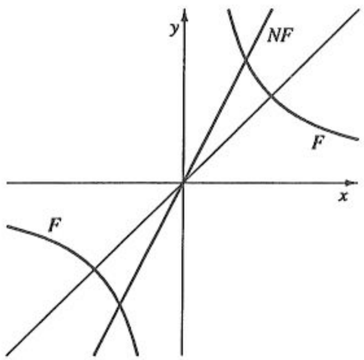
$$\frac{3x^3 + x}{4x^2 + 2} = \frac{3x + \frac{1}{x}}{4 + \frac{2}{x^2}} \to \frac{3x}{4} \text{ as } x \to \infty,$$

and since

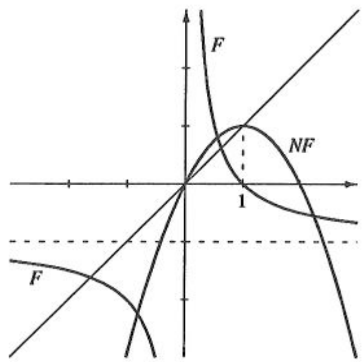
$$\begin{aligned} (NF)'(x) &= \frac{F(x)F''(x)}{[F'(x)]^2} \\ &= \frac{x^2(x^2+1)(12x^2+2)}{4x^2(2x^2+1)^2} \\ &= \frac{(x^2+1)(6x^2+1)}{2(2x^2+1)^2}, \end{aligned}$$

we have that $(NF)'(0) = (1)(1)/2 = 1/2$. Recall, however, that if p is a root of multiplicity m, then $(NF)'(p) = (m-1)/m$, and since $m = 2$ in this case, the previous calculation is verified.

1e) $F(x) = 1/x = x^{-1} \Rightarrow F'(x) = -x^{-2} \Rightarrow F''(x) = 2x^{-3}$. But F has



(a) The Newton I.F. for $F(x) = 1/x$ is linear.



(b) The Newton I.F. for $F(x) = 1/x - 1$ is a parabola.

Figure 13.5: Two members of the family $F(x) = 1/x + c$.

no roots, and

$$\begin{aligned} NF(x) &= x - \frac{x^{-1}}{-x^{-2}} \\ &= x + x \\ &= 2x. \end{aligned}$$

Hence, all orbits, except the orbit of 0, diverge. (See Figure 13.5a.)

1f) $F(x) = 1/x - 1 = (1-x)/x$. (Note that $F(x)$ vanishes when $x = 1$.)
$F'(x) = -1/x^2 = -x^{-2}$.
$F''(x) = 2/x^3 = 2x^{-3}$.

$$\begin{aligned} NF(x) &= x - \frac{x^{-1} - 1}{-x^{-2}} \\ &= x + x - x^2 \\ &= 2x - x^2 = x(2-x). \end{aligned}$$

In Figure 13.5b, graphical analysis suggests that

$$\{\, x \mid (NF)^n(x) \to 1 \text{ as } n \to \infty \,\} = \{\, x \mid 0 < x < 2 \,\}.$$

1g) $F(x) = x/\sqrt{1+x^2} = x(1+x^2)^{-1/2}$.
By the product rule,

$$\begin{aligned} F'(x) &= (1+x^2)^{-1/2} + x(-\tfrac{1}{2})(1+x^2)^{-3/2}(2x) \\ &= (1+x^2)^{-3/2}[(1+x^2) - x^2] \\ &= (1+x^2)^{-3/2}. \end{aligned}$$

The Newton iteration function for F is

$$\begin{aligned} NF(x) &= x - \frac{x(1+x^2)^{-1/2}}{(1+x^2)^{-3/2}} \\ &= x - x(1+x^2) \\ &= -x^3, \end{aligned}$$

and so $DNF(x) = -3x^2$. By inspection of Figure 13.6, we see that $(NF)^2(1) = 1$, and so 1 is a period 2 point for NF. But the corresponding 2-cycle is repelling since

$$\begin{aligned} D(NF)^2(1) &= DNF(1) \cdot DNF(-1) \\ &= -3(1)^2 \cdot (-3)(-1)^2 \\ &= 9. \end{aligned}$$

From the graph, it appears that the open interval $(-1, 1)$ is the immediate basin of 0. Indeed, observe that

$$\begin{aligned} NF(x) &= -x^3 \\ (NF)^2(x) &= -(-x^3)^3 = x^9 \\ (NF)^3(x) &= -(-x^9)^3 = -x^{27} \\ &\vdots \\ (NF)^n(x) &= (-1)^n x^{3^n} \end{aligned}$$

and $|(NF)^n(x)| = |(-1)^n x^{3^n}| = |x^{3^n}| \to 0$ as $n \to \infty$ provided $|x| < 1$.

1h) $F(x) = xe^x \;\Rightarrow\; F'(x) = e^x + xe^x = e^x + F(x)$. Consequently,

$$\begin{aligned} NF(x) &= x - \frac{xe^x}{e^x + xe^x} \\ &= \frac{xe^x + x^2e^x - xe^x}{e^x + xe^x} \\ &= \frac{x^2e^x}{e^x + xe^x} = \frac{x^2}{1+x}. \end{aligned}$$

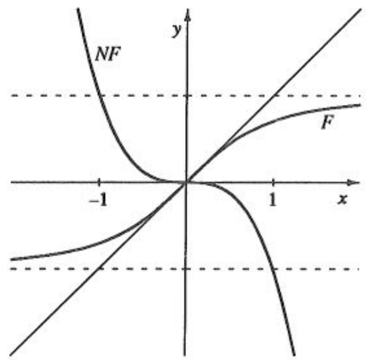


Figure 13.6: The Newton I.F. for $F(x) = x/\sqrt{1+x^2}$ is a simple cubic polynomial.

The reader should verify the following observations taken from Figure 13.7, some of which are trivial:

1. F has a minimum at $x = -1$.
2. $x = -1$ is an asymptote for NF.
3. $F(x) \to 0^-$ as $x \to -\infty$.
4. $NF(x) \to x$ as $x \to -\infty$.
5. $NF(x) \to x$ as $x \to +\infty$.
6. For $-\infty < x_0 < -1$, $(NF)^n(x_0) \to -\infty$ as $n \to \infty$.
7. For $-1 < x_0 < \infty$, $(NF)^n(x_0) \to 0$ as $n \to \infty$.

2. What happens when Newton's method is applied to $R(x) = \sqrt{x}$?

For $R(x) = \sqrt{x}$, we have $R'(x) = 1/(2\sqrt{x})$. Thus,

$$\begin{aligned} NR(x) &= x - \frac{\sqrt{x}}{1/(2\sqrt{x})} \\ &= x - 2x \\ &= -x. \end{aligned}$$

See Figure 13.8. Now we know that $\mathrm{per}_2\, NR = \mathbb{R}$ since $(NR)^2(x) = x$ for all x, and so Newton's method fails miserably in this case. Why? Most likely because $R'(0)$ does not exist.

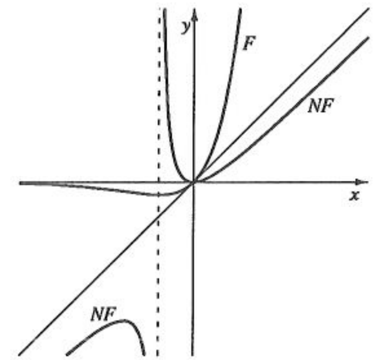


Figure 13.7: The Newton I.F. for $F(x) = xe^x$.

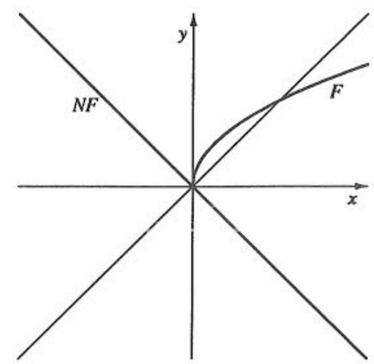


Figure 13.8: The Newton I.F. for $R(x) = \sqrt{x}$ is linear.

3. Find all fixed points for the associated Newton iteration function for $F(x) = x/(x-1)^m$ when $m = 1, 2, 3, \ldots$. Which are attracting and which are repelling?

We write $F(x) = x(x-1)^{-m}$ and compute via the product rule

$$\begin{aligned} F'(x) &= (x-1)^{-m} - mx(x-1)^{-m-1} \\ &= (x-1)^{-m-1}(x-1-mx) \\ &= (x(1-m)-1)/(x-1)^{m+1} \end{aligned}$$

which vanishes when $x(1-m)-1=0$ or when $x = 1/(1-m)$, provided $m \neq 1$. Likewise,

$$\begin{aligned} F''(x) &= (1-m)(x-1)^{-(m+1)} - (m+1)(x(1-m)-1)(x-1)^{-(m+2)} \\ &= (x-1)^{-(m+2)}((1-m)(x-1) - (m+1)(x(1-m)-1)) \\ &= (x-1)^{-(m+2)}(x(m^2-m)+2m) \end{aligned}$$

which we'll use later in this exercise. Now the Newton iteration function for F is

$$\begin{aligned} NF(x) &= x - \frac{F(x)}{F'(x)} \\ &= x - \frac{x/(x-1)^m}{(x(1-m)-1)/(x-1)^{m+1}} \\ &= x - \frac{x(x-1)}{x(1-m)-1} \\ &= \frac{x(x(1-m)-1) - x(x-1)}{x(1-m)-1} \\ &= \frac{mx^2}{1-x(1-m)}. \end{aligned} \tag{13.1}$$

Setting Equation (13.1) equal to x and solving, we find that fix $NF = \{0, 1\}$. Note carefully that *these fixed points are independent of m*. (See also Figures 13.9 and 13.10.) But are they attracting, superattracting, repelling, or what? To answer this question, we compute

$$\begin{aligned} DNF(x) &= \frac{F(x)F''(x)}{(F'(x))^2} \\ &= \frac{\frac{x}{(x-1)^m}\frac{x(m^2-m)+2m}{(x-1)^{m+2}}}{\left(\frac{x(1-m)-1}{(x-1)^{m+1}}\right)^2} \end{aligned}$$

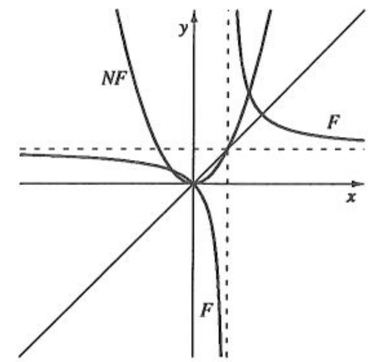


Figure 13.9: The Newton I.F. for $F(x) = x/(x-1)$ is the parabola $x \mapsto x^2$.

$$= \frac{x^2(m^2-m)+2mx}{(x(1-m)-1)^2}. \tag{13.2}$$

Thus, $DNF(0) = 0$ and

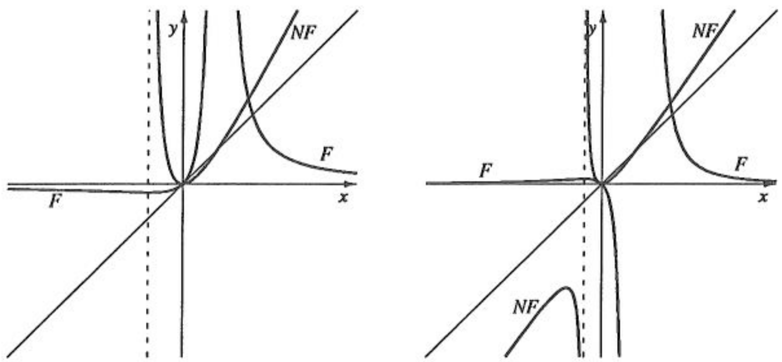
$$\begin{aligned} DNF(1) &= \frac{m^2-m+2m}{(1-m)^2 - 2(1-m)+1} \\ &= \frac{m^2+m}{(1-m)((1-m)-2+1)} \\ &= \frac{m+1}{m-1} \end{aligned}$$

which is greater than unity for all $m > 1$. So the latter fixed point is *always* repelling.

Now it appears as though NF has *two* critical points when $m = 3$ but only a single such point for $m = 2$ (see Figure 13.10). Let's examine this conjecture more closely. Setting the derivative in (13.2) equal to 0, we have

$$\begin{aligned} &\frac{mx(x(m-1)+2)}{(x(1-m)-1)^2} = 0 \\ \Rightarrow\ & mx(x(m-1)+2) = 0 \\ \Rightarrow\ & x = 0 \quad \text{or} \quad x = 2/(1-m), \end{aligned}$$

and so indeed, there are two critical points for all $m > 1$. But why isn't



(a) $m = 2$ (b) $m = 3$

Figure 13.10: Two more members of the family $F(x) = x/(x-1)^m$.

the second one visible in Figure 13.10a? Well, for $x = 2/(1-m)$, we have

$$NF(x) = -\frac{4m}{(1-m)^2},$$

and so for $m = 2$, the critical point is at $(-2, -8)$. But this point is beyond the scope of the graph in Figure 13.10a, and sure enough, zooming out we get the more complete picture seen in Figure 13.11.

4. Consider the Newton iteration function for $F(x) = \sec x$. What are the fixed points for NF? Does this contradict the Newton Fixed Point Theorem? Why or why not?

We begin with the following calculations:

$$F(x) = \sec x; \qquad F''(x) = \sec x \sec^2 x + \sec x \tan x \tan x$$
$$F'(x) = \sec x \tan x; \qquad = \sec x(\sec^2 x + \tan^2 x).$$

Hence,

$$NF(x) = x - \frac{F(x)}{F'(x)}$$

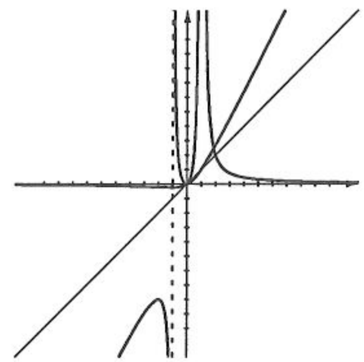


Figure 13.11: A distant view of $F(x) = x/(x-1)^2$.

$$= x - \frac{\sec x}{\sec x \tan x}$$
$$= x - \cot x \tag{13.3}$$

has fixed points at the zeros of the cotangent function, that is, at $x = k\pi/2$ for nonzero integers k. This result is unexpected however, since F itself has no zeros whatsoever. But observe that these are precisely those values for which secant is undefined. Moreover,

$$\begin{aligned}(NF)'(x) &= \frac{F(x)F''(x)}{(F'(x))^2} \\ &= \frac{\sec^2 x(\sec^2 x + \tan^2 x)}{(\sec x \tan x)^2} \\ &= 1 + \csc^2 x\end{aligned}$$

which may also be obtained by direct differentiation of (13.3). Now, cosecant obtains relative optima at the fixed points of NF (or the zeros of F). In fact, $\csc(k\pi/2) = 1$ for nonzero integers k. So, $(NF)'(k\pi/2) = 2 > 1$, and so these fixed points are repelling under iteration of NF.

5. Suppose P and Q are polynomials and let $F(x) = P(x)/Q(x)$. What can be said about the fixed points of the associated Newton iteration function for F? Which fixed points are attracting and which are repelling?

If $F(x) = P(x)/Q(x)$, then

$$F'(x) = \frac{Q(x)P'(x) - Q'(x)P(x)}{(Q(x))^2}$$

and so

$$\begin{aligned} NF(x) &= x - \frac{F(x)}{F'(x)} \\ &= x - \frac{P(x)/Q(x)}{(Q(x)P'(x) - Q'(x)P(x))/(Q(x))^2} \\ &= x - \frac{P(x)Q(x)}{Q(x)P'(x) - Q'(x)P(x)}. \end{aligned}$$

The fixed points for NF are found by solving

$$x - \frac{P(x)Q(x)}{Q(x)P'(x) - Q'(x)P(x)} = x$$

which implies that

$$P(x)Q(x) = 0.$$

In other words, we have found that fix $NF = \{ x \mid P(x)Q(x) = 0 \}$.

Throughout the rest of this exercise, define $\Delta_1(x) = Q(x)P'(x) - Q'(x)P(x)$ and $\Delta_2(x) = Q(x)P''(x) - Q''(x)P(x)$ for convenience. Now it can be shown that

$$F''(x) = \frac{Q(x)\Delta_2(x) - 2Q'(x)\Delta_1(x)}{(Q(x))^3}$$

and so

$$\begin{aligned} (NF)'(x) &= \frac{F(x)F''(x)}{(F'(x))^2} \\ &= \frac{\frac{P(x)}{Q(x)} \cdot \frac{Q(x)\Delta_2(x) - 2Q'(x)\Delta_1(x)}{(Q(x))^3}}{\frac{\Delta_1^2(x)}{(Q(x))^4}} \\ &= \frac{P(x)Q(x)\Delta_2(x) - 2P(x)Q'(x)\Delta_1(x)}{\Delta_1^2(x)}. \end{aligned} \tag{13.4}$$

But when $P(x)Q(x) = 0$, Equation (13.4) reduces to

$$\begin{aligned} (NF)'(x) &= \frac{-2P(x)Q'(x)\Delta_1(x)}{\Delta_1^2(x)} \\ &= \frac{-2P(x)Q'(x)}{\Delta_1(x)}. \end{aligned}$$

We may summarize all of these results as follows. For an arbitrary rational function $F(x) = P(x)/Q(x)$, we have that fix $NF = \{ x \mid P(x)Q(x) = 0 \}$, each of which is attracting if and only if

$$\left| \frac{-2P(x)Q'(x)}{Q(x)P'(x) - Q'(x)P(x)} \right| < 1. \tag{13.5}$$

Inverting both sides of (13.5) and simplifying, we have the equivalent condition

$$\left| 1 - \frac{Q(x)P'(x)}{P(x)Q'(x)} \right| > 2.$$

6. *A bifurcation.* Consider the family of functions $F_\mu(x) = x^2 + \mu$. Clearly, $F_\mu(x) = 0$ has two roots when $\mu < 0$, one root when $\mu = 0$, and no real roots when $\mu > 0$. Your goal in these exercises is to investigate how the dynamics of the associated Newton iteration function changes as μ changes.

6a) Sketch the graphs of the associated Newton iteration function NF_μ in the three cases $\mu < 0$, $\mu = 0$, and $\mu > 0$.

We handle the simplest case first. When $\mu = 0$,

$$NF_0(x) = x - \frac{x^2}{2x} = x - \frac{x}{2} = \frac{x}{2}.$$

See Figure 13.12 for the case $\mu = 0$. In general,

$$\begin{aligned} NF_\mu(x) &= x - \frac{x^2 + \mu}{2x} \\ &= \frac{x^2 - \mu}{2x}. \end{aligned}$$

Typical members of this family of I.F.s are shown in Figure 13.13.

6b) Use graphical analysis to explain the dynamics of NF_μ when $\mu < 0$ and $\mu = 0$.

The dynamics of NF_0 are simple: the origin attracts all orbits, but not quadratically. On the other hand, the Newton I.F. for F_μ for $\mu < 0$ is more interesting. The orbits of all positive numbers quadratically converge to the positive fixed point, whereas the orbits of negative numbers converge to the negative fixed point. See the graphical analyses depicted in Figure 13.14.

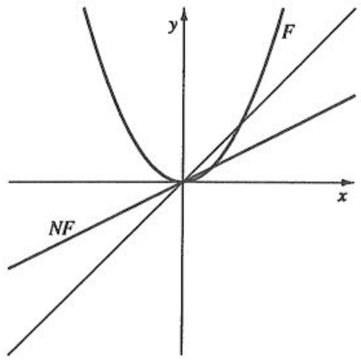
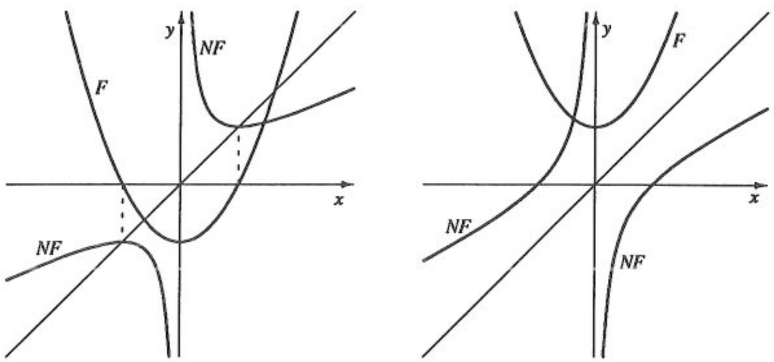
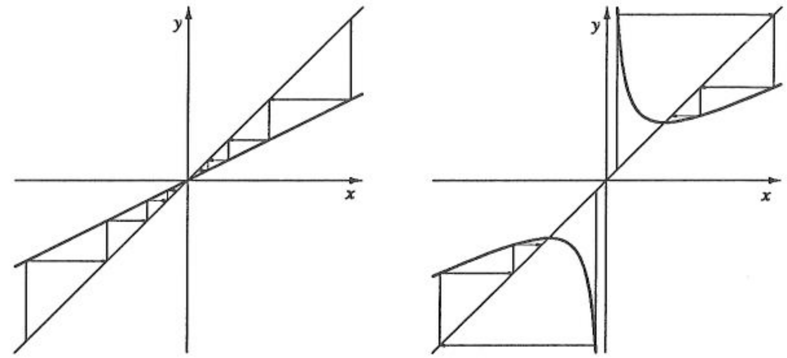


Figure 13.12: The Newton iteration function for $F(x) = x^2$ is linear.



(a) $\mu < 0$ (b) $\mu > 0$

Figure 13.13: Typical members of the family $F_\mu(x) = x^2 + \mu$ and their associated Newton iteration functions.



(a) $\mu = 0$ (b) $\mu < 0$

Figure 13.14: Dynamics of NF_μ.

6c) Prove that, if $\mu > 0$, the Newton iteration function for $x \mapsto x^2 + 1$ is conjugate to the Newton iteration function for F_μ via the conjugacy $H(x) = \sqrt{\mu}\,x$. Conclude that the Newton iteration function is chaotic for $\mu > 0$.

As suggested, we let $\mu > 0$ and show that

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{NF_1} & \mathbb{R} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ \mathbb{R} & \xrightarrow{NF_\mu} & \mathbb{R} \end{array}$$

commutes. This follows since

$$H \circ NF_1(x) = H\left(\frac{x^2-1}{2x}\right) = \sqrt{\mu}\,\frac{x^2-1}{2x}$$

and

$$NF_\mu \circ H(x) = NF_\mu(\sqrt{\mu}\,x)$$

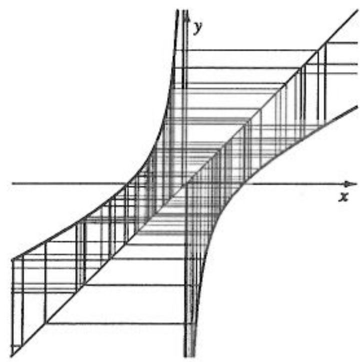


Figure 13.15: Chaotic dynamics of NF_μ for $\mu > 0$.

$$= \frac{(\sqrt{\mu}\,x)^2 - \mu}{2\sqrt{\mu}\,x} = \frac{\mu(x^2-1)}{2\sqrt{\mu}\,x} = \sqrt{\mu}\,\frac{x^2-1}{2x}.$$

6d) Find an analogous conjugacy when $\mu < 0$.

We will show that

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{NF_{-1}} & \mathbb{R} \\ \Big\downarrow H & & \Big\downarrow H \\ \mathbb{R} & \xrightarrow{NF_\mu} & \mathbb{R} \end{array}$$

commutes, where $H(x) = \sqrt{-\mu}\,x$. Again, this is so because

$$H \circ NF_{-1}(x) = H\left(\frac{x^2+1}{2x}\right) = \sqrt{-\mu}\,\frac{x^2+1}{2x}$$

and

$$NF_\mu \circ H(x) = NF_\mu(\sqrt{-\mu}\,x)$$

$$= \frac{(\sqrt{-\mu}\,x)^2 - \mu}{2\sqrt{-\mu}\,x} = \frac{-\mu(x^2+1)}{2\sqrt{-\mu}\,x} = \sqrt{-\mu}\,\frac{x^2+1}{2x}.$$

7. ***A more complicated bifurcation.*** Consider the family of functions given by $G_\mu(x) = x^2(x-1) + \mu$.

7a) Sketch the graphs of the Newton iteration function NG_μ in the three cases $\mu < 0$, $\mu = 0$, and $\mu > 0$.

We have that $G'_\mu(x) = x(3x-2)$ and $G''_\mu(x) = 6x - 2$, and in fact, all of G_μ's derivatives are independent of μ. Now,

$$NG_\mu(x) = x - \frac{x^2(x-1)+\mu}{x(3x-2)} = \frac{x^2(3x-2) - x^2(x-1) - \mu}{x(3x-2)} = \frac{x^2(2x-1)-\mu}{x(3x-2)}.$$

In particular, when $\mu = 0$,

$$NG_0(x) = \frac{x(2x-1)}{3x-2}$$

which is shown in Figure 13.16. Other representatives of this family of I.F.s are shown in Figure 13.17.

Let's also compute

$$(NG_\mu)'(x) = \frac{G_\mu(x)G''_\mu(x)}{(G'_\mu(x))^2} = \frac{(x^2(x-1)+\mu)(6x-2)}{(x(3x-2))^2} \qquad (13.6)$$

which we'll use below.

7b) Use graphical analysis to discuss the fate of all orbits in case $\mu = 0$.

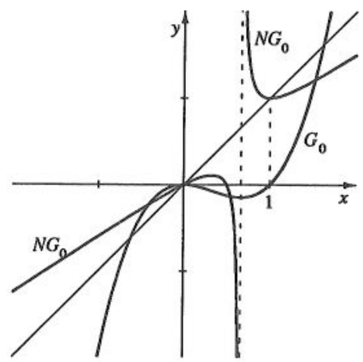
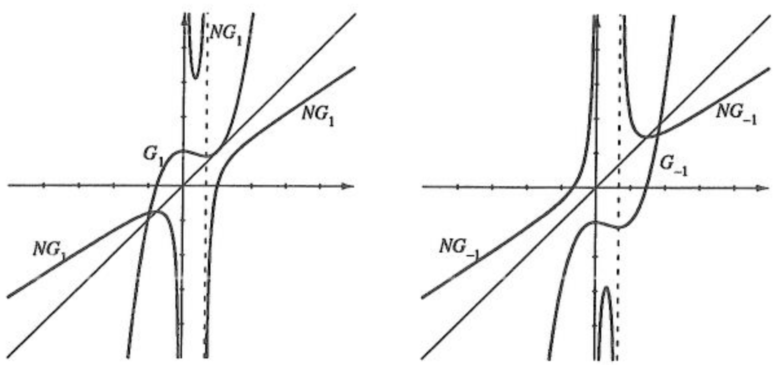


Figure 13.16: The Newton iteration function for $G_0(x) = x^2(x-1)$.



(a) $\mu > 0$ (b) $\mu < 0$

Figure 13.17: Typical members of the family $G_\mu(x) = x^2(x-1) + \mu$ and their associated Newton iteration functions.

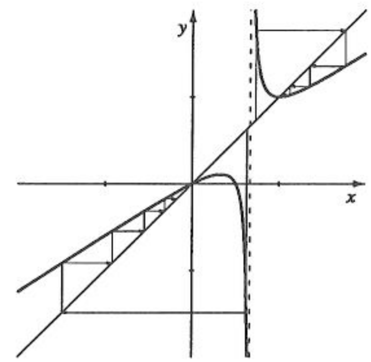


Figure 13.18: Dynamics of $NG_0(x) = (x(2x-1))/(3x-2)$.

It's clear from Figure 13.16 that fix $NG_0 = \{0, 1\}$. (These fixed points are most easily obtained as the zeros of G_0.) Now the derivative of NG_0 may be computed directly from (13.6):

$$(NG_0)'(x) = \frac{(x-1)(6x-2)}{(3x-2)^2}.$$

Thus, $x = 1$ is superattracting since $(NG_0)'(1) = 0$, but $x = 0$ is not since $(NG_0)'(0) = 1/2$. This is because 0 is a root of multiplicity two of $G_0(x) = 0$.

It can be seen from Figure 13.18 that the orbits of all points to the right of the vertical asymptote converge (quadratically) to $x = 1$, while those to the left of the asymptote converge (linearly) to $x = 0$. Since $x = 2/3$ has no preimage, these statements are true without exception.

7c) Show that NG_μ has exactly one critical point which is not fixed for all but one μ-value.

The critical points of NG_μ are precisely the zeros of G_μ, as well as the zeros of its second derivative. The latter gives rise to the critical point $x = 1/3$ which is clearly independent of μ. The remaining critical points, the zeros of G_μ, are solutions to

$$x^3 - x^2 + \mu = 0, \tag{13.7}$$

a nontrivial cubic equation. By Descartes' rule of signs, it's clear that

(13.7) has but a single real root, but a closed form solution is difficult. Maple claims that

$$x = 1/3 + w_1 + w_2$$

where

$$w_1 = (1/27 - \mu/2 - w_3/18)^{1/3},$$
$$w_2 = (1/27 - \mu/2 + w_3/18)^{1/3},$$

and

$$w_3 = \sqrt{3\mu(27\mu - 4)}.$$

Note when $\mu = 0$, w_3 vanishes, and so $w_1 = w_2 = 1/3$. Thus the root in this case is unity, which agrees with that obtained by a direct manipulation of (13.7).

8. Consider the function $G(x) = x^4 - x^2 - 11/36$.

8a) Compute the inflection points of G. Show that they are critical points for the associated Newton function.

Observe that G is an even function with two real roots and two complex conjugate roots. (See Figure 13.19a.) The latter are investigated in Chapter ?? of the text; the former are calculated below. Now the derivative of G is $G'(x) = 4x^3 - 2x$, and so

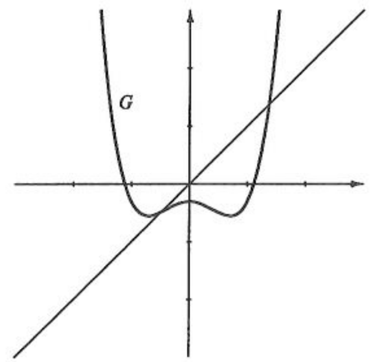
$$\begin{aligned} NG(x) &= x - \frac{x^4 - x^2 - 11/36}{2x(2x^2-1)} \\ &= \frac{2x^2(2x^2-1) - x^4 + x^2 + 11/36}{2x(2x^2-1)} \\ &= \frac{3x^4 - x^2 + 11/36}{2x(2x^2-1)}. \end{aligned}$$

We remark that NG is an odd function. Next we compute $G''(x) = 12x^2 - 2$ (another even function) which has zeros at $x = \pm 1/\sqrt{6}$. From this it follows that

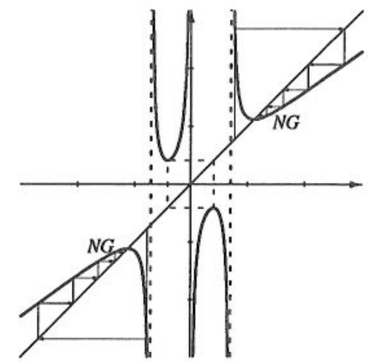
$$(NG)'(x) = \frac{(x^4 - x^2 - 11/36)(12x^2 - 2)}{4x^2(2x^2-1)^2}$$

which obviously has critical points at $\pm 1/\sqrt{6}$. Indeed, the zeros of G'' are always critical points for NG since

$$(NG)'(x) = \frac{G(x)G''(x)}{(G'(x))^2}.$$



(a) Graph of $G(x) = x^4 - x^2 - 11/36$.



(b) Dynamics of NG.

Figure 13.19: A quartic polynomial and its associated Newton I.F.

Note that the zeros of G are also critical points for NG, and in this case they are roots of

$$x^4 - x^2 - 11/36 = 0$$

which, by the quadratic formula, satisfy

$$x^2 = \frac{3 \pm 2\sqrt{5}}{6}.$$

Now only two of these are real (let's denote them by p_+ and p_-), and are given by

$$p_\pm = \pm\sqrt{\frac{3 + 2\sqrt{5}}{6}}.$$

8b) Prove that these two points lie on a 2-cycle.

A single calculation takes care of both cases:

$$\begin{aligned} NG(\pm 1/\sqrt{6}) &= \frac{3(\pm 1/\sqrt{6})^4 - (\pm 1/\sqrt{6})^2 + 11/36}{2(\pm 1/\sqrt{6})(2(\pm 1/\sqrt{6})^2 - 1)} \\ &= \frac{3(1/36) - 1/6 + 11/36}{2(\pm 1/\sqrt{6})(1/3 - 1)} \end{aligned}$$

$$= \frac{8/36}{(4/3)(\mp 1/\sqrt{6})}$$
$$= \frac{1/6}{\mp 1/\sqrt{6}}$$
$$= \mp\sqrt{6}/6 = \mp 1/\sqrt{6}.$$

8c) What can you say about the convergence of Newton's method for this function?

Graphical analysis (see Figure 13.19b) suggests that for $\sqrt{2}/2 < x_0 < \infty$, $(NG)^n(x_0) \to p_+$ as $n \to \infty$. Similarly, for $-\infty < x_0 < -\sqrt{2}/2$, $(NG)^n(x_0) \to p_-$ as $n \to \infty$. But what happens in the interval $-\sqrt{2}/2 < x_0 < \sqrt{2}/2$? Since $(NG)'(\pm 1/\sqrt{6}) = 0$, this 2-cycle is superattracting and so Newton's method fails to converge for some x_0, and instead oscillates back and forth in the vicinity of $1/\sqrt{6}$ and $-1/\sqrt{6}$. The immediate basin of this 2-cycle is pretty small however, since experiments indicate that the orbits of all x_0 outside the pair of intervals $0.29 \le |x_0| \le 0.51$ converge.

9. Use calculus to sketch the graph of the Newton iteration function for $F(x) = x(x^2+1)$. For which x-values does this iteration converge to a root?

Observe that the origin is a zero for F, and in fact, it's the only real root of $F(x) = 0$. Now we also have that $F'(x) = 3x^2 + 1$ and $F''(x) = 6x$. Thus,

$$NF(x) = x - \frac{x(x^2+1)}{3x^2+1} = \frac{x(3x^2+1) - x(x^2+1)}{3x^2+1} = \frac{2x^3}{3x^2+1}$$

and

$$(NF)'(x) = \frac{F(x)F''(x)}{(F'(x))^2} = \frac{x(x^2+1)\cdot 6x}{(3x^2+1)^2} = \frac{6x^2(x^2+1)}{(3x^2+1)^2}.$$

Since this derivative is always positive, the Newton function never oscillates (can it ever?). Indeed, we have that $(NF)'(0) = 0$ and so the origin is superattracting. See Figure 13.20 for the graphs of F and NF, and some typical orbits.

10. Prove that the critical points of the Newton iteration function associated with

$$F(x) = (x^2-1)(x^2+A)$$

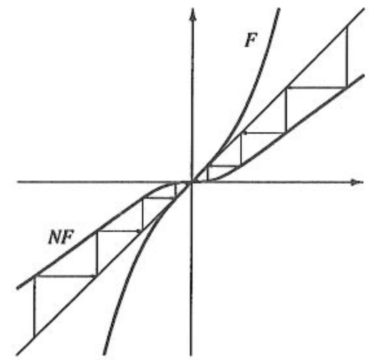


Figure 13.20: The Newton iteration function for $F(x) = x(x^2+1)$ has a globally attracting fixed point at the origin.

lie on a 2-cycle when $A = (29 - \sqrt{720})/11$.

First, some preliminary calculations:

$$F(x) = (x^2-1)(x^2+A);$$
$$F'(x) = (2x)(x^2+A) + (x^2-1)(2x) = 2x(2x^2+A-1);$$
$$F''(x) = 2(2x^2+A-1) + 2x(4x) = 4x^2 + 2(A-1) + 8x^2 = 12x^2 + 2(A-1).$$

Now the Newton iteration function for F is

$$NF(x) = x - \frac{(x^2-1)(x^2+A)}{2x(2x^2+A-1)}$$
$$= \frac{2x^2(2x^2+A-1) - (x^2-1)(x^2+A)}{2x(2x^2+A-1)}$$
$$= \frac{4x^4 + (A-1)2x^2 - (x^4 - x^2 + Ax^2 - A)}{2x(2x^2+A-1)}$$
$$= \frac{3x^4 + (A-1)x^2 + A}{2x(2x^2+A-1)}$$

which agrees with the equation given on p. 170 of the text. Since

$$NF(-x) = \frac{3(-x)^4 + (A-1)(-x)^2 + A}{2(-x)(2(-x)^2 + A - 1)}$$

$$= -\frac{3x^4 + (A-1)x^2 + A}{2x(2x^2 + A - 1)} = -NF(x),$$

we have that NF is an odd function and may therefore solve $NF(x) = -x$ to find the 2-cycles for NF:

$$\begin{aligned} NF(x) &= \frac{3x^4 + (A-1)x^2 + A}{2x(2x^2 + A - 1)} = -x \\ &\Rightarrow \frac{3x^4 + (A-1)x^2 + A + 2x^2(2x^2 + A - 1)}{2x(2x^2 + A - 1)} = 0 \\ &\Rightarrow 7x^4 + 3(A-1)x^2 + A = 0. \end{aligned} \tag{13.8}$$

Recall that the critical points of NF are either zeros of F or F''. In particular, the latter are roots of $12x^2 + 2(A-1) = 0$ which has the pair of solutions

$$x = \pm\sqrt{\frac{1-A}{6}}. \tag{13.9}$$

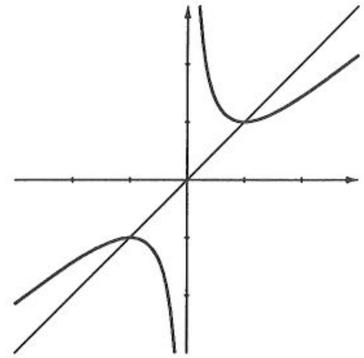
Substituting these values into (13.8), we obtain

$$\begin{aligned} &7\left(\pm\sqrt{\frac{1-A}{6}}\right)^4 + 3(A-1)\left(\pm\sqrt{\frac{1-A}{6}}\right)^2 + A = 0 \\ &\Rightarrow 7\left(\frac{1-A}{6}\right)^2 - 3(1-A)\left(\frac{1-A}{6}\right) + A = 0 \\ &\Rightarrow \left(\frac{7}{36} - \frac{1}{2}\right)(1-A)^2 + A = 0 \\ &\Rightarrow \frac{-11}{36}\left(1 - \frac{58}{11}A + A^2\right) = 0 \\ &\Rightarrow 11 - 58A + 11A^2 = 0 \end{aligned}$$

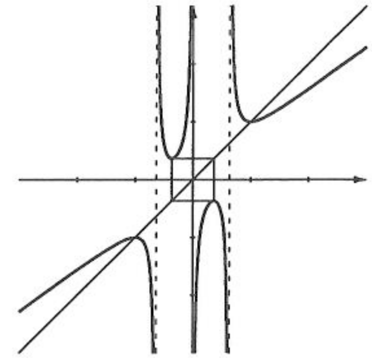
which, by the quadratic equation, has solutions

$$\begin{aligned} A_{\pm} &= \frac{58 \pm \sqrt{58^2 - 4 \cdot 11^2}}{22} \\ &= \frac{29 \pm \sqrt{720}}{11}. \end{aligned}$$

Of these two roots, only A_- produces a real value when plugged into (13.9). See Figure 13.21 for the corresponding Newton iteration functions. Notice



(a) $A = (29 + \sqrt{720})/11$



(b) $A = (29 - \sqrt{720})/11$

Figure 13.21: A pair of Newton I.F.s for $F(x) = (x^2 - 1)(x^2 + A)$.

the striking similarity between Figure 13.21b and Figure 13.19b—there's a conjugacy lurking about!

11. Prove that the equation $F(x) = 0$ has a root of multiplicity m at x_0 if and only if $F(x)$ may be written in the form

$$F(x) = (x - x_0)^m G(x)$$

where G does not vanish at x_0. *Hint:* Use the Taylor expansion of F about x_0.

The hint will be useful going in one direction only. By definition, $D^j F(x_0)$ vanishes for $j = 0, 1, \ldots, m-1$, whereas $D^m F(x_0)$ does not. Now by Taylor's theorem, there exists a ξ (which depends on x) between x and x_0 such that

$$\begin{aligned} F(x) &= \frac{D^m F(x_0)}{m!}(x - x_0)^m + \frac{D^{m+1} F(\xi)}{(m+1)!}(x - x_0)^{m+1} \\ &= (x - x_0)^m \left(\frac{D^m F(x_0)}{m!} + \frac{D^{m+1} F(\xi)}{(m+1)!}(x - x_0)\right) \\ &= (x - x_0)^m G(x) \end{aligned}$$

where $G(x_0) = D^m F(x_0)/m!$ does not vanish.

Going the other way, suppose $F(x) = (x - x_0)^m G(x)$ with $G(x_0) \neq 0$. By Leibniz's rule,

$$D^n F(x) = \sum_{j=0}^{n} \binom{n}{j} D^j (x - x_0)^m D^{n-j} G(x).$$

But for $n = 0, 1, \ldots, m-1$, we have that $D^n F(x_0) = 0$ since

$$D^j (x - x_0)^m = (x - x_0)^{m-j} \prod_{i=0}^{j-1} (k - i)$$

vanishes at $x = x_0$ for $j = 0, 1, \ldots, m-1$. On the other hand, $D^m (x - x_0)^m = \prod_{i=0}^{m-1}(m - i) = m!$, and so $D^m F(x_0) = m!\, G(x_0)$ which doesn't vanish by hypothesis.

12. Let $G(x) = \exp(-1/x^2)$ if $x \neq 0$ and set $G(0) = 0$. Compute the Newton iteration function for G. What can be said about the fixed point of NG? Why does this occur?

Straightforward calculation yields

$$G'(x) = \frac{2\exp(-1/x^2)}{x^3} = \frac{2G(x)}{x^3}$$

and

$$\begin{aligned} G''(x) &= \frac{2x^3 G'(x) - 6x^2 G(x)}{x^6} \\ &= \frac{4G(x) - 6x^2 G(x)}{x^6} \\ &= G(x)\frac{4 - 6x^2}{x^6}. \end{aligned} \tag{13.10}$$

Thus,

$$NG(x) = x - \frac{G(x)}{\frac{2G(x)}{x^3}} = x - \frac{x^3}{2}.$$

Note that $x = 0$ is the only fixed point of NG, and

$$(NG)'(x) = 1 - \frac{3x^2}{2}$$

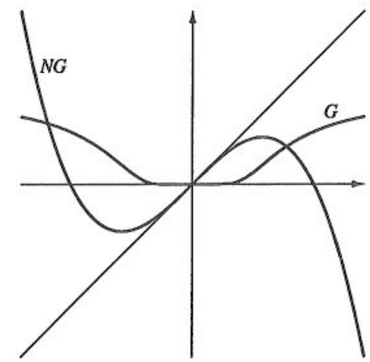


Figure 13.22: The Newton I.F. of $G(x) = \exp(-1/x^2)$ has a weakly attracting fixed point at the origin.

by direct calculation. But $(NG)'(0) = 1$ and so we must apply the results of Chapter 5. To this end, note that $(NG)''(0) = 0$ and $(NG)'''(0) = -3 < 0$. Hence, by Theorem 5.2, the origin is weakly attracting. But why does this happen?

It should be clear from the graph in Figure 13.22 that $G'(0) = 0$. Indeed, the graph appears *very* flat in the vicinity of the origin. In fact, $D^n G(0) = 0$ for all n, an amazing fact which we now show analytically.

By definition,

$$\begin{aligned} G'(0) &= \lim_{h\to 0} \frac{G(0+h) - G(0)}{h} \\ &= \lim_{h\to 0} \frac{\exp(-1/h^2)}{h} \\ &= \lim_{h\to 0} \frac{1}{h\exp(1/h^2)} \\ &= \lim_{t\to\infty} \frac{1}{\exp(t^2)/t} \qquad \text{where } t = 1/h \\ &= \lim_{t\to\infty} \frac{t}{\exp(t^2)} \end{aligned}$$

which has the indeterminate form $\frac{\infty}{\infty}$. By L'Hôpital's rule,

$$\lim_{t\to\infty} \frac{t}{\exp(t^2)} = \lim_{t\to\infty} \frac{1}{2t\exp(t^2)} = 0$$

which shows that $G'(0) = 0$. A similar calculation shows that $G''(0) = 0$. That $D^n G(0)$ vanishes for all n follows by induction, for $D^n G(x)$ is a finite sum of terms of the form

$$\frac{kG(x)}{x^m}$$

each of which vanishes as $x \to 0$. Indeed,

$$\begin{aligned} D\left(\frac{kG(x)}{x^m}\right) &= D(kx^{-m}G(x)) \\ &= kx^{-m}G'(x) - mkx^{-(m+1)}G(x) \\ &= \frac{kG'(x)}{x^m} - \frac{mkG(x)}{x^{m+1}} \\ &= \frac{2kG(x)}{x^{m+3}} - \frac{mkG(x)}{x^{m+1}}. \end{aligned}$$

For example, when $k = 2$ and $m = 3$,

$$D\left(\frac{2G(x)}{x^3}\right) = \frac{4G(x)}{x^6} - \frac{6G(x)}{x^4}$$

which agrees with (13.10) above.

We leave the details of this proof to the interested reader. The upshot of all this is that the origin is a zero of infinite multiplicity, and so G is not analytic at the origin. This explains why it's only weakly attracting for *NG*.

The reader may also enjoy verifying the following facts:

1. G has inflection points at $x = \pm\sqrt{6}/3$.
2. $\lim_{x\to 0} G(x) = 0$, and hence our choice of $G(0)$ is justified.
3. $\lim_{x\to\pm\infty} G(x) = 1$.
4. $G''(0) = 0$, by arguments similar to the above.