Chapter 14

Fractals

Exercises

1. Without using the computer, predict the structure of the attractor generated by the iterated function system with contraction factor β and fixed points p_i:

1a) $\beta = 1/3$; $p_0 = \binom{0}{0}$, $p_1 = \binom{1}{0}$, $p_2 = \binom{0}{1}$.

The given data correspond to the iterated function system

$$\begin{aligned} A_0\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} \\ A_1\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3}\begin{pmatrix} x-1 \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ A_2\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3}\begin{pmatrix} x \\ y-1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{14.1}$$

which gives rise to a type of Sierpinski triangle which, quite surprisingly, has fractal dimension $D = \log 3/\log 3 = 1$. The attractor of this IFS is

$$\{\, (x, y) \in K \times K \mid y \leq 1 - x \,\}$$

where K is the Cantor middle-thirds set. See Figure 14.1 for the first two iterations of (14.1).

1b) $\beta = 1/2$; $p_0 = \binom{0}{0}$, $p_1 = \binom{1}{0}$.

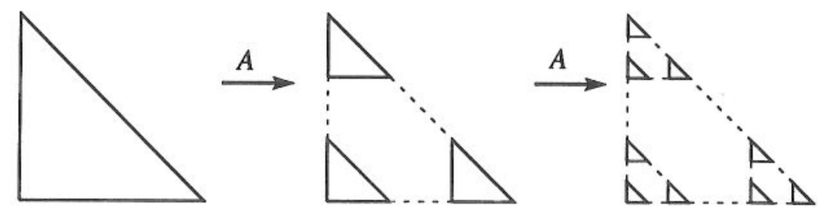


Figure 14.1: Two iterations of Equation 14.1.

The corresponding iterated function system is

$$\begin{aligned} A_0\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix} \\ A_1\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} x-1 \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

Note that the unit interval is fixed with respect to this transformation and, hence, is its attractor with fractal dimension $D = \log 2/\log 2 = 1$ (as expected).

1c) $\beta = 1/3$; $p_0 = \binom{0}{0}$, $p_1 = \binom{1}{0}$, $p_2 = \binom{0}{1}$, $p_3 = \binom{1}{1}$.

The attractor of the iterated function system

$$\begin{aligned} A_0\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} \\ A_1\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3}\begin{pmatrix} x-1 \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ A_2\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3}\begin{pmatrix} x \\ y-1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ A_3\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

is precisely $K \times K$, and its fractal dimension is $D = \log 4/\log 3$ which is exactly twice that of K. What do you think the fractal dimension of the *three*-dimensional analogue of the Cantor set K will be?

2. Give explicitly the iterated function system that generates the **Cantor middle-fifths set**. This set is obtained by the same process that generated

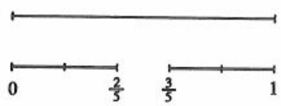


Figure 14.2: The first stage in the construction of the Cantor middle-fifths set.

the Cantor middle-thirds set, except that the middle fifth of each interval is removed at each stage. What is the fractal dimension of this set?

Recall that the Cantor middle-thirds set K is generated either by the planar IFS

$$A_0\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{3}\begin{pmatrix}x\\y\end{pmatrix}$$
$$A_1\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{3}\begin{pmatrix}x-1\\y\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}$$

or the one-dimensional iterated function system

$$A_0(x) = \frac{1}{3}x$$
$$A_1(x) = \frac{1}{3}x + \frac{2}{3}.$$

Now the Cantor middle-fifths set is generated by removing the middle fifth of any remaining closed interval. In particular, the open interval $(2/5, 3/5)$ is removed during the first stage of the construction (see Fig. 14.2). This gives rise to the iterated function system

$$A_0(x) = \frac{2}{5}x$$
$$A_1(x) = \frac{2}{5}x + \frac{3}{5}$$

which also has the planar form

$$A_0\begin{pmatrix}x\\y\end{pmatrix} = \frac{2}{5}\begin{pmatrix}x\\y\end{pmatrix}$$
$$A_1\begin{pmatrix}x\\y\end{pmatrix} = \frac{2}{5}\begin{pmatrix}x-1\\y\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}.$$

Finally, the Cantor middle-fifths set has fractal dimension

$$D = \frac{\log 2}{\log 5/2} = 0.75647\ldots$$

which is greater than the fractal dimension of the Cantor middle-thirds set since less is being taken away at each stage of the construction.

The following seven exercises deal with the Sierpinski right triangle (see Figure 14.3) generated by the following contractions:

$$\begin{aligned} A_0\begin{pmatrix}x\\y\end{pmatrix} &= \frac{1}{2}\begin{pmatrix}x\\y\end{pmatrix} \\ A_1\begin{pmatrix}x\\y\end{pmatrix} &= \frac{1}{2}\begin{pmatrix}x-1\\y\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} \\ A_2\begin{pmatrix}x\\y\end{pmatrix} &= \frac{1}{2}\begin{pmatrix}x\\y-1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}. \end{aligned} \tag{14.2}$$

4. What are the fixed points for A_0, A_1, and A_2?

The fixed points are $p_0 = \binom{0}{0}$, $p_1 = \binom{1}{0}$, and $p_2 = \binom{0}{1}$, respectively.

5. Show that $A_1^n\binom{x_0}{y_0}$ converges to $\binom{1}{0}$.

Since the y-coordinate is divided by 2 each iteration, it clearly goes to 0. Let's look at the x-coordinate more closely which boils down to iterating the linear function

$$F(x) = \frac{x+1}{2}.$$

This function has a fixed point at $x = 1$, and of course this fixed point is attracting since $F'(x) \equiv 1/2$.

Another approach involves showing p_1 is an attracting fixed point using the Jacobian of A_1, but we haven't developed the appropriate machinery to do this.[1]

6. To which point does the sequence

$$A_2^2\left(A_1^n\begin{pmatrix}x_0\\y_0\end{pmatrix}\right)$$

[1] See *Encounters with Chaos* by D. Gulick (McGraw-Hill, 1992) or R.L. Devaney's *An Introduction to Chaotic Dynamical Systems* (Second Edition, Addison-Wesley, 1989).

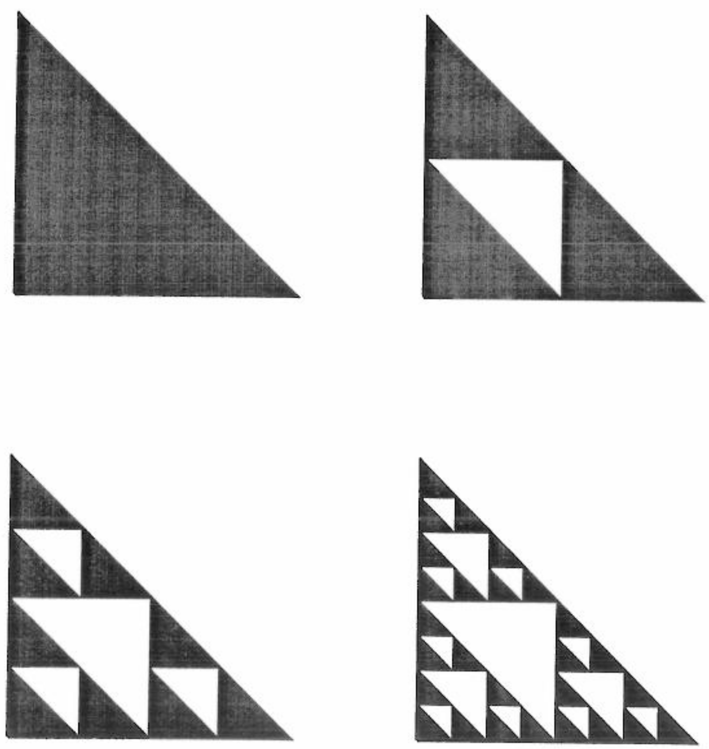


Figure 14.3: Three iterations of Equation 14.2.

converge?

Since

$$A_1^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

as $n \to \infty$, it must be true that

$$\begin{aligned} A_2^2\left(A_1^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) \to A_2^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= A_2 \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix} \end{aligned}$$

as $n \to \infty$.

7. Show that the sequence $A_1\left(A_0^n \binom{x_0}{y_0}\right)$ converges to $\binom{1/2}{0}$.

Since

$$A_0^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \frac{1}{2^n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

we have that

$$\begin{aligned} A_1\left(A_0^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) &= A_1 \begin{pmatrix} \frac{x_0}{2^n} \\ \frac{y_0}{2^n} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \frac{x_0}{2^n} - 1 \\ \frac{y_0}{2^n} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{x_0 - 2^n}{2^{n+1}} \\ \frac{y_0}{2^{n+1}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{x_0 - 2^n + 2^{n+1}}{2^{n+1}} \\ \frac{y_0}{2^{n+1}} \end{pmatrix}. \end{aligned}$$

The y-coordinate clearly goes to 0, but what about the x-coordinate? We see that

$$\lim_{n\to\infty} \frac{x_0 - 2^n + 2^{n+1}}{2^{n+1}} = \lim_{n\to\infty} \frac{x_0/2^{n+1} - \frac{1}{2} + 1}{1} = \frac{1}{2}$$

as required.

8. Show that the sequence

$$(A_1 \circ A_0)^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

accumulates on the two points

$$\begin{pmatrix} 1/3 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}.$$

First compute

$$\begin{aligned} (A_1 \circ A_0)\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} x/2-1 \\ y/2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (x-2)/4 \\ y/4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (x+2)/4 \\ y/4 \end{pmatrix} \end{aligned} \tag{14.3}$$

and then

$$\begin{aligned} (A_0 \circ A_1)\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{2}\left(\frac{1}{2}\begin{pmatrix} x-1 \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\ &= \frac{1}{2}\begin{pmatrix} (x+1)/2 \\ y/2 \end{pmatrix} \\ &= \begin{pmatrix} (x+1)/4 \\ y/4 \end{pmatrix}. \end{aligned} \tag{14.4}$$

Calculating the fixed points of the x-coordinates (the y-coordinates obviously go to zero) in (14.3) and (14.4), one obtains the desired results. Thus

$$\begin{aligned} (A_1 \circ A_0)^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} &\to \begin{pmatrix} 2/3 \\ 0 \end{pmatrix} \\ \text{and} \quad (A_0 \circ A_1)^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} &\to \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} \end{aligned}$$

as $n \to \infty$. It's interesting to note that these accumulation points are the period 2 points of the doubling map.

9. On which points does the sequence

$$(A_2 \circ A_1)^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

accumulate?

Since

$$\begin{aligned} (A_2 \circ A_1)\begin{pmatrix} x \\ y \end{pmatrix} &= A_2\begin{pmatrix} (x+1)/2 \\ y/2 \end{pmatrix} \\ &= \begin{pmatrix} (x+1)/4 \\ (y+2)/4 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} (A_1 \circ A_2)\begin{pmatrix} x \\ y \end{pmatrix} &= A_1\begin{pmatrix} x/2 \\ (y+1)/2 \end{pmatrix} \\ &= \begin{pmatrix} (x+2)/4 \\ (y+1)/4 \end{pmatrix}, \end{aligned}$$

we solve the equations

$$\begin{pmatrix} (x+1)/4 \\ (y+2)/4 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} (x+2)/4 \\ (y+1)/4 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

to obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix},$$

respectively.

10. Show that the sequence

$$(A_2 \circ A_1 \circ A_0)^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

accumulates on the points

$$\begin{pmatrix}2/7\\4/7\end{pmatrix}, \begin{pmatrix}4/7\\1/7\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}1/7\\2/7\end{pmatrix}.$$

As in the preceding two problems, we calculate

$$\begin{aligned}(A_2 \circ A_1 \circ A_0)\begin{pmatrix}x\\y\end{pmatrix} &= (A_2 \circ A_1)\begin{pmatrix}x/2\\y/2\end{pmatrix}\\ &= \begin{pmatrix}(x/2+1)/4\\(y/2+2)/4\end{pmatrix}\\ &= \begin{pmatrix}(x+2)/8\\(y+4)/8\end{pmatrix}\end{aligned} \tag{14.5}$$

and

$$\begin{aligned}(A_1 \circ A_0 \circ A_2)\begin{pmatrix}x\\y\end{pmatrix} &= (A_1 \circ A_0)\begin{pmatrix}x/2\\(y+1)/2\end{pmatrix}\\ &= \begin{pmatrix}(x/2+2)/4\\((y+1)/2)/4\end{pmatrix}\\ &= \begin{pmatrix}(x+4)/8\\(y+1)/8\end{pmatrix}\end{aligned} \tag{14.6}$$

and

$$\begin{aligned}(A_0 \circ A_2 \circ A_1)\begin{pmatrix}x\\y\end{pmatrix} &= (A_0 \circ A_2)\begin{pmatrix}(x+1)/2\\y/2\end{pmatrix}\\ &= A_0\begin{pmatrix}(x+1)/4\\(y+2)/4\end{pmatrix}\\ &= \begin{pmatrix}(x+1)/8\\(y+2)/8\end{pmatrix},\end{aligned} \tag{14.7}$$

and then compute the fixed point in each of (14.5–14.7). It's interesting to note that $1/7, 2/7$, and $4/7$ constitute a 3-cycle for the doubling map.

11. Consider the fractal generated by replacing a line segment with the smaller segments shown in Figure 14.4, where each new segment is exactly one-third as long as the original. Draw carefully the next two iterations of this process. What are the fractal and topological dimensions of the resulting fractal?

Figure 14.4: A Koch-like transformation.

Three iterations of the transformation in Fig. 14.4 give the geometric structures in Fig. 14.5. Note that the third iterate has $5 \cdot 25 = 125$ edges. The fourth iterate shown in Fig. 14.6 has 625 edges! The fractal dimension of this attractor is $D = \log 5/\log 3 \approx 1.465$. It has topological dimension 1 since any disk intersects the curve at a set of discrete points with topological dimension 0.

14. Show that the rational numbers form a subset of the real line that has topological dimension 0. What is the topological dimension of the set of irrationals?

A set has topological dimension 0 if every point has arbitrarily small neighborhoods whose boundaries do not intersect the set. The rationals are such a set since the boundary of a disk with irrational radius fails to intersect the rationals (the sum of a rational and an irrational is irrational).

It's also true that the irrationals have topological dimension 0. To see this, suppose we had the following lemmas:

Lemma 14.1 $A \subseteq X \;\Rightarrow\; t\text{−}\dim A \leq t\text{−}\dim X$.

Lemma 14.2 *For* $A \subseteq \mathbb{R}^n$, $t\text{−}\dim A = n \iff$ *A contains a non-empty open subset of* $\mathbb{R}^n$.

Then by Lemma 14.1, the irrationals have t−dim ≤ 1. But by Lemma 14.2 the irrationals can not have t−dim $= 1$ since they contain no open subset. Therefore, the t−dim of the irrationals is zero.

15. Compute exactly the area of the Koch snowflake.

Koch's snowflake may be enclosed within a rectangle having length 1 and width $2\sqrt{3}/3$, and hence its area is no larger than $2\sqrt{3}/3$ square units.

Let A_0 be the area of an equilateral triangle with sides of length 1. Since the height of this triangle is $\sqrt{3}/2$ units, we have that $A_0 = \sqrt{3}/4$ square units. Now at the kth step of the Koch process, $3 \cdot 4^{k-1}$ equilateral triangles

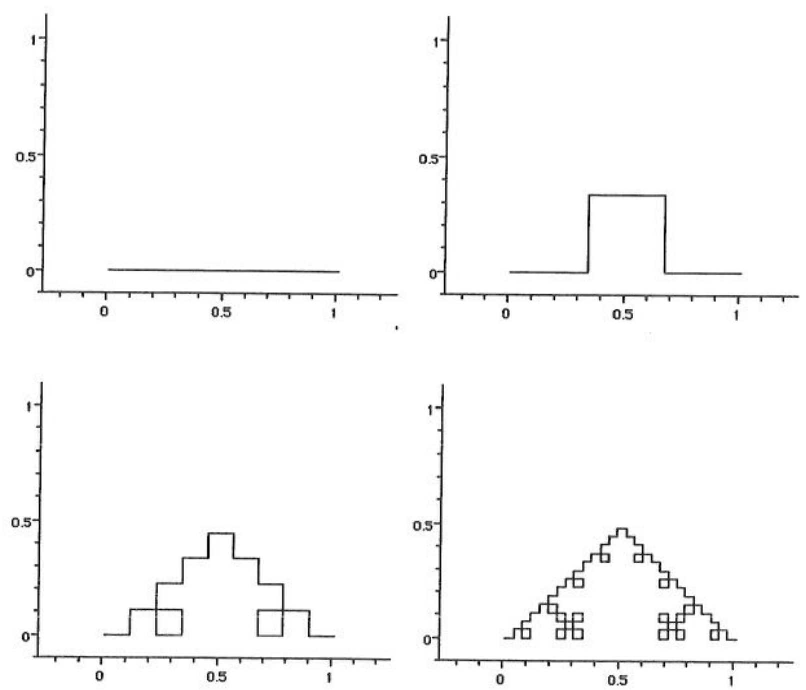


Figure 14.5: Three iterations of a Koch-like curve.

each with area $\sqrt{3}/(4\cdot 3^{2k})$ are added to the snowflake. The combined area of these triangles is

$$3\cdot 4^{k-1}\frac{\sqrt{3}}{4\cdot 3^{2k}} = 3\cdot\frac{4^k}{4}\cdot\frac{\sqrt{3}}{4\cdot 9^k} = \frac{3\sqrt{3}}{16}\left(\frac{4}{9}\right)^k.$$

Thus, the area of *all* these triangles is

$$\sum_{k=1}^{\infty} 3\cdot 4^{k-1}\frac{\sqrt{3}}{4\cdot 3^{2k}} = \frac{3\sqrt{3}}{16}\sum_{k=1}^{\infty}\left(\frac{4}{9}\right)^k$$
$$= \frac{3\sqrt{3}}{16}\cdot\frac{4}{5}$$

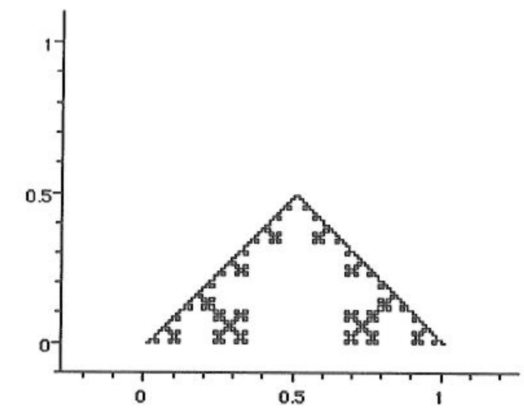


Figure 14.6: The fourth iterate of the transformation in Fig. 14.4.

$$= \frac{3\sqrt{3}}{20},$$

and so the area of the Koch snowflake is

$$A = A_0 + \sum_{k=1}^{\infty} A_k = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20} = \frac{2\sqrt{3}}{5}$$

square units.

17. Consider the standard Pascal's triangle generated by binomial coefficients. In this triangle, replace each odd number by a black dot and each even number by a white dot. Describe the figure that results.

See Figure 14.7 which resembles a Sierpinski triangle.

18. Rework Exercise 17, this time replacing each number by a black dot if it is congruent to 1 mod 3 and a white dot otherwise. (That is, points that yield a remainder of 1 upon division by 3 are colored black.) Now describe the resulting figure. How does it compare with the figure generated in Exercise 17?

Figures 14.8 and 14.9 display the points congruent to 1 mod 3 and 1 mod 5, respectively.

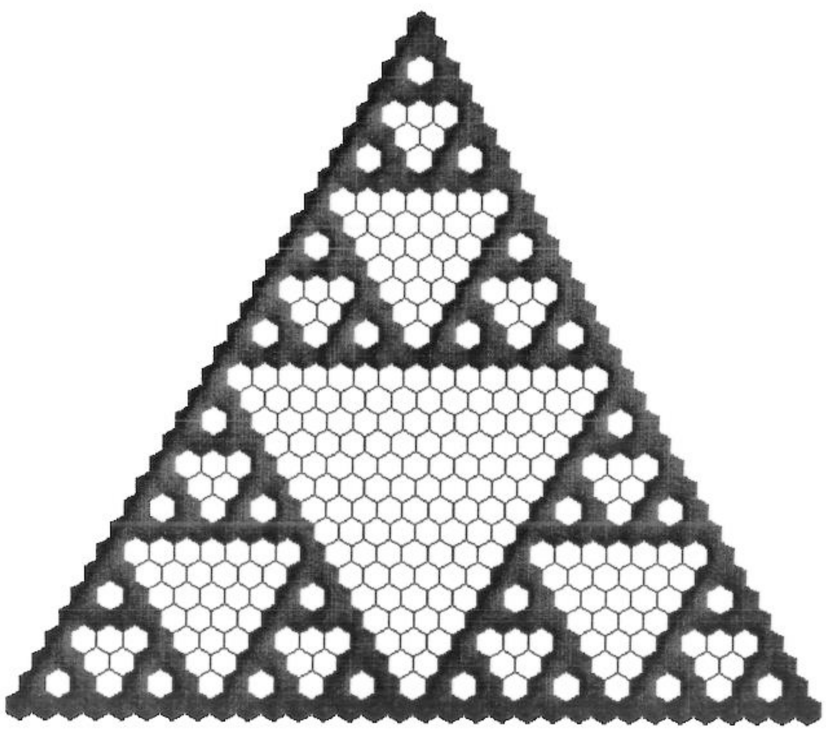


Figure 14.7: A coloring of Pascal's triangle.

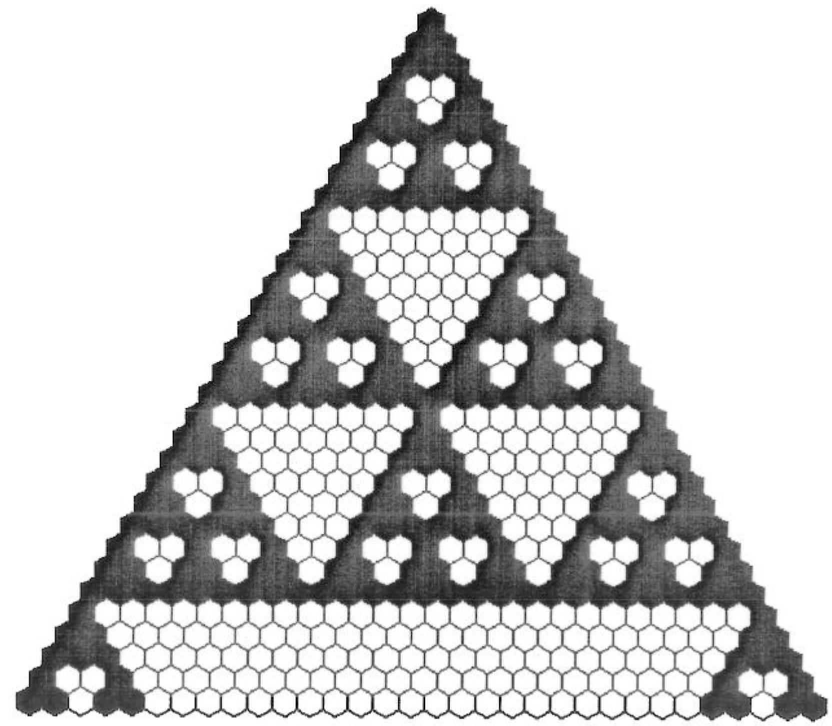


Figure 14.8: Another coloring of Pascal's triangle.

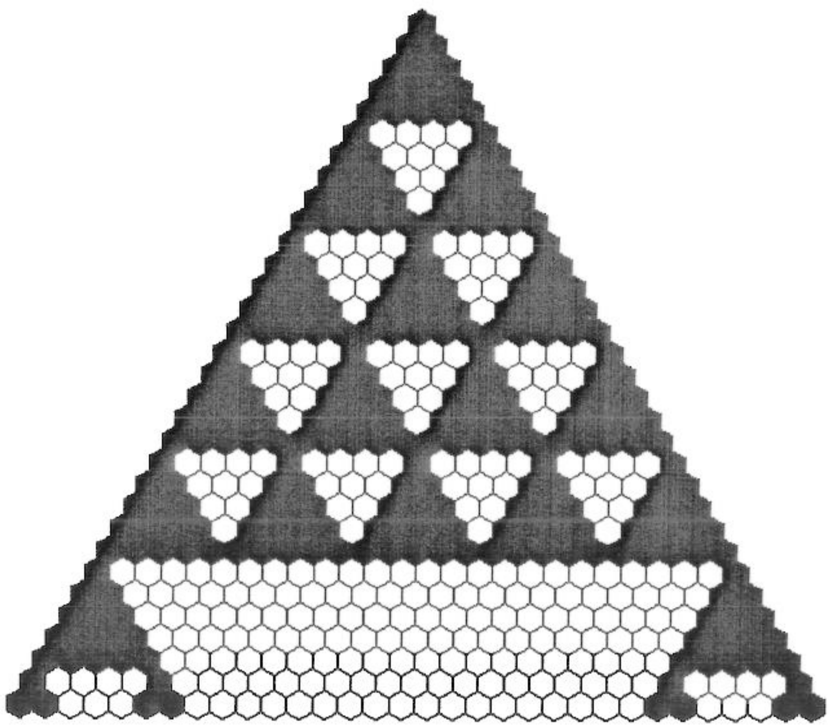


Figure 14.9: Yet another coloring of Pascal's triangle.

Chapter 15

Complex Functions

Exercises

1. Compute the following:

1a) $(3+7i)^2 = 9 + 49i^2 + 42i = -40 + 42i.$

1b) $(4-2i)^3 = (4-2i)^2(4-2i) = (12-16i)(4-2i) = 16 - 88i.$

1c) $(7+2i)(5-3i)(4i) = (41-11i)(4i) = 44 + 164i.$

1d) $\dfrac{3+2i}{6-5i} = \dfrac{3+2i}{6-5i}\cdot\dfrac{6+5i}{6+5i} = \dfrac{18+10i^2+15i+12i}{36+25} = \dfrac{8+27i}{61}.$

1e) $\dfrac{1}{(3+2i)^2} = \left(\dfrac{1}{3+2i}\cdot\dfrac{3-2i}{3-2i}\right)^2 = \left(\dfrac{3-2i}{9+4}\right)^2 = \dfrac{(9-4)-12i}{13^2} = \dfrac{5-12i}{169}.$

or $\dfrac{1}{(3+2i)^2} = \dfrac{1}{(9-4)+12i} = \dfrac{1}{5+12i}\cdot\dfrac{5-12i}{5-12i} = \dfrac{5-12i}{25+144} = \dfrac{5-12i}{169}.$

2. Find the polar representation of each of the following complex numbers:

2a) $-7i$

By inspection, $r = 7$ and $\theta = 3\pi/2$.

Therefore, $-7i = 7(\cos(3\pi/2) + i\sin(3\pi/2)) = 7(0 - i) = -7i.$

2b) -6

Again by inspection, $r = 6$ and $\theta = \pi$.

Thus, $-6 = 6(\cos\pi + i\sin\pi) = 6(-1 + i\cdot 0) = -6.$

2c) $2 + 2i$

In this case, $r = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$ and by inspection, $\theta = \pi/4$.

So, $2+2i = 2\sqrt{2}(\cos(\pi/4)+i\sin(\pi/4)) = 2\sqrt{2}(\sqrt{2}/2+i\sqrt{2}/2) = 2+2i.$

2d) $-2 + 2i$

Similarly, $r = \sqrt{(-2)^2+2^2} = 2\sqrt{2}$, but this time $\theta = 3\pi/4$.

So, $-2+2i = 2\sqrt{2}(\cos(3\pi/4)+i\sin(3\pi/4)) = 2\sqrt{2}(-\sqrt{2}/2+i\sqrt{2}/2) = -2+2i.$

2e) $1 + \sqrt{3}\,i$

Here, $r = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$ and $\theta = \pi/3$.

Thus, $1+\sqrt{3}\,i = 2(\cos(\pi/3)+i\sin(\pi/3)) = 2(1/2+i\sqrt{3}/2) = 1+\sqrt{3}\,i.$

2f) $-1 + \sqrt{3}\,i$

Likewise, $r = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$, but $\theta = 2\pi/3$.

Therefore, $-1+\sqrt{3}\,i = 2(\cos(2\pi/3)+i\sin(2\pi/3)) = 2(-1/2+i\sqrt{3}/2) = -1+\sqrt{3}\,i.$

3. Find the complex square roots of each of the complex numbers in the previous exercise.

3a) $\sqrt{-7i} = \pm\sqrt{7}(\cos(3\pi/4) + i\sin(3\pi/4))$
$= \pm\sqrt{7}(-\sqrt{2}/2 + i\sqrt{2}/2)$
$= \pm\left(\dfrac{-\sqrt{14}+i\sqrt{14}}{2}\right).$

Check: $\left(\frac{-\sqrt{14}+i\sqrt{14}}{2}\right)^2 = \frac{(14-14)-2\sqrt{14}\sqrt{14}\,i}{4} = \frac{-28i}{4} = -7i.$ ✓

3b) $\sqrt{-6} = \pm\sqrt{6}(\cos(\pi/2) + i\sin(\pi/2)) = \pm\sqrt{6}(0 + i\cdot 1) = \pm\sqrt{6}\,i$ which is obvious when you stop to think about it!

3c) $\sqrt{2+2i} = \pm\sqrt{2\sqrt{2}}\,(\cos(\pi/8) + i\sin(\pi/8))$

$$= \pm\sqrt{2\sqrt{2}}\left(\sqrt{\tfrac{2+\sqrt{2}}{4}} + i\sqrt{\tfrac{2-\sqrt{2}}{4}}\right)$$
$$= \pm\left(\sqrt{\tfrac{2\sqrt{2}(2+\sqrt{2})}{4}} + i\sqrt{\tfrac{2\sqrt{2}(2-\sqrt{2})}{4}}\right)$$
$$= \pm\left(\sqrt{\tfrac{4\sqrt{2}+4}{4}} + i\sqrt{\tfrac{\sqrt{4\sqrt{2}-4}}{4}}\right)$$
$$= \pm\left(\sqrt{\sqrt{2}+1} + i\sqrt{\sqrt{2}-1}\right).$$

Check: $\left(\sqrt{\sqrt{2}+1} + i\sqrt{\sqrt{2}-1}\right)^2$
$$= (\sqrt{2}+1) - (\sqrt{2}-1) + 2i\sqrt{\sqrt{2}+1}\,\sqrt{\sqrt{2}-1}$$
$$= 2+2i. \quad ✓$$

Note: The above value for $\sin(\pi/8)$ is obtained from the half-angle formula for sine:

$$\sin\left(\frac{\pi}{8}\right) = \pm\sqrt{\frac{1-\cos(\pi/4)}{2}} = \pm\sqrt{\frac{1-\sqrt{2}/2}{2}} = \pm\sqrt{\frac{2-\sqrt{2}}{4}}$$

and so

$$\cos^2\left(\frac{\pi}{8}\right) = 1 - \sin^2(\pi/8) = 1 - \frac{2-\sqrt{2}}{4} = \frac{2+\sqrt{2}}{4}.$$

3d) $\sqrt{-2+2i} = \pm\sqrt{2\sqrt{2}}\,(\cos(3\pi/8) + i\sin(3\pi/8))$

$$= \pm\sqrt{2\sqrt{2}}\left(\sqrt{\tfrac{2-\sqrt{2}}{4}} + i\sqrt{\tfrac{2+\sqrt{2}}{4}}\right)$$
$$= \pm\left(\sqrt{\sqrt{2}-1} + i\sqrt{\sqrt{2}+1}\right).$$

Check: $\left(\sqrt{\sqrt{2}-1} + i\sqrt{\sqrt{2}+1}\right)^2$
$$= (\sqrt{2}-1) - (\sqrt{2}+1) + 2i\sqrt{\sqrt{2}-1}\,\sqrt{\sqrt{2}+1}$$
$$= -2+2i. \quad ✓$$

3e) $\sqrt{1+\sqrt{3}\,i} = \pm\sqrt{2}\,(\cos(\pi/6) + i\sin(\pi/6))$

$$= \pm\sqrt{2}\left(\tfrac{\sqrt{3}}{2} + i\tfrac{1}{2}\right) = \pm\left(\tfrac{\sqrt{6}}{2} + i\tfrac{\sqrt{2}}{2}\right).$$

Check: $\left(\tfrac{\sqrt{6}}{2} + i\tfrac{\sqrt{2}}{2}\right)^2 = \tfrac{6}{4} - \tfrac{2}{4} + 2i\tfrac{\sqrt{6}}{2}\tfrac{\sqrt{2}}{2} = 1 + \sqrt{3}\,i.$ ✓

3f) $\sqrt{-1+\sqrt{3}\,i} = \pm\sqrt{2}\,(\cos(2\pi/6) + i\sin(2\pi/6))$

$$= \pm\sqrt{2}\left(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right) = \pm\left(\tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{6}}{2}\right).$$

Check: $\left(\tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{6}}{2}\right)^2 = \tfrac{2}{4} - \tfrac{6}{4} + 2i\tfrac{\sqrt{2}}{2}\tfrac{\sqrt{6}}{2} = -1 + \sqrt{3}\,i.$ ✓

4. What is the formula for the quotient of two complex numbers given in polar representation?

Let $z_1 = r_1 e^{\theta_1 i}$ and $z_2 = r_2 e^{\theta_2 i}$. Then

$$\frac{z_1}{z_2} = \frac{r_1 e^{\theta_1 i}}{r_2 e^{\theta_2 i}} = \frac{r_1}{r_2} e^{(\theta_1-\theta_2)i}$$

assuming $r_2 \neq 0$ (which is the same as saying that $z_2 \neq 0$). In other words, the ratio of two complex numbers is a third complex number whose modulus is the ratio of the two moduli and whose argument is the difference of the two arguments.

5. Let $L_\alpha(z) = \alpha z$. Sketch the orbit of 1 in the plane for each of the following values of α:

5a) $\alpha = i/2$

Facts: $\operatorname{re}\alpha = 0$; $\operatorname{im}\alpha = 1/2$; $r = \sqrt{0^2 + (1/2)^2} = 1/2 < 1$; $\theta = \pi/2$.

$$L_\alpha^1(1) = L_\alpha(1) = \frac{i}{2}(1) = \frac{i}{2}$$
$$L_\alpha^2(1) = L_\alpha\left(\frac{i}{2}\right) = \frac{i}{2}\left(\frac{i}{2}\right) = -\frac{1}{4}$$
$$L_\alpha^3(1) = L_\alpha\left(-\frac{1}{4}\right) = \frac{i}{2}\left(-\frac{1}{4}\right) = -\frac{i}{8}$$
$$L_\alpha^4(1) = L_\alpha\left(-\frac{i}{8}\right) = \frac{i}{2}\left(-\frac{i}{8}\right) = \frac{1}{16}$$
$$\vdots$$
$$L_\alpha^n(1) = \left(\frac{i}{2}\right)^n$$

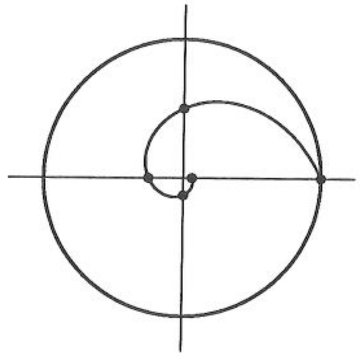


Figure 15.1: The orbit of 1 under iteration of $L(z) = iz/2$.

As seen in Figure 15.1,

$$|L_\alpha^n(1)| = \left|\left(\frac{i}{2}\right)^n\right| = \left|\frac{i}{2}\right|^n = (1/2)^n \to 0$$

as $n \to \infty$.

5b) $\alpha = 2i$

Facts: $\operatorname{re}\alpha = 0$; $\operatorname{im}\alpha = 2$; $r = \sqrt{0^2 + 2^2} = 2 > 1$; $\theta = \pi/2$.

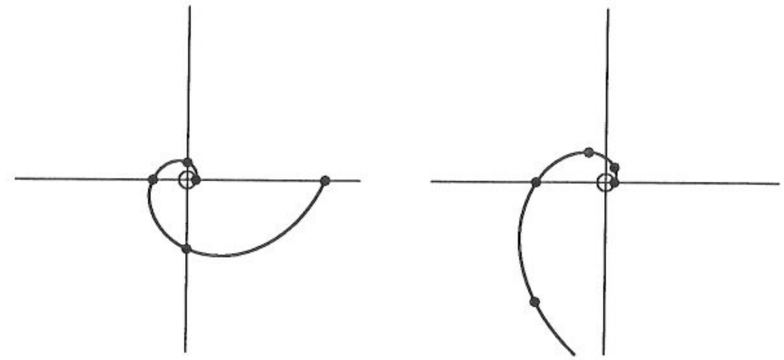
$$\begin{aligned}
L_\alpha^1(1) &= L_\alpha(1) = 2i(1) = 2i \\
L_\alpha^2(1) &= L_\alpha(2i) = 2i(2i) = -4 \\
L_\alpha^3(1) &= L_\alpha(-4) = 2i(-4) = -8i \\
L_\alpha^4(1) &= L_\alpha(-8i) = 2i(-8i) = 16 \\
&\vdots \\
L_\alpha^n(1) &= (2i)^n
\end{aligned}$$

As indicated in Figure 15.2a,

$$|L_\alpha^n(1)| = |2i|^n = (2)^n \to \infty$$

as $n \to \infty$.

5c) $\alpha = 1 + \sqrt{3}\,i$



(a) $L(z) = 2iz$. (b) $L(z) = (1 + \sqrt{3}\,i)z$.

Figure 15.2: The orbit of 1 under iteration of complex linear maps with a repelling fixed point at the origin.

Relevant facts: $\operatorname{re}\alpha = 1$; $\operatorname{im}\alpha = \sqrt{3}$; $r = \sqrt{1^2 + (\sqrt{3})^2} = 2 > 1$; $\theta = \pi/3$.

$$\begin{aligned}
L_\alpha^1(1) &= L_\alpha(1) = (1 + \sqrt{3}\,i)(1) = 1 + \sqrt{3}\,i \\
L_\alpha^2(1) &= L_\alpha(1 + \sqrt{3}\,i) = (1 + \sqrt{3}\,i)(1 + \sqrt{3}\,i) = -2 + 2\sqrt{3}\,i \\
L_\alpha^3(1) &= L_\alpha(-2 + 2\sqrt{3}\,i) = (1 + \sqrt{3}\,i)(-2 + 2\sqrt{3}\,i) = -12 \\
L_\alpha^4(1) &= L_\alpha(-12) = (1 + \sqrt{3}\,i)(-12) = -12 - 12\sqrt{3}\,i \\
&\vdots \\
L_\alpha^n(1) &= (1 + \sqrt{3}\,i)^n
\end{aligned}$$

As in the previous exercise,

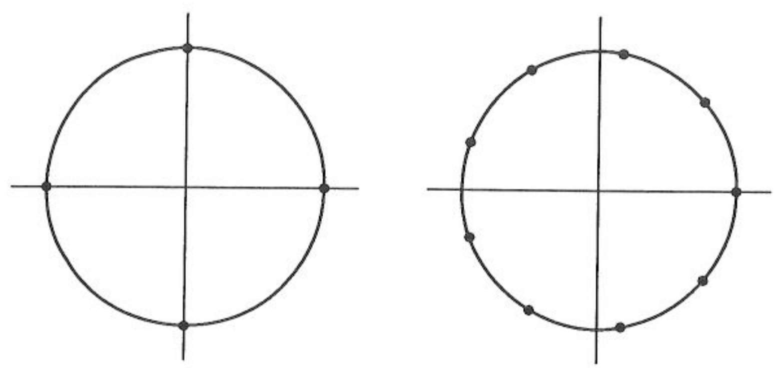
$$|L_\alpha^n(1)| = |1 + \sqrt{3}\,i|^n = 2^n \to \infty$$

as $n \to \infty$. See Figure 15.2b.

5d) $\alpha = i$

Facts: $\operatorname{re}\alpha = 0$; $\operatorname{im}\alpha = 1$; $r = \sqrt{0^2 + 1^2} = 1$; $\theta = \pi/2$.

$$L_\alpha^1(1) = L_\alpha(1) = i(1) = i$$



(a) $L(z) = iz$. (b) $L(z) = e^{2\pi i/9}z$.

Figure 15.3: The orbit of 1 under iteration of complex linear maps with a neutral fixed point at the origin.

$$\begin{aligned} L_\alpha^2(1) &= L_\alpha(i) = i(i) = -1 \\ L_\alpha^3(1) &= L_\alpha(-1) = i(-1) = -i \\ L_\alpha^4(1) &= L_\alpha(-i) = i(-i) = 1 \\ &\vdots \\ L_\alpha^n(1) &= (i)^n \end{aligned}$$

In this case, as indicated in Figure 15.3a, $|L_\alpha^n(1)| = |i|^n = 1$ for all n.

5e) $\alpha = e^{2\pi i/9}$

Observe that $r = 1$ and $\theta = 2\pi/9$ (that is, $\tau = 1/9$).

$$\begin{aligned} L_\alpha^1(1) &= L_\alpha(1) = e^{2\pi i/9}(1) = e^{2\pi i/9} \\ L_\alpha^2(1) &= L_\alpha(e^{2\pi i/9}) = e^{2\pi i/9}(e^{2\pi i/9}) = e^{4\pi i/9} \\ L_\alpha^3(1) &= L_\alpha(e^{4\pi i/9}) = e^{2\pi i/9}(e^{4\pi i/9}) = e^{6\pi i/9} \\ L_\alpha^4(1) &= L_\alpha(e^{6\pi i/9}) = e^{2\pi i/9}(e^{6\pi i/9}) = e^{8\pi i/9} \\ &\vdots \\ L_\alpha^9(1) &= L_\alpha(e^{16\pi i/9}) = e^{2\pi i/9}(e^{16\pi i/9}) = e^{18\pi i/9} = e^{2\pi i} = 1 \end{aligned}$$

See Figure 15.3b.

5f) $\alpha = e^{\sqrt{2}\,\pi i}$

Observe that $r = 1$ and $\theta = \sqrt{2}\,\pi$ (that is, $\tau = \sqrt{2}/2$).

$$\begin{aligned} L_\alpha^1(1) &= L_\alpha(1) = e^{\sqrt{2}\,\pi i}(1) = e^{\sqrt{2}\,\pi i} \\ L_\alpha^2(1) &= L_\alpha(e^{\sqrt{2}\,\pi i}) = e^{\sqrt{2}\,\pi i}(e^{\sqrt{2}\,\pi i}) = e^{2\sqrt{2}\,\pi i} \\ L_\alpha^3(1) &= L_\alpha(e^{2\sqrt{2}\,\pi i}) = e^{\sqrt{2}\,\pi i}(e^{2\sqrt{2}\,\pi i}) = e^{3\sqrt{2}\,\pi i} \\ L_\alpha^4(1) &= L_\alpha(e^{3\sqrt{2}\,\pi i}) = e^{\sqrt{2}\,\pi i}(e^{3\sqrt{2}\,\pi i}) = e^{4\sqrt{2}\,\pi i} \\ &\vdots \\ L_\alpha^n(1) &= e^{n\sqrt{2}\,\pi i} \end{aligned}$$

This irrational rotation of the circle is tough to sketch. Each point on the orbit of 1 is rotated $\sqrt{2}\,\pi$ radians around the circle ($\sqrt{2}/2$ of a revolution) and the orbit is dense.

6. Prove that the complex function $F(z) = \alpha z + \beta$ where α and β are complex is conjugate to a linear function of the form $L_\gamma(z) = \gamma z$. Determine γ in terms of α and β. What happens when $\alpha = 1$?

We hope to find a homeomorphism H of the form

$$H(z) = z + b$$

such that

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\;F\;} & \mathbb{C} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ \mathbb{C} & \xrightarrow{\;L\;} & \mathbb{C} \end{array}$$

commutes. That is, we will find an H such that $H \circ F = L \circ H$. But $H \circ F(x) = (\alpha z + \beta) + b$ and $L \circ H(z) = \gamma(z + b)$, and so $\alpha z + \beta + b = \gamma z + \gamma b$. Equating coefficients, we have that $\alpha = \gamma$ and $\beta + b = \gamma b$. The latter reduces to $b = \beta/(\alpha - 1)$ where of course $\alpha \neq 1$ (when $\alpha = 1$, the dynamics of F are atypical in the sense that $z \mapsto z + \beta$ is not structurally stable). We conclude that F and L have the same slope (see Figure 15.4).

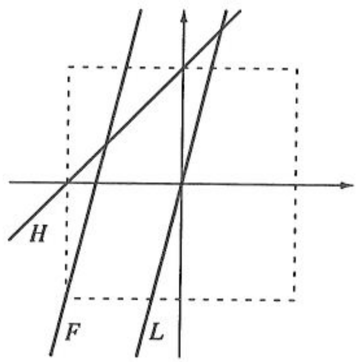


Figure 15.4: An affine map F is conjugate to a linear map L via another affine map H.

The following fact is useful when proving a function is *not* complex differentiable (and will be used in the sequel): If $\lim_{t\to 0}\gamma(t) = z_0$, then

$$\lim_{z\to z_0}\frac{F(z)-F(z_0)}{z-z_0} = \lim_{t\to 0}\frac{F(\gamma(t))-F(z_0)}{\gamma(t)-z_0}. \quad (15.1)$$

For example, let $\gamma(t) = z_0 + t$ so that $\gamma^{-1}(t) = t - z_0$. Now make the substitution $z := \gamma(t)$ in the definition of $F'(z)$. As $z \to z_0$, we see that $t \to \gamma^{-1}(z_0) = 0$, and so (15.1) holds. Similarly, if we let $\eta(t) = z_0 + it$, then $\eta^{-1}(t) = (t - z_0)/i$, and the same thing happens if we make the substitution $z := \eta(t)$. In general, let $\nu(t) = z_0 + ct$ where c is any complex number. Then $\nu^{-1}(t) = (t - z_0)/c$ and substituting $z := \nu(t)$ in $F'(z)$ we have

$$\lim_{t\to 0}\frac{F(\nu(t))-F(z_0)}{\nu(t)-z_0}$$

which is a particular branch of $F'(z)$.

7. For which of the following complex functions does the complex derivative exist?

7a) $F(x+iy) = (x+iy)^3$

This function is actually $z \mapsto z^3$ in disguise. As expected, it has a

complex derivative, and in fact, $F'(z) = 3z^2$ since

$$\begin{aligned}\lim_{z\to z_0}\frac{F(z)-F(z_0)}{z-z_0} &= \lim_{z\to z_0}\frac{z^3-z_0^3}{z-z_0}\\ &= \lim_{z\to z_0}\frac{(z-z_0)(z^2+z_0z+z_0^2)}{z-z_0}\\ &= \lim_{z\to z_0}(z^2+z_0z+z_0^2)\\ &= z_0^2+z_0z_0+z_0^2\\ &= 3z_0^2.\end{aligned}$$

7b) $F(x+iy) = x^2+iy^2$

This function is not complex differentiable. Let $\gamma(t) = (x_0+t)+iy_0$ and $\eta(t) = x_0 + i(y_0+t)$. Then

$$\begin{aligned}\lim_{t\to 0}\frac{F(\gamma(t))-F(z_0)}{\gamma(t)-z_0} &= \lim_{t\to 0}\frac{(x_0+t)^2+iy_0^2-(x_0^2+iy_0^2)}{(x_0+t)+iy_0-(x_0+iy_0)}\\ &= \lim_{t\to 0}\frac{x_0^2+2tx_0+t^2+iy_0^2-x_0^2-iy_0^2}{x_0+t+iy_0-x_0-iy_0}\\ &= \lim_{t\to 0}\frac{t(2x_0+t)}{t}\\ &= 2x_0\end{aligned}$$

but

$$\begin{aligned}\lim_{t\to 0}\frac{F(\eta(t))-F(z_0)}{\eta(t)-z_0} &= \lim_{t\to 0}\frac{x_0^2+i(y_0+t)^2-(x_0^2+iy_0^2)}{x_0+i(y_0+t)-(x_0+iy_0)}\\ &= \lim_{t\to 0}\frac{x_0^2+i(y_0^2+2ty_0+t^2)-x_0^2-iy_0^2}{x_0+iy_0+it-x_0-iy_0}\\ &= \lim_{t\to 0}\frac{it(2y_0+t)}{it}\\ &= 2y_0.\end{aligned}$$

7d) $F(z) = \bar{z}^2 + c$

Recall that the function $z \mapsto \bar{z}$ does *not* have a complex derivative (see p. 213 in the text). We'll employ a similar approach here.

Let $\gamma(t) = z_0 + t$ and $\eta(t) = z_0 + it$. Note that

$$\lim_{t\to 0}\gamma(t) = \lim_{t\to 0}\eta(t) = z_0$$

and so it makes sense to write

$$\begin{aligned}\lim_{t\to 0}\frac{F(\gamma(t)) - F(z_0)}{\gamma(t) - z_0} &= \lim_{t\to 0}\frac{(\overline{z_0+t})^2 + c - (\bar{z}_0^2 + c)}{z_0 + t - z_0}\\ &= \lim_{t\to 0}\frac{(\bar{z}_0^2 + 2t\bar{z}_0 + t^2) + c - \bar{z}_0^2 - c}{t}\\ &= \lim_{t\to 0}\frac{t(2\bar{z}_0 + t)}{t}\\ &= 2\bar{z}_0\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to 0}\frac{F(\eta(t)) - F(z_0)}{\eta(t) - z_0} &= \lim_{t\to 0}\frac{(\overline{z_0+it})^2 + c - (\bar{z}_0^2 + c)}{z_0 + it - z_0}\\ &= \lim_{t\to 0}\frac{(\bar{z}_0^2 - 2it\bar{z}_0 - t^2) + c - \bar{z}_0^2 - c}{it}\\ &= \lim_{t\to 0}\frac{-t(2i\bar{z}_0 + t)}{it}\\ &= -2\bar{z}_0\end{aligned}$$

which are negatives of one another. So F does not have a complex derivative, and we begin to suspect any complex function with a conjugate in its definition.

7e) $F(x+iy) = ix - y$

Since $ix - y = i(x+iy)$, this function is actually $z \mapsto iz$ in disguise, and so we guess that $F'(z) = i$, a constant. Sure enough,

$$\begin{aligned}\lim_{z\to z_0}\frac{F(z) - F(z_0)}{z - z_0} &= \lim_{z\to z_0}\frac{iz - iz_0}{z - z_0}\\ &= \lim_{z\to z_0}\frac{i(z - z_0)}{z - z_0}\\ &= i.\end{aligned}$$

7f) $F(z) = 2z(i - z)$

Observe that $2z(i-z) = 2iz - 2z^2$, and so we shouldn't be surprised if $F'(z) = 2i - 4z$. Indeed,

$$\begin{aligned}\lim_{z\to z_0}\frac{F(z) - F(z_0)}{z - z_0} &= \lim_{z\to z_0}\frac{2z(i-z) - 2z_0(i - z_0)}{z - z_0}\\ &= \lim_{z\to z_0}\frac{2zi - 2z_0 i - 2z^2 + 2z_0^2}{z - z_0}\\ &= \lim_{z\to z_0}\frac{2i(z - z_0) - 2(z^2 - z_0^2)}{z - z_0}\\ &= \lim_{z\to z_0}\frac{2i(z - z_0) - 2(z + z_0)(z - z_0)}{z - z_0}\\ &= \lim_{z\to z_0}(2i - 2(z + z_0))\\ &= 2i - 4z_0.\end{aligned}$$

7g) $F(z) = z^3 + (i+1)z$

The answer is just what you'd expect by now:

$$\begin{aligned}\lim_{z\to z_0}\frac{F(z) - F(z_0)}{z - z_0} &= \lim_{z\to z_0}\frac{z^3 + (i+1)z - (z_0^3 + (i+1)z_0)}{z - z_0}\\ &= \lim_{z\to z_0}\frac{z^3 - z_0^3 + (i+1)(z - z_0)}{z - z_0}\\ &= \lim_{z\to z_0}\frac{(z^2 + z_0 z + z_0^2)(z - z_0) + (i+1)(z - z_0)}{z - z_0}\\ &= \lim_{z\to z_0}(z^2 + z_0 z + z_0^2 + (i+1))\\ &= 3z_0^2 + (i+1).\end{aligned}$$

8. Find all fixed points for each of the following complex functions and determine whether they are attracting, repelling, or neutral.

8a) $Q_2(z) = z^2 + 2$

$$\begin{aligned}&z^2 + 2 = z\\ \Rightarrow\ &z^2 - z + 2 = 0\\ \Rightarrow\ &z = \tfrac{1\pm\sqrt{1-4(1)(2)}}{2} = \tfrac{1\pm\sqrt{-7}}{2}.\end{aligned}$$

Check: $$Q_2\left(\tfrac{1\pm\sqrt{-7}}{2}\right) = \left(\tfrac{1\pm\sqrt{-7}}{2}\right)^2 + 2 = \tfrac{1-7\pm2\sqrt{-7}}{4} + \tfrac{8}{4} = \tfrac{2\pm2\sqrt{-7}}{4} = \tfrac{1\pm\sqrt{-7}}{2}. \checkmark$$

Since $Q_2'(z) = 2z$,

$$\left|Q_2'\left(\frac{1\pm\sqrt{-7}}{2}\right)\right| = |1\pm\sqrt{-7}| = \sqrt{8} = 2\sqrt{2}$$

which is greater than one, and so the fixed point is repelling.

8b) $F(z) = z^2 + z + 1$

$z^2 + z + 1 = z \Rightarrow z^2 + 1 = 0 \Rightarrow z = \pm i.$

Check: $F(\pm i) = (\pm i)^2 \pm i + 1 = -1 \pm i + 1 = \pm i.$ ✓

$F'(z) = 2z + 1 \Rightarrow |F'(\pm i)| = |1 \pm 2i| = \sqrt{5} > 1$, and so both fixed points are repelling.

8c) $F(z) = iz^2$

$iz^2 = z \Rightarrow iz^2 - z = 0 \Rightarrow (iz-1)z = 0 \Rightarrow z = 0$ or $z = 1/i = -i.$

Check: $F(-i) = i(-i)^2 = i(-1) = -i$, and 0 is obviously fixed. ✓

Since $F'(z) = 2iz$, we have $F'(-i) = 2i(-i) = -2i^2 = 2$. In other words, this fixed point is repelling. The origin, on the other hand, is superattracting since $F'(0) = 0$.

8d) $F(z) = -1/z$

$-1/z = z \Rightarrow -1 = z^2 \Rightarrow z = \pm i.$

Check: $F(\pm i) = -1/\pm i = \pm i.$ ✓

$F'(z) = 1/z^2 \Rightarrow F'(\pm i) = 1/(\pm i)^2 = -1$ which implies that both of these fixed points are neutral.

8e) $F(z) = 2z(i - z)$

$2z(i-z) = z \Rightarrow 2z(i-z) - z = 0 \Rightarrow (2(i-z)-1)z = 0 \Rightarrow z = 0$ or $z = i - 1/2$.

Check: $F(i-1/2) = 2(i-1/2)(i-(i-1/2)) = (2i-1)(1/2) = i - 1/2$, and again 0 is obviously fixed. ✓

Since $F(z) = 2zi - 2z^2$, we have $F'(z) = 2i - 4z$, and so $|F'(0)| = |2i| = 2$ and $|F'(i-1/2)| = |2i - 4(i-1/2)| = |2-2i| = 2\sqrt{2} > 1$. Hence, both fixed points are repelling.

8f) $F(z) = -iz(1-z)/2$

$-iz(1-z)/2 = z \Rightarrow -iz(1-z)/2 - z = 0 \Rightarrow -z(i(1-z)/2 + 1) = 0 \Rightarrow z = 0$ or $z = 1 + 2/i$.

Check: $F(1+2/i) = -i(1+2/i)(1-(1+2/i))/2 = -(i+2)(-2/i)/2 = 1 + 2/i$, and the origin is obviously fixed. ✓

Since $-iz(1-z)/2 = (iz^2 - iz)/2$, we have that $F'(z) = (2iz - i)/2$. Hence, $|F'(0)| = |-i/2| = 1/2 < 1$ and $|F'(1+2/i) = (2i(1+2/i) - i)/2| = |(2i+4-i)/2| = |(i+4)/2| = \sqrt{17}/2 > 1$. Thus, 0 is attracting and the other fixed point is repelling.

8g) $F(z) = z^3 + (i+1)z$

$z^3 + (i+1)z = z \Rightarrow z^3 + (i+1)z - z = 0 \Rightarrow z(z^2 + (i+1) - 1) = 0 \Rightarrow z = 0$ or $z^2 + i = 0$, that is, $z = \pm\sqrt{-i}$.

Check: $F(\pm\sqrt{-i}) = (\pm\sqrt{-i})^3 + (i+1)(\pm\sqrt{-i}) = \mp i\sqrt{-i} \pm i\sqrt{-i} \pm \sqrt{-i} = \pm\sqrt{-i}$. The origin is obviously fixed. ✓

$F'(z) = 3z^2 + i + 1$ implies that $|F'(0)| = |i+1| = \sqrt{2} > 1$, and $F'(\pm\sqrt{-i}) = |3(\pm\sqrt{-i})^2 + i + 1| = |3(-i) + i + 1| = |1 - 2i| = \sqrt{5} > 1$. Again, both fixed points are repelling.

9. Show that $z_0 = -1 + i$ lies on a cycle of period 2 for $Q_i(z) = z^2 + i$. Is this cycle attracting, repelling, or neutral?

Since $Q_i(i-1) = (i-1)^2 + i = i^2 + 1 - 2i + i = -i$ and $Q_i(-i) = (-i)^2 + i = i - 1$, these points constitute a 2-cycle for Q_i. Note that $Q_i'(z) = 2z$ and

$$(Q_i^2)'(i-1) = Q_i'(i-1)\cdot Q_i'(-i) = 2(i-1)\cdot 2(-i) = 4(1+i).$$

Now, since $|4(1+i)| = 4|1+i| = 4\sqrt{2} > 1$, this 2-cycle is repelling.

10. Show that $z_0 = e^{2\pi i/3}$ lies on a cycle of period 2 for $Q_0(z) = z^2$. Is this cycle attracting, repelling, or neutral?

It's true that $e^{2\pi i/3}$ lies on a 2-cycle since $Q_0(e^{2\pi i/3}) = (e^{2\pi i/3})^2 = e^{4\pi i/3}$ while $Q_0(e^{4\pi i/3}) = (e^{4\pi i/3})^2 = e^{8\pi i/3} = e^{2\pi i}\cdot e^{2\pi i/3} = e^{2\pi i/3}$. Observe that

$Q_0'(z) = 2z$. The 2-cycle is repelling since

$$\begin{aligned}(Q_0^2)'(e^{2\pi i/3}) &= Q_0'(e^{2\pi i/3}) \cdot Q_0'(e^{4\pi i/3}) \\ &= 2e^{2\pi i/3} \cdot 2e^{4\pi i/3} \\ &= 4e^{6\pi i/3}\end{aligned}$$

and $|4e^{6\pi i/3}| = 4 > 1$.

11. Show that $z_0 = e^{2\pi i/7}$ lies on a cycle of period 3 for $Q_0(z) = z^2$. Is this cycle attracting, repelling, or neutral?

Straightforward computation yields $Q_0(e^{2\pi i/7}) = (e^{2\pi i/7})^2 = e^{4\pi i/7}$ and $Q_0(e^{4\pi i/7}) = (e^{4\pi i/7})^2 = e^{8\pi i/7}$ and $Q_0(e^{8\pi i/7}) = (e^{8\pi i/7})^2 = e^{16\pi i/7} = e^{2\pi i} \cdot e^{2\pi i/7} = e^{2\pi i/7}$. Since $Q_0'(z) = 2z$,

$$\begin{aligned}(Q_0^3)'(e^{2\pi i/7}) &= Q_0'(e^{2\pi i/7}) \cdot Q_0'(e^{4\pi i/7}) \cdot Q_0'(e^{8\pi i/7}) \\ &= 2e^{2\pi i/7} \cdot 2e^{4\pi i/7} \cdot 2e^{8\pi i/7} \\ &= 8e^{14\pi i/7}\end{aligned}$$

and $|8e^{14\pi i/3}| = 8 > 1$.

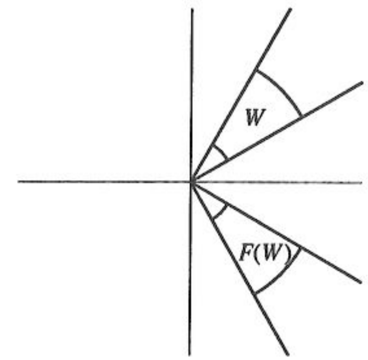
It's not surprising that the periodic orbits in Exercises 10 and 11 are repelling since the squaring function is known to exhibit chaotic behavior on the unit circle and so *all* orbits must be repelling. And it appears that the rate of divergence is greater for cycles with larger periods—indeed, the previous two exercises suggest that the derivative along a cycle of period n is 2^n. Can you prove this?

13. Does the Boundary Mapping Principle hold for $F(z) = \bar{z}$? Why or why not?

Yes, it does, despite the fact that F does not have a complex derivative. Consider the chunk $W = \{\, re^{i\theta} \mid r_1 < r < r_2 \text{ and } \theta_1 < \theta < \theta_2 \,\}$. The image of W with respect to F is $F(W) = \{\, re^{i\theta} \mid r_1 < r < r_2 \text{ and } -\theta_2 < \theta < -\theta_1 \,\}$, which is also a chunk (see Figure 15.5a). Hence, F satisfies the Boundary Mapping Principle by the argument given on the bottom of p. 217 of the text.

14. Does the Boundary Mapping Principle hold for $F(x + iy) = x^2 + iy^2$? Why or why not?

Note that F maps the plane onto the first quadrant $\{\, x + iy \mid x, y \geq 0 \,\}$. Now there are points in the plane (which itself has no boundary) that get mapped onto the positive x- or y-axis. In fact, any point on the x-axis (that is, $y = 0$) gets mapped onto the positive x-axis which is part of the boundary of $F(\mathbb{C})$, and similarly for the y-axis. Hence, F does not satisfy the Boundary Mapping Principle.



(a) The Boundary Mapping Principle holds for $z \mapsto \bar{z}$.

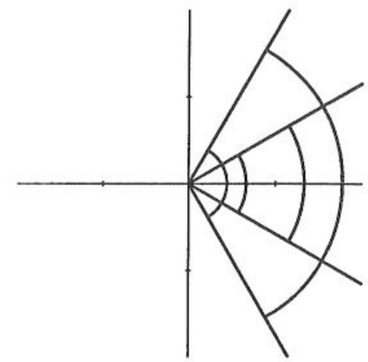

(b) This particular application of the squaring function does not contradict the Boundary Mapping Principle.

Figure 15.5: Two illustrations of the Boundary Mapping Principle.

15. Give an example of a region R in the plane that has the property that, under $Q_0(z) = z^2$, there is a boundary point of R that is mapped into the interior of $Q_0(R)$. Does this contradict the Boundary Mapping Principle? Why or why not?

Take R to be any chunk which contains the fixed point 1. Then $R \subset Q_0(R)$, and *every* point on the boundary of R is contained in the interior of $Q_0(R)$ (see Figure 15.5b).

This does *not* contradict the Boundary Mapping Principle which says that interior points in W get mapped to interior points in $Q_0(W)$, which is the same as saying that boundary points in $Q_0(W)$ have pre-images in W which are also boundary points. It says nothing about the boundary points of W themselves.

Chapter 16

The Julia Set

Exercises

1. Describe the filled Julia set for $F(z) = z^3$.

Like the quadratic map Q_0, the filled Julia set for $z \mapsto z^3$ is the closed unit disk. Observe that

$$\begin{aligned} F(z) &= z^3 \\ F^2(z) &= z^9 \\ F^3(z) &= z^{27} \\ &\vdots \\ F^n(z) &= z^{3^n} \end{aligned}$$

where the latter may be proved by induction. Now recall that

$$z^k = r^k(\cos k\theta + i \sin k\theta) = r^k e^{ik\theta}$$

where $r = |z|$ and $\tan\theta = \operatorname{im} z / \operatorname{re} z$. Thus we have

$$F^n(z) = z^{3^n} = r^{3^n} e^{i3^n\theta}$$

which is analogous to the formula for $Q_0^n(z)$ given on p. 221 of the text. The rest of the analysis mirrors that of Q_0. For instance, it follows that $|z_0| = 1 \Rightarrow |F(z_0)| = 1$, that is, the unit circle is invariant, since

$$|F(z_0)| = |z_0^3| = |z_0|^3 = 1^3 = 1.$$

2. Describe the filled Julia set for $F(z) = z^d$ for $d \geq 4$.

This is a straightforward generalization of Exercise 1. In this case,

$$F^n(z) = z^{d^n} = r^{d^n} e^{id^n\theta}$$

and

$$\begin{aligned} |F^n(z)| &= \left| r^{d^n} e^{id^n\theta} \right| \\ &= |r^{d^n}| \to \begin{cases} 0 & \text{if } 0 \leq r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases} \quad \text{as } n \to \infty. \end{aligned}$$

Thus, the filled Julia set is once again the closed unit disk.

4. Use the techniques of Section 16.2 to conjugate $F(z) = z^3$ to a polynomial $P(z)$ via $H(z) = z + \frac{1}{z}$. What is $P(z)$?

In Section 16.2, it was shown that Q_0 is conjugate to Q_{-2} via H. We summarize this result below:

$$\begin{aligned} H(Q_0(z)) &= Q_c(H(z)) \\ \Rightarrow \quad z^2 + \frac{1}{z^2} &= \left(z + \frac{1}{z}\right)^2 + c \\ \Rightarrow \quad \frac{z^4+1}{z^2} &= \left(\frac{z^2+1}{z}\right)^2 + c \\ &= \frac{z^4 + 2z^2 + 1}{z^2} + c \\ &= \frac{z^4 + (2+c)z^2 + 1}{z^2}. \end{aligned}$$

Equating coefficients, we see that $2 + c = 0$, that is, $c = -2$. Thus,

$$\begin{array}{ccc} \mathbb{C} - D & \xrightarrow{Q_0} & \mathbb{C} - D \\ \big\downarrow H & & \big\downarrow H \\ \mathbb{C} - [-2,2] & \xrightarrow{Q_{-2}} & \mathbb{C} - [-2,2] \end{array}$$

commutes, where D is the closed unit disk.

For $F(z) = z^3$, we have

$$\begin{aligned} H(F(z)) &= P(H(z)) \\ \Rightarrow \quad z^3 + \frac{1}{z^3} &= P\left(z + \frac{1}{z}\right). \end{aligned}$$

Now, if we let $w = z + 1/z$, then

$$z_\pm = \frac{w \pm \sqrt{w^2-4}}{2}.$$

But $z_- = 1/z_+$ (and vice versa), and so

$$\begin{aligned} P(w) &= \left(\frac{w+\sqrt{w^2-4}}{2}\right)^3 + \left(\frac{w-\sqrt{w^2-4}}{2}\right)^3 \\ &= \frac{1}{8}(2w^3 + 6w(w^2-4)) \end{aligned}$$

after expanding and combining terms. Simplifying, we obtain

$$P(w) = w^3 - 3w$$

which is a more-or-less familiar face (see pp. 27, 50, and 81 of the text, for instance).

5. The saddle-node bifurcation.

5a) Find all complex c-values for which $Q_c(z) = z^2 + c$ has a fixed point z_0 with $Q_c'(z_0) = 1$.

There is only one such point, namely, $c = 1/4$, obtained from the pair of equations

$$\begin{cases} z^2 + c = z \\ 2z = 1 \end{cases}.$$

The corresponding fixed point is at $z = 1/2$.

5b) Consider the fixed points of Q_c in the complex plane for $c < 1/4$, $c = 1/4$, and $c > 1/4$. Determine whether these fixed points are attracting, repelling, or neutral. This is the complex saddle-node bifurcation.

The case when $c = 1/4$ has already been discussed in Exercise 5a. Reviewing pp. 52–55 in the text, the fixed points of Q_c satisfy

$$z^2 + c = z,$$

that is,

$$z = \frac{1 \pm \sqrt{1-4c}}{2}.$$

Call these fixed points p_+ and p_- as in the text. Since $Q_c'(z) = 2z$,

$$Q_c'\left(\frac{1 \pm \sqrt{1-4c}}{2}\right) = 1 \pm \sqrt{1-4c}.$$

Now when $c < 1/4$, it follows that $1 - 4c > 0$, and so $Q_c'(p_+) = 1 + \sqrt{1-4c} > 1$. Hence, p_+ is repelling for $c < 1/4$. For the other fixed point, we have

$$\begin{aligned} & \left|1 - \sqrt{1-4c}\right| < 1 \\ \Rightarrow\ & -1 < 1 - \sqrt{1-4c} < 1 \\ \Rightarrow\ & -2 < -\sqrt{1-4c} < 0 \\ \Rightarrow\ & 4 > 1 - 4c > 0 \\ \Rightarrow\ & 3 > -4c > -1 \\ \Rightarrow\ & -3/4 < c < 1/4 \end{aligned}$$

which shows that p_- is attracting in this range. For $c < -3/4$, both fixed points are repelling and give way to an attracting 2-cycle (see Exercise 6).

When $c > 1/4$, we have $1-4c < 0$, and so the fixed points are complex. But they are not attracting since

$$\begin{aligned} & \left|1 \pm \sqrt{1-4c}\right| < 1 \\ \Rightarrow\ & \left|1 \pm i\sqrt{4c-1}\right| < 1 \\ \Rightarrow\ & \sqrt{1 + 4c - 1} < 1 \\ \Rightarrow\ & 2\sqrt{c} < 1 \\ \Rightarrow\ & \sqrt{c} < 1/2 \\ \Rightarrow\ & c < 1/4 \end{aligned}$$

which is a contradiction. To prove that the fixed points of Q_c for $c > 1/4$ are repelling, simply reverse the above steps *and* the sense of the inequalities.

5c) Determine the set of all complex c-values for which Q_c has an attracting fixed point. Sketch this set of c-values in the plane. Where does this set meet the real axis?

Any such fixed point must satisfy the two equations

$$\begin{cases} z^2 + c = z \\ |2z| = 1 \end{cases}$$

simultaneously. Hence, the set of c-values for which Q_c has an attracting fixed point is given by

$$\{ c \mid c = z - z^2 \text{ and } |z| < 1/2 \}.$$

The boundary of this set satisfies $|z| = 1/2$ which is the equation of a circle centered at the origin with radius $1/2$, that is,

$$z = \frac{1}{2}e^{i\theta}.$$

Hence, the boundary of the set of c-values for which Q_c has an attracting fixed point satisfies

$$c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta}$$

which is the equation of a cardioid in the c-plane. (See Figure 16.1.) The intersection of this polar curve with the real axis is obtained by setting $\theta = 0$ and $\theta = \pi$. This gives $c = 1/4$ and $c = -3/4$, respectively, and agrees with the results of Exercise 5b.

5d) Determine the set of complex c-values at which the function $F_c(z) = z^3 + c$ has a saddle-node bifurcation similar to that in part b.

Likely candidates are those c for which F_c has a fixed point z_0 satisfying $F_c'(z_0) = 1$. This leads to the pair of equations

$$\begin{cases} z^3 + c = z \\ 3z^2 = 1 \end{cases}.$$

Since $z^2 = 1/3$ from the second equation, we have $z = \pm\sqrt{3}/3$ and so $z^3 = z \cdot z^2 = \pm\sqrt{3}/9$. Therefore,

$$\begin{aligned} c = z - z^3 &= \pm\sqrt{3}/3 - \pm\sqrt{3}/9 \\ &= \pm\sqrt{3}/3 \mp \sqrt{3}/9 \\ &= \pm 2\sqrt{3}/9. \end{aligned}$$

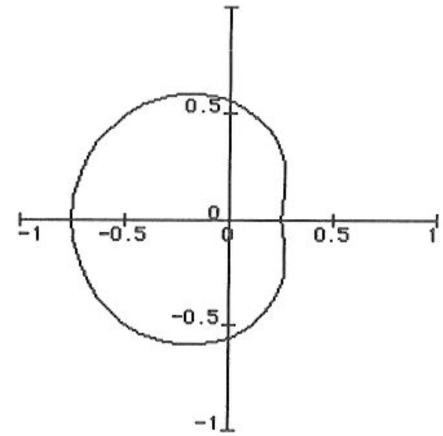


Figure 16.1: The boundary of the set of c-values for which Q_c has an attracting fixed point is a cardioid.

But what about the fixed points of F_c for real c in a neighborhood of these c-values? Are they attracting, repelling, or neutral? Looking ahead at Fig. 17.11 on p. 262 of the text, we observe that the pair of cusps on either side of the degree-3 bifurcation set correspond to the two c-values computed above. On the real axis between these two cusps, there is an attracting fixed point. On either side of these two values, the fixed points are repelling. Showing this rigorously is nontrivial, however.

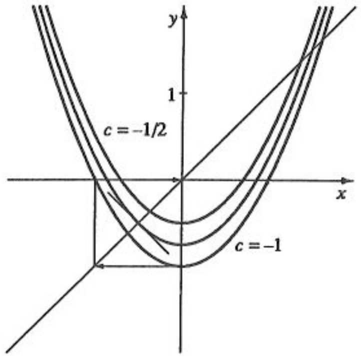
6. The period-doubling bifurcation.

6a) Find all complex c-values for which $Q_c(z) = z^2 + c$ has a fixed point z_0 with $Q_c'(z_0) = -1$.

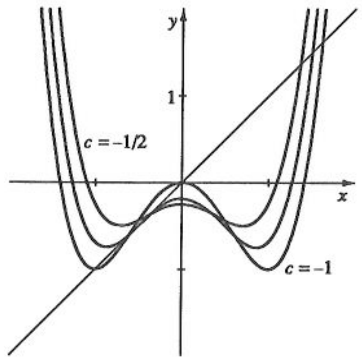
Similar to Exercise 5a, we must solve the system of equations

$$\begin{cases} z^2 + c = z \\ 2z = -1 \end{cases}$$

which has the solution $z = -1/2$ and $c = -3/4$. Representatives of the quadratic family in a neighborhood of this bifurcation point are shown in Figure 16.2.



(a) The 2-cycle of Q_{-1} is attracting and real, while that of $Q_{-1/2}$ is repelling and complex.



(b) The second iterates of Q_c for $c = -1$, $-3/4$, and $-1/2$ clearly show the bifurcation.

Figure 16.2: Quadratic maps in a neighborhood of the period-doubling bifurcation point $c = -3/4$.

6b) Show that the points

$$q_{\pm}(c) = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3-4c}$$

lie on a 2-cycle unless $c = -3/4$.

Straightforward calculation yields

$$\begin{aligned} Q_c\left(-\frac{1}{2} \pm \frac{1}{2}\sqrt{-3-4c}\right) &= \left(-\frac{1}{2} \pm \frac{1}{2}\sqrt{-3-4c}\right)^2 + c \\ &= \frac{1}{4} + \frac{1}{4}(-3-4c) \mp \frac{1}{2}\sqrt{-3-4c} + c \\ &= \frac{1}{4} - \frac{3}{4} - c \mp \frac{1}{2}\sqrt{-3-4c} + c \\ &= -\frac{1}{2} \mp \frac{1}{2}\sqrt{-3-4c}. \end{aligned}$$

Note that

$$q_{\pm}(-3/4) = -\frac{1}{2} \mp \frac{1}{2}\sqrt{-3-4(-3/4)} = -1/2$$

which agrees with the results of Exercise 6a.

6c) Determine whether this cycle is attracting, repelling, or neutral in the two real cases $-5/4 < c < -3/4$ and $c > -3/4$.

Using the Chain Rule Along a Cycle given on p. 47 of the text, plus the fact that $Q_c'(z) = 2z$, we have

$$\begin{aligned} (Q_c^2)'(q_+) &= Q_c'(q_+) \cdot Q_c'(q_-) \\ &= (-1+\sqrt{-3-4c})(-1-\sqrt{-3-4c}) \\ &= 1-(-3-4c) \\ &= 4+4c \\ &= 4(1+c) \end{aligned}$$

and similarly for $(Q_c^2)'(q_-)$. But

$$\begin{aligned} c > -3/4 &\Rightarrow c+1 > 1/4 \\ &\Rightarrow 4(c+1) > 1 \end{aligned}$$

which says the 2-cycle is repelling in this range. When $-5/4 < c < -3/4$, we have

$$\begin{aligned} -5/4 < c < -3/4 &\Rightarrow -1/4 < c+1 < 1/4 \\ &\Rightarrow -1 < 4(c+1) < 1 \end{aligned}$$

and so the cycle is attracting.

6d) Sketch the locations of the fixed points and $q_{\pm}$ for Q_c in the three real cases $-5/4 < c < -3/4$, $c = -3/4$, and $c > -3/4$. This is the complex period-doubling bifurcation.

The bifurcation diagram in Figure 16.3 depicts a period-doubling bifurcation as the parameter decreases through $-3/4$.

7. Consider the complex function $G_{\lambda}(z) = \lambda(z - z^3)$. Show that the points

$$q_{\pm}(\lambda) = \pm\sqrt{\frac{\lambda+1}{\lambda}}$$

lie on a cycle of period 2 unless $\lambda = 0$ or -1. Discuss the bifurcation that occurs at $\lambda = -1$ by sketching the relative positions of $q_{\pm}$ and the fixed points of G_{λ} as λ increases through -1, assuming only real values.

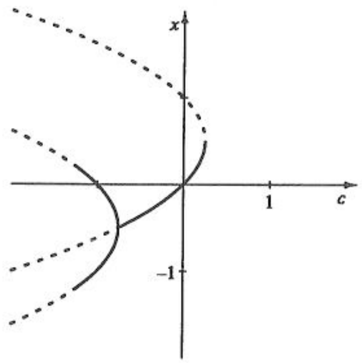


Figure 16.3: A bifurcation diagram for Q_c.

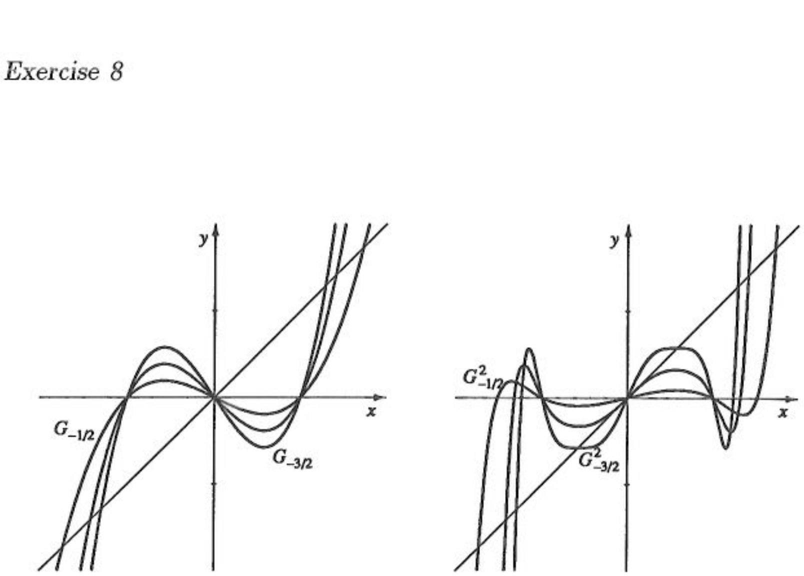
It's convenient to write $G_\lambda(z) = \lambda(z - z^3) = \lambda z(1 - z^2)$. (See Figure 16.4 for representatives from this family.) Now,

$$G_\lambda\left(\sqrt{\frac{\lambda+1}{\lambda}}\right) = \lambda\sqrt{\frac{\lambda+1}{\lambda}}\left(1 - \left(\sqrt{\frac{\lambda+1}{\lambda}}\right)^2\right)$$
$$= \lambda\sqrt{\frac{\lambda+1}{\lambda}}\left(\frac{\lambda-(\lambda+1)}{\lambda}\right)$$
$$= \lambda\sqrt{\frac{\lambda+1}{\lambda}}\left(\frac{-1}{\lambda}\right)$$
$$= -\sqrt{\frac{\lambda+1}{\lambda}}$$

and vice versa. Of course, when $\lambda = 0$, $q_\pm(\lambda)$ is undefined. And when $\lambda = -1$, $q_\pm(\lambda) = 0$, but 0 is fixed.

Computing the fixed points of G_λ, we have

$$\lambda z(1 - z^2) = z$$
$$\Rightarrow \lambda(1 - z^2) = 1 \qquad \text{for } z \neq 0$$
$$\Rightarrow 1 - z^2 = 1/\lambda$$
$$\Rightarrow 1 - 1/\lambda = z^2$$
$$\Rightarrow \frac{\lambda-1}{\lambda} = z^2$$



(a) Three maps... (b) ...and their second iterates.

Figure 16.4: Representatives of G_λ in a neighborhood of $\lambda = -1$.

$$\Rightarrow \pm\sqrt{\frac{\lambda-1}{\lambda}} = z$$

which we'll denote as p_+ and p_-. Note the similarity between $q_\pm$ and $p_\pm$. The bifurcation diagram for this family, shown in Figure 16.5, illustrates a bizarre relationship. By the way, it appears from the graph that the 2-cycle of $G_{-3/2}$ is superattracting. Is this true?

Also notice the relationship between G_λ and the familiar logistic equation $F_\lambda(z) = \lambda z(1 - z)$. There's a rather obvious generalization to made here, namely,

$$G_{\lambda,d}(z) = \lambda z(1 - z^d).$$

Exercise. Completely analyze the dynamics of $G_{\lambda,d}$. (One result along these lines is that $G_{\lambda,d}$ has $d+1$ fixed points, namely, 0 and the d solutions to $z^d = (\lambda - 1)/\lambda$.)

8. Let $Q_i(z) = z^2 + i$. Prove that the orbit of 0 is eventually periodic. Is this cycle attracting or repelling? Use the Backward Iteration Algorithm to compute the filled Julia set for Q_i. Does this set appear to be connected or disconnected?

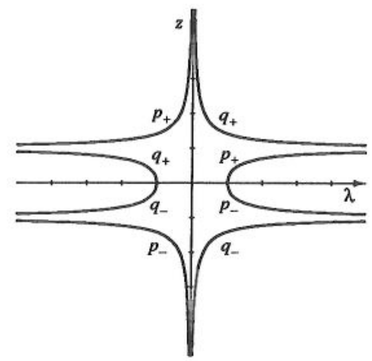


Figure 16.5: A bifurcation diagram for G_λ.

The orbit of 0 given by

$$0 \mapsto i \mapsto i-1 \mapsto -i \mapsto i-1 \mapsto \cdots$$

is eventually periodic with period 2 after just two iterations, that is, $0 \in \text{per}_2^2 Q_i$. Since $Q_i'(z) = 2z$ (which it is for *all* c, by the way) we have

$$\begin{aligned}(Q_i^2)'(i-1) &= Q_i'(i-1)\cdot Q_i'(-i)\\ &= 2(i-1)\cdot 2(-i)\\ &= -4i(i-1)\\ &= 4+4i\\ &= 4(1+i)\end{aligned}$$

and $|4(1+i)| = 4|1+i| = 4\sqrt{2} > 1$. Hence, this cycle is repelling.

We're unable to use *Chaos and Dynamics* to compute the Julia set of Q_i since the software has no option to perform inverse iteration, but we know what's going to happen. The Julia set of Q_i *is* connected (since the orbit of 0 is bounded) and is something called a **dendrite**.[1]

9. The Logistic Functions. The following exercises deal with the family of logistic functions $F_\lambda(z) = \lambda z(1-z)$ where both λ and z are complex numbers.

[1] See Figure 5 of the paper "Julia sets" by Linda Keen in the book *Chaos and Fractals: The Mathematics Behind the Computer Graphics* (R. L. Devaney and L. Keen, editors), American Mathematical Society, Providence, RI, 1989.

9a) Prove that $F_\lambda'(z) = \lambda(1-2z)$ using complex differentiation.

By definition,

$$\begin{aligned}F_\lambda'(z_0) &= \lim_{z\to z_0}\frac{F_\lambda(z)-F_\lambda(z_0)}{z-z_0}\\ &= \lim_{z\to z_0}\frac{\lambda z(1-z)-\lambda z_0(1-z_0)}{z-z_0}\\ &= \lambda\lim_{z\to z_0}\frac{(z-z^2)-(z_0-z_0^2)}{z-z_0}\\ &= \lambda\lim_{z\to z_0}\frac{(z-z_0)-(z^2-z_0^2)}{z-z_0}\\ &= \lambda\lim_{z\to z_0} 1-(z+z_0)\end{aligned}$$

since $(z^2-z_0^2) = (z-z_0)(z+z_0)$. Hence,

$$F_\lambda'(z_0) = \lambda(1-2z_0)$$

upon evaluating the limit.

9b) Find all fixed points for F_λ.

The fixed points of F_λ are solutions to

$$\lambda z(1-z) = z.$$

Assuming $z \neq 0$, we have

$$\begin{aligned}\lambda z(1-z) = z &\Rightarrow \lambda(1-z) = 1\\ &\Rightarrow 1-z = 1/\lambda\\ &\Rightarrow 1-1/\lambda = z\\ &\Rightarrow \frac{\lambda-1}{\lambda} = z.\end{aligned}$$

(A similar calculation was performed in Exercise 7.) Thus,

$$\text{fix}\, F_\lambda = \left\{0, \frac{\lambda-1}{\lambda}\right\}$$

since 0 is also a fixed point by inspection.

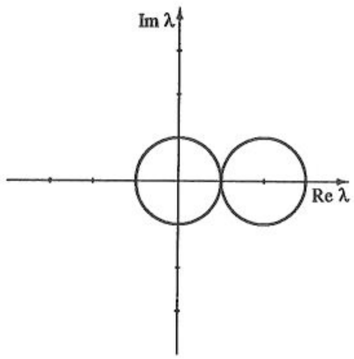


Figure 16.6: The boundary of the fixed point region for F_λ.

9c) Find all parameter values λ for which F_λ has an attracting fixed point.

Using the result in Exercise 9a, we see that $F'_\lambda(0) = \lambda$. So the origin is attracting when $|\lambda| < 1$, that is, inside the circle of radius 1 in the λ-plane.

We also have

$$\begin{aligned} F'_\lambda\left(\frac{\lambda-1}{\lambda}\right) &= \lambda\left(1 - 2\,\frac{\lambda-1}{\lambda}\right) \\ &= \lambda\left(\frac{\lambda - 2(\lambda-1)}{\lambda}\right) \\ &= 2 - \lambda, \end{aligned}$$

and so the other fixed point is attracting when $|2 - \lambda| < 1$, that is, inside the circle of radius 1 centered at $\lambda = 2$. See Figure 16.6.

Chapter 17

The Mandelbrot Set

Exercises

1. Prove that Q_c has a periodic point of prime period 2 at each root of the equation $z^2 + z + c + 1 = 0$. *Hint:* Recall a similar result proved in the real case in Section 6.1.

The second iterate of Q_c is

$$\begin{aligned} Q_c^2(z) &= Q_c(Q_c(z)) \\ &= (z^2 + c)^2 + c \\ &= z^4 + 2cz^2 + c^2 + c \end{aligned}$$

and so the period 2 points of Q_c satisfy

$$z^4 + 2cz^2 + c^2 + c = z$$

which is a fourth degree polynomial having four roots. Two of these roots are the fixed points of Q_c and therefore satisfy $z^2 + c = z$. Thus we may factor out this quadratic from the fourth degree polynomial and get another quadratic, the solutions of which constitute a cycle of prime period 2. In other words, all we need to do is compute

$$\frac{z^4 + 2cz^2 - z + c^2 + c}{z^2 - z + c} = z^2 + z + c + 1$$

by polynomial long division, say.

2. Show that Q_c has an attracting 2-cycle inside the circle of radius 1/4 centered at -1.

From Exercise 1, the second iterate of Q_c is

$$Q_c^2(z) = z^4 + 2cz^2 + c^2 + c$$

which has fixed points

$$\begin{aligned} \operatorname{fix} Q_c^2 &= \{\, z \mid z^4 + 2cz^2 - z + c^2 + c = 0 \,\} \\ &= \{\, z \mid (z^2 - z + c)(z^2 + z + c + 1) = 0 \,\} \\ &= \left\{ \frac{1 \pm \sqrt{1-4c}}{2}, \frac{-1 \pm \sqrt{-4c-3}}{2} \right\}. \end{aligned}$$

Call these points p_1, p_2, q_1, and q_2, respectively. Note that p_1 and p_2 are fixed points for Q_c while q_1 and q_2 constitute a cycle of prime period 2. (Compare with the results of Section 6.1 in the text.) Also note that the points $-p_1$ and $-p_2$ are eventually fixed since Q_c is an even function. Additionally, the following relations hold:[1]

1. $p_1 + p_2 = 1$
2. $c = (1 - p_1^2 - p_2^2)/2$
3. $q_1 + q_2 = -1$
4. $c = q_1 q_2 - 1$

This latter fact is particularly important to our problem, as we shall see.

Using the Chain Rule Along a Cycle given on p. 47 of the text, plus the fact that $Q_c'(z) = 2z$, we have that

$$\begin{aligned} (Q_c^2)'(q_1) &= Q_c'(q_1) \cdot Q_c'(q_2) \\ &= 2q_1 \cdot 2q_2, \end{aligned}$$

and similarly for $(Q_c^2)'(q_2)$. Now we want this 2-cycle to be attracting, so we must have

$$|2q_1 \cdot 2q_2| < 1$$

which implies that

$$|q_1 q_2| < 1/4.$$

[1] See Appendix A of Hans Lauwerier's *Fractals: Endlessly Repeated Geometrical Figures* published by Princeton University Press in 1991.

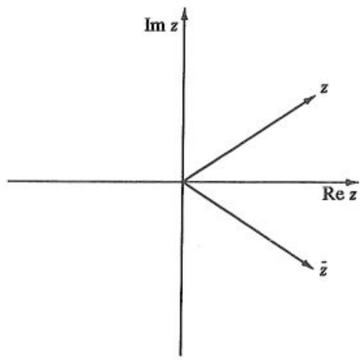


Figure 17.1: Complex conjugates are symmetric with respect to the real axis.

Applying the fact that $c = q_1 q_2 - 1$ from above, we obtain

$$|c+1| < 1/4$$

which is a disk of radius 1/4 centered at -1 in the c-plane. All of these points are in the Mandelbrot set since the orbit of the critical point 0 is attracted to this 2-cycle by the complex analogue of the theorem in Section 12.2 of the text.

3. Prove that the Mandelbrot set is symmetric about the real axis. *Hint:* Show this by proving that Q_c is conjugate to $Q_{\bar{c}}$. Show that your conjugacy takes 0 to 0. Therefore the orbit of 0 has similar fates for both Q_c and $Q_{\bar{c}}$.

Since a complex number and its conjugate are symmetric to one another with respect to the real axis (see Figure 17.1), the hint is justified.

Define $H: \mathbb{C} \to \mathbb{C}$ such that $H(z) = \bar{z}$. Then

$$\begin{aligned} Q_{\bar{c}} \circ H(z) &= \bar{z}^2 + \bar{c} \\ &= \overline{z^2} + \bar{c} \qquad \text{since } \bar{z}_1 \bar{z}_2 = \overline{z_1 z_2} \\ &= \overline{z^2 + c} \qquad \text{since } \bar{z}_1 + \bar{z}_2 = \overline{z_1 + z_2} \\ &= H \circ Q_c(z). \end{aligned}$$

Therefore, Q_c is conjugate to $Q_{\bar{c}}$ via H. In other words,

$$Q_{\bar{c}}(\bar{z}) = \overline{Q_c(z)},$$

and in fact,

$$Q_{\bar{c}}^n(\bar{z}) = \overline{Q_c^n(z)}.$$

(Why?) In particular, since the complex conjugate of 0 is 0, we have that

$$Q_{\bar{c}}^n(0) = \overline{Q_c^n(0)}$$

which shows that the Mandelbrot set is symmetric with respect to the real axis. This is because

$$|Q_{\bar{c}}^n(0)| = |\overline{Q_c^n(0)}| = |Q_c^n(0)|,$$

and so, as $n \to \infty$, $|Q_{\bar{c}}^n(0)| \to \infty$ whenever $|Q_c^n(0)| \to \infty$.

Finally, since H is its own inverse, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{Q_c} & \mathbb{C} \\ H \downarrow \uparrow H & & H \uparrow \downarrow H \\ \mathbb{C} & \xrightarrow{Q_{\bar{c}}} & \mathbb{C} \end{array}$$

We remark that an alternative, but equivalent, approach to this problem would be to show that

$$Q_c^n(z) = \overline{Q_{\bar{c}}(\bar{z})}$$

by induction, which further motivates the previous choice of H.

The Logistic Functions. The following seven exercises deal with the logistic family $F_\lambda(z) = \lambda z(1-z)$ where both λ and z are complex numbers.

4. Prove that, for $\lambda \neq 0$, $|F_\lambda(z)| > |z|$ provided $|z| > \frac{1}{|\lambda|} + 1$. Use this to give the analogue of the escape criterion for the logistic family.

We have

$$\begin{aligned} |F_\lambda(z)| &= |\lambda z(1-z)| \\ &= |\lambda||z||1-z| \\ &= |\lambda||z||z-1| \\ &\geq |\lambda||z|(|z|-1) \end{aligned}$$

by the triangle inequality. Now, since $|z| - 1 > \frac{1}{|\lambda|}$ by supposition,

$$|\lambda||z|(|z| - 1) > |\lambda||z|\frac{1}{|\lambda|} = |z|$$

and so $|F_\lambda(z)| > |z|$.

The key idea used in this proof is the fact that $|z - 1| \geq |z| - 1$, a special case of the more general formula

$$|z_1 - z_2| \geq |z_1| - |z_2|.$$

The latter is derived from the familiar form of the triangle inequality as follows:

$$|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|.$$

Moving $|z_2|$ to the far left gives the desired result.

6. Show that, if $\lambda \neq 0$, the logistic function F_λ is conjugate to the complex quadratic function $Q_c(z) = z^2 + c$ where $c = \frac{\lambda}{2} - \frac{\lambda^2}{4}$. Let $c = V(\lambda)$ be this correspondence between λ and c. Why does this result fail if $\lambda = 0$?

Suppose $F_\lambda(z) = \lambda z(1 - z)$ and $Q_c(z) = z^2 + c$. We seek a conjugacy between F_λ and Q_c of the form $L(z) = az + b$ such that

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F_\lambda} & \mathbb{C} \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle L} \\ \mathbb{C} & \xrightarrow{Q_c} & \mathbb{C} \end{array}$$

commutes. In other words, we want

$$L \circ F_\lambda(z) = Q_c \circ L(z)$$

which in our case is equivalent to

$$a\lambda z(1 - z) + b = (az + b)^2 + c.$$

Expanding both sides and equating coefficients, we arrive at the following system of equations:

$$\begin{cases} -a\lambda = a^2 \\ a\lambda = 2ab \\ b = b^2 + c \end{cases} . \tag{17.1}$$

Provided $a \neq 0$, the first two of these equations yield

$$\begin{cases} a = -\lambda \\ b = \lambda/2 \end{cases}$$

and from the last equation in (17.1) we obtain

$$c = b - b^2 = \lambda/2 - \lambda^2/4$$

as required. The parameter λ can not be zero because $0 \neq a = -\lambda$.

Higher Degree Polynomials. The following eight exercises deal with polynomials of the form $P_{c,d}(z) = z^d + c$.

11. Prove that $P'_{c,d}(z) = dz^{d-1}$. Conclude that, for each integer $d > 1$, $P_{c,d}$ has a single critical point at 0.

First we need

Lemma 17.1

$$\frac{z^d - z_0^d}{z - z_0} = \sum_{k=1}^{d} z_0^{k-1} z^{d-k}.$$

Proof:

$$\begin{aligned} (z - z_0)\sum_{k=1}^{d} z_0^{k-1} z^{d-k} &= \sum_{k=1}^{d} z_0^{k-1} z^{d-k+1} - \sum_{k=1}^{d} z_0^{k} z^{d-k} \\ &= \sum_{k=0}^{d-1} z_0^{k} z^{d-k} - \sum_{k=1}^{d} z_0^{k} z^{d-k} \\ &= z^d - z_0^d. \end{aligned}$$

□

And now we can prove that $P'_{c,d}(z) = dz^{d-1}$. Dropping subscripts for the moment, we have by definition

$$\begin{aligned} P'(z_0) &= \lim_{z \to z_0} \frac{P(z) - P(z_0)}{z - z_0} \\ &= \lim_{z \to z_0} \frac{(z^d + c) - (z_0^d + c)}{z - z_0} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{z \to z_0} \frac{z^d - z_0^d}{z - z_0} \\
&= \lim_{z \to z_0} \sum_{k=1}^{d} z_0^{k-1} z^{d-k} \qquad \text{by Lemma 17.1} \\
&= \sum_{k=1}^{d} \lim_{z \to z_0} z_0^{k-1} z^{d-k} \\
&= \sum_{k=1}^{d} z_0^{d-1} \\
&= d z_0^{d-1}.
\end{aligned}
$$

12. Prove the following escape criterion for $P_{c,d}$. Show that if $|z| \geq |c|$ and $|z|^{d-1} > 2$, then $|P_{c,d}^n(z)| \to \infty$ as $n \to \infty$.

We know that

$$|P_{c,d}(z)| = |z^d + c| \geq |z^d| - |c|$$

by the triangle inequality. But

$$|z^d| - |c| = |z|^d - |c| \geq |z|^d - |z|$$

since $|z| \geq |c|$ by supposition. And so

$$|z|^d - |z| = |z|(|z|^{d-1} - 1) > |z|$$

since $|z|^{d-1} - 1 > 1$ by hypothesis. Hence, $|P_{c,d}(z)| > |z|$. Similarly, $|P_{c,d}^2(z)| = |P_{c,d}(P_{c,d}(z))| > |P_{c,d}(z)|$, and in general, we have that

$$\cdots > |P_{c,d}^n(z)| > \cdots > |P_{c,d}^2(z)| > |P_{c,d}(z)| > |z|.$$

In fact, $|P_{c,d}^n(z)|$ is unbounded since

$$|P_{c,d}^n(z)| > |z|(|z|^{d-1} - 1)^n \tag{17.2}$$

for $n \geq 2$. We remark that (17.2) is also true for $n = 1$ provided $|z| > |c|$.

Corollary 17.2 *If $|z| \geq |c|$ and $|z| > 2$, then $|P_{c,d}^n(z)| \to \infty$ as $n \to \infty$.*

Proof: Simply observe that $|z|^{d-1} > |z|^{d-2} > \cdots > |z|^2 > |z| > 2$. □

13. Show that if $|c| > 2^{1/(d-1)}$, the orbit of the critical point of $P_{c,d}$ escapes to infinity.

The following gives a minimal escape criterion for $P_{c,d}$:

Proposition 17.3 *If $|z| \geq |c|$ and $|z| > 2^{1/(d-1)}$, then $|P_{c,d}^n(z)| \to \infty$ as $n \to \infty$.*

As a consequence of Proposition 17.3 (which is merely a restatement of Exercise 12) we have[2]

Corollary 17.4 *The filled Julia set of $P_{c,3}$ lies inside the circle of radius $\sqrt{2}$ centered at the origin.*

as well as

Corollary 17.5 *If $|c| > 2^{1/(d-1)}$, then the critical orbit of $P_{c,d}$ escapes to infinity.*

Proof: Observe that $P_{c,d}(0) = c$. Now put $z := c$ in Proposition 17.3. □

In other words, the degree d bifurcation set of $P_{c,d}$ lies entirely inside the circle of radius $2^{1/(d-1)}$ centered at the origin.

16. For a fixed value of d, find the set of c-values for which $P_{c,d}$ has an attracting fixed point. Find an expression for the boundary of this region. Identify this region in the images generated by the previous exercise.

From Exercise 11, we have that $P_{c,d}'(z) = dz^{d-1}$. Now the boundary of the fixed point region of the degree d bifurcation set satisfies $|P_{c,d}'(z)| = 1$. But

$$
\begin{aligned}
&|P_{c,d}'(z)| = 1 \\
\Rightarrow\ &|dz^{d-1}| = 1 \\
\Rightarrow\ &d|z|^{d-1} = 1 \qquad \text{assuming } d > 0 \\
\Rightarrow\ &|z| = d^{1/(1-d)}.
\end{aligned}
$$

In other words, z lies on a circle of radius $d^{1/(1-d)}$ centered at the origin, that is,

$$z = d^{1/(1-d)} e^{i\theta}. \tag{17.3}$$

[2]Thanks to Scott Huddleston for showing this in alt.fractals.

Now, the fixed points of $P_{c,d}$ satisfy

$$\begin{aligned} z^d + c &= z \\ \Rightarrow c &= z - z^d. \end{aligned} \tag{17.4}$$

Substituting (17.3) into (17.4), we obtain

$$c = d^{1/(1-d)}e^{i\theta} - d^{d/(1-d)}e^{di\theta}. \tag{17.5}$$

Note when $d = 2$, we get

$$c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta}$$

as in the text. Equation 17.5 gives the boundary of the fixed point region for a fixed d in (c, θ) coordinates, whereas the fixed point region itself is given by

$$c = z - z^d \quad \text{for} \quad |z| < d^{1/(1-d)}.$$

Fixed point regions for various values of d are shown in Figure 17.2. Note the symmetry unfolding as the value of d increases. It's natural to wonder what happens for other values of d, or as d approaches infinity. The latter question is answered below.

Motivated by the form of (17.5), let us suppose

$$f(x) = \begin{cases} x^{1/(1-x)} & \text{if } x \geq 0 \text{ and } x \neq 1 \\ e^{-1} & \text{if } x = 1 \end{cases}.$$

Then f is continuous at $x = 1$ since

$$\begin{aligned} \lim_{x\to 1} x^{1/(1-x)} &= \lim_{x\to 1} \exp\left(\log x^{1/(1-x)}\right) \\ &= \exp\left(\lim_{x\to 1} \frac{\log x}{1-x}\right) \\ &= \exp\left(\lim_{x\to 1} \frac{1/x}{-1}\right) \qquad \text{by L'Hôpital's rule} \\ &= e^{-1}. \end{aligned}$$

Similarly, we may show that

$$\lim_{x\to\infty} x^{1/(1-x)} = 1,$$

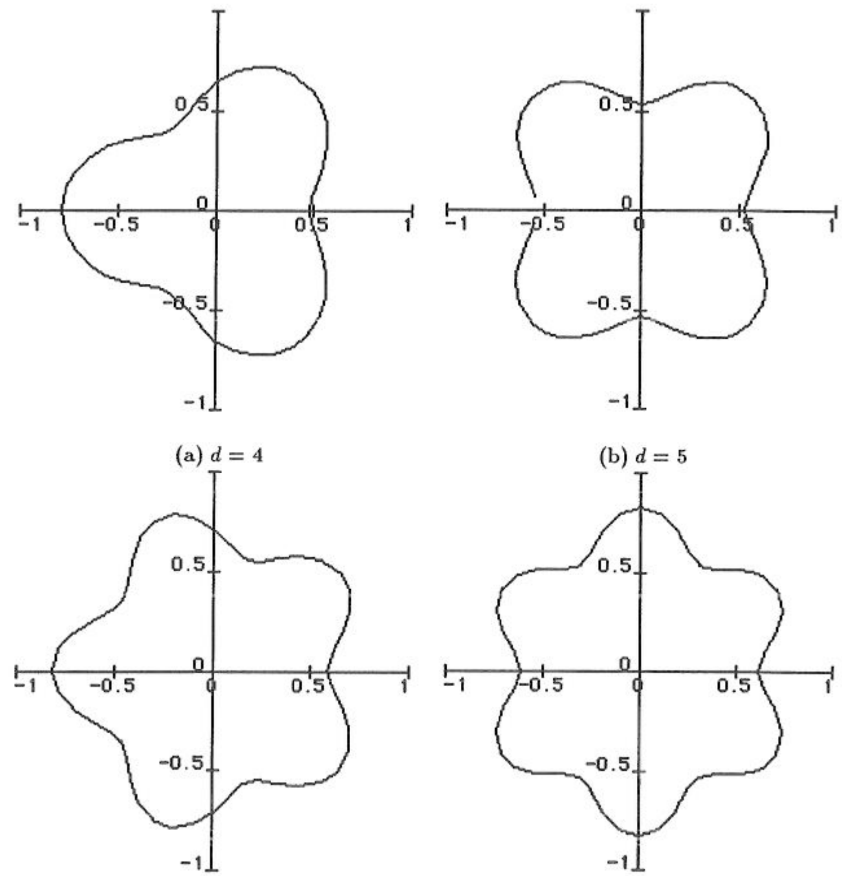


Figure 17.2: Fixed point regions for various values of d.

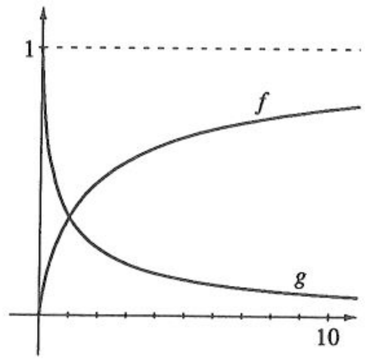


Figure 17.3: The graphs of f and g intersect at $x = e^{-1}$.

and so we have a pretty good picture of this function taking the nonnegative reals onto $[0, 1)$. See Figure 17.3.

Now consider

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \\ x^{x/(1-x)} & \text{if } x > 0 \text{ and } x \neq 1 \\ e^{-1} & \text{if } x = 1 \end{cases}.$$

Then g is continuous at $x = 1$ (exercise), and its value at $x = 0$ is justified since

$$\begin{aligned} \lim_{x\to 0+} x^{x/(1-x)} &= \exp\left(\lim_{x\to 0+} \frac{\log x}{\frac{1-x}{x}}\right) \\ &= \exp\left(\lim_{x\to 0+} \frac{1/x}{-1/x^2}\right) \qquad \text{by L'Hôpital's rule} \\ &= \exp\left(\lim_{x\to 0+} -x\right) \\ &= e^0 \\ &= 1. \end{aligned}$$

See Figure 17.3 for the graph of g.

We can use these results to analyze (17.5). When $d = 1$, we have

$$c = e^{-1}e^{i\theta} - e^{-1}e^{i\theta}$$

since both $d^{1/(1-d)}$ and $d^{d/(1-d)}$ approach e^{-1} as $d \to 1$. Thus c vanishes at $d = 1$, and so does the degree d bifurcation set. And as $d \to \infty$, the quantity $d^{d/(1-d)} \to 0$, and so $c \to e^{i\theta}$, that is, the degree d bifurcation set approaches the unit circle.

The previous results suggest the following series of experiments and projects:

Experiment. As an extension of Exercise 13, compute the degree d bifurcation set of $P_{c,d}$ for various *real* values of d between 2 and 5, say.

Experiment. Repeat the previous experiment for $1 \leq d \leq 2$.

Project. Produce a 2-d animation of the degree d bifurcation set of $P_{c,d}$ as d varies between 2 and 5, or for d between 1 and 2.

Project. Render a 3-d image of the degree d bifurcation set of $P_{c,d}$ in $(\operatorname{re} c, \operatorname{im} c, d)$ coordinates with $-2 \leq \operatorname{re} c, \operatorname{im} c \leq 2$ and $2 \leq d \leq 5$, say. *Hint:* Position the viewer's eye well away from the negative d axis since the slice at $d = 2$ will most certainly obscure the rest of the image.

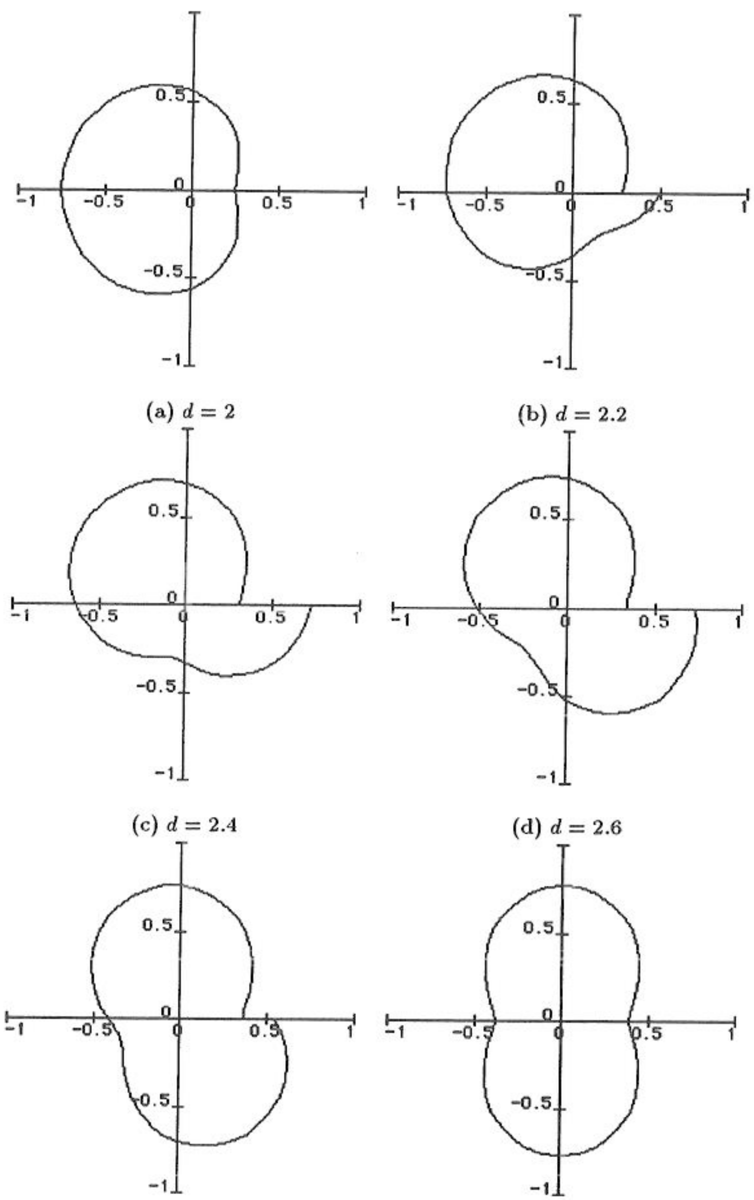
For reference, the fixed point regions for a sequence of equally spaced d values between 2 and 3 are shown in Figure 17.4. The Maple program used to generate these images is given below:

```
drawFixedPointRegion :=
  proc(d)
    local term1, term2, theta;
    term1 := evalc(exp(I*theta))/d^(1/(d-1));
    term2 := evalc(exp(d*I*theta))/d^(d/(d-1));
    plot([abs(term1 - term2), theta, theta=0..2*Pi],
                              -1..1, -1..1, coords=polar);
  end:
for i from 2 by 0.2 to 3 do  drawFixedPointRegion(i)  od;
```

17. Prove that the degree 3 bifurcation set is symmetric with respect to reflection through the origin. *Hint:* See Exercise 3 above.

First of all, observe that the degree d bifurcation set is symmetric with respect to the real axis for *all* d (the proof is similar to that given in Exercise 3). To show that it's symmetric with respect to reflection through the origin, define $H: \mathbb{C} \to \mathbb{C}$ with $H(z) = -z$ (see Figure 17.5 for justification). Then

$$\begin{aligned} P_{-c,3} \circ H(z) &= (-z)^3 - c \\ &= -(z^3 + c) \\ &= H \circ P_{c,3}(z). \end{aligned}$$



(a) $d = 2$ (b) $d = 2.2$

(c) $d = 2.4$ (d) $d = 2.6$

(e) $d = 2.8$ (f) $d = 3$

Figure 17.4: The evolution of the fixed point region as d varies between 2

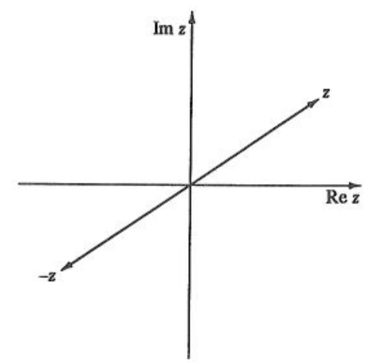


Figure 17.5: More geometry of complex numbers.

Therefore, $P_{c,3}$ is conjugate to $P_{-c,3}$ via H. In other words,

$$P_{-c,3}(-z) = -P_{c,3}(z),$$

and in fact,

$$P^n_{-c,3}(-z) = -P^n_{c,3}(z).$$

(Why?) In particular, we have that

$$P^n_{-c,3}(0) = -P^n_{c,3}(0)$$

which shows that the degree 3 bifurcation set is symmetric with respect to the origin since

$$|P^n_{-c,3}(0)| = |-P^n_{c,3}(0)| = |P^n_{c,3}(0)|,$$

and so, as $n \to \infty$, $|P^n_{-c,3}(0)| \to \infty$ whenever $|P^n_{c,3}(0)| \to \infty$.
An alternative, but equivalent, approach is to show that

$$P^n_{c,3}(z) = -P^n_{-c,3}(-z)$$

by induction (which may even be more straightforward!).
A similar result holds for any odd positive integer d. That is, since H is its

own inverse, the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{P_{c,d}} & \mathbb{C} \\ H \Big\downarrow \Big\uparrow H & & H \Big\uparrow \Big\downarrow H \\ \mathbb{C} & \xrightarrow{P_{-c,d}} & \mathbb{C} \end{array}$$

commutes for any odd positive integer d.

As an afterthought, it should be possible to find H_1 and H_2 with $H = H_2 \circ H_1$ so that the following diagram commutes:

$$\begin{array}{ccccc} \mathbb{C} & \xrightarrow{H_1} & \mathbb{C} & \xrightarrow{H_2} & \mathbb{C} \\ P_{c,3} \Big\downarrow & & P_{\bar{c},3} \Big\downarrow & & \Big\downarrow P_{-c,3} \\ \mathbb{C} & \xrightarrow{H_1} & \mathbb{C} & \xrightarrow{H_2} & \mathbb{C} \end{array}$$

Exercise. Find H_1 and H_2 such that

$$\begin{aligned} H_1 \circ P_{c,3} &= P_{\bar{c},3} \circ H_1, \\ H_2 \circ P_{\bar{c},3} &= P_{-c,3} \circ H_2, \quad \text{and} \\ H \circ P_{c,3} &= P_{-c,3} \circ H \end{aligned}$$

with $H = H_2 \circ H_1$.

18. Prove that the degree 4 bifurcation set is symmetric with respect to rotation through the angle $2\pi/3$. *Hint:* Show that $P_{c,4}$ is conjugate to $P_{e^{2\pi i/3}c,4}$. Generalize this and the result of the previous exercise to the degree d bifurcation set.

The proof is similar to the one used in Exercise 3 and again in Exercise 17. Let $H\colon \mathbb{C} \to \mathbb{C}$ with $H(z) = e^{2\pi i/3}z$, and then proceed as before. Note that

H is a homeomorphism with $H^{-1}(z) = e^{4\pi i/3}z$.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{P_{c,4}} & \mathbb{C} \\ H \Big\downarrow & & \Big\downarrow H \\ \mathbb{C} & \xrightarrow{P_{e^{2\pi i/3}c,4}} & \mathbb{C} \end{array}$$

In general, the degree d bifurcation set is symmetric with respect to rotation through an angle of $2\pi/(d-1)$ radians, or any integer multiple thereof. In other words, there are $d-1$ axes of rotation. The degree 6 bifurcation set, for example, has a five-way symmetry with respect to the angles $2k\pi/5$ for $0 \le k < d-1$. Other examples are given in the following table:

d	2	3	4	5	6	...
$2\pi/(d-1)$	2π	π	$2\pi/3$	$\pi/2$	$2\pi/5$	...

These results agree with previous statements claiming that

1. the degree d bifurcation set is symmetric with respect to the real axis *for all d*;
2. the degree d bifurcation set is symmetric with respect to reflection through the origin *for odd d*.

The proof of these claims amounts to showing that $P_{c,d}$ is conjugate to $P_{e^{2\pi i/(d-1)}c,d}$ via $H(z) = e^{2\pi i/(d-1)}z$, the details of which are left as an exercise.

19. Show that $c = -2$ and $c = i$ are Misiurewicz points for Q_c, that is, 0 is eventually periodic for Q_c. On a sketch of the Mandelbrot set, locate these two c-values as accurately as possible.

For $Q_{-2}(z) = z^2 - 2$, we know that

$$0 \mapsto -2 \mapsto 2 \mapsto 2 \mapsto \cdots$$

and so $0 \in \mathrm{per}_1^2\, Q_{-2}$. Similarly, for $Q_i(z) = z^2 + i$, we have

$$0 \mapsto i \mapsto (i-1) \mapsto -i \mapsto (i-1) \mapsto \cdots$$

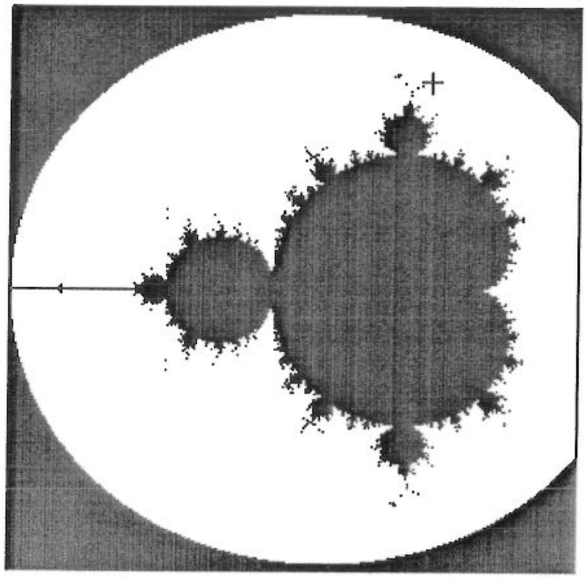


Figure 17.6: The point $z = i$ is a Misiurewicz point.

which shows that $0 \in \text{per}_2^2 Q_i$. This latter c-value has been indicated with a crosshair on a copy of the Mandelbrot set in Figure 17.6. The value $c = -2$ lies on the negative real axis, at the leftmost tip of the spine of $\mathcal{M}$.

20. For c and z complex, consider the functions[3]

$$F_c(z) = c\left(z^2 + \frac{1}{z^2}\right).$$

20a) What are the critical points of F_c?

Since the derivative of F_c is $F_c'(z) = c(2z - 2/z^3)$, we have

$$\begin{aligned} 2c\left(z - \frac{1}{z^3}\right) &= 0 \\ \Rightarrow z - \frac{1}{z^3} &= 0 \\ \Rightarrow z &= \frac{1}{z^3} \end{aligned}$$

[3] Thanks to Kerry Mitchell for suggesting this family of functions.

$$\begin{aligned} &\Rightarrow z^4 = 1 \\ &\Rightarrow z = \pm 1 \quad \text{and} \quad z = \pm i. \end{aligned}$$

In other words, the critical points of F_c are the 4th roots of unity.

20b) Prove that the orbits of all critical points have the same fate.

Evaluating the function at each of its critical points,

$$F_c(\pm 1) = c\left((\pm 1)^2 + \frac{1}{(\pm 1)^2}\right) = 2c$$

$$F_c(\pm i) = c\left((\pm i)^2 + \frac{1}{(\pm i)^2}\right) = -2c$$

and again at the resulting critical values,

$$F_c(\pm 2c) = c\left((\pm 2c)^2 + \frac{1}{(\pm 2c)^2}\right) = 4c^3 + \frac{1}{4c}$$

we find that each critical point leads to the same critical *value*. (Note that F_c is an even function.)

20c) Show that

$$|z| + \frac{1}{|z|^3} > \frac{1}{|c|}$$

is an escape criterion for F_c, that is, if any point of the orbit of z_0 satisfies this condition, then the orbit of z_0 escapes to infinity.

We have

$$\begin{aligned} |F_c(z)| &= \left|c\left(z^2 + \frac{1}{z^2}\right)\right| \\ &= \left|cz\left(z + \frac{1}{z^3}\right)\right| \\ &= |c||z|\left|z + \frac{1}{z^3}\right| \\ &\geq |c||z|\left(|z| + \frac{1}{|z|^3}\right) \\ &> |c||z|\frac{1}{|c|} \qquad \text{since } |z| + \frac{1}{|z|^3} > \frac{1}{|c|} \\ &= |z| \end{aligned}$$

which shows that $|F_c(z)| > |z|$. It remains to be shown that $|F_c^n(z)|$ is unbounded as $n \to \infty$.

20d) Which orbit would you use to compute the analogue of the Mandelbrot set for this family of functions?

Any of the critical points suffice since they all have the same critical *value* (see part b).

Appendix A

Mathematical Notation

Sets:

$\emptyset$ — the empty set; the set containing no elements; sometimes written as $\{\}$ but never as $\{\emptyset\}$; the latter is a set—it is just not the empty set.

$\in$ — is an element of; for example,

$$\frac{1}{2} \in \left\{ \frac{1}{n+1} \;\middle|\; n \in \mathbb{N} \right\},$$

but there is no x such that $x \in \emptyset$.

$[a, b]$ — the closed interval with endpoints a and b; formally,

$$[a,b] = \{\, x \in \mathbb{R} \mid a \le x \le b \,\}.$$

(a, b) — the open interval with "endpoints" a and b; in this case,

$$(a,b) = \{\, x \in \mathbb{R} \mid a < x < b \,\};$$

some authors use the notation $]a, b[$ to avoid ambiguity with the ordered pair (a, b); for example, the complex number $x + iy$ is associated with the point (x, y) in the complex plane.

$\subset$ — is a proper subset of; when we write $A \subset B$, we mean that every element of the set A is also an element of B, but $A \neq B$; in other words, A is properly contained in B; for example, the open interval $(a, b) \subset \mathbb{R}$ for any finite real numbers a and b.

$\subseteq$ — is a subset of; $A \subseteq B$ means that every element of A is also an element of B, but here A and B could be equal; for example, the only open interval which is not a proper subset of $\mathbb{R}$ is $\mathbb{R}$ itself; yes, $\mathbb{R}$ is open, and we sometimes write $(-\infty, \infty) = \mathbb{R}$.

$\times$ — cross product; new sets are sometimes built up from existing sets by means of cross products; for example, the cross product of any two closed intervals is a rectangle in the plane, which itself is denoted by $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

$|S|$ — the size of the set S; for a finite set, $|S|$ is simply the number of elements in S; for example, if $|S| = n$, then the set of all subsets of S (called the **power set** of S) contains 2^n elements, that is, $|\mathcal{P}(S)| = 2^{|S|}$.

Number Systems and Constants:

$\mathbb{N}$ — the natural numbers; $\mathbb{N} = \{0, 1, 2, \ldots\}$.

$\mathbb{Z}$ — the integers; the natural numbers together with their negatives; $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

$\mathbb{Q}$ — the rational numbers; ratios of integers; for nonzero q, $\mathbb{Q} = \{\, p/q \mid p, q \in \mathbb{Z} \,\}$.

$\mathbb{R}$ — the real numbers; includes the rational numbers together with those that are not (that is, the irrational numbers).

$\mathbb{C}$ — the complex numbers; $\mathbb{C} = \{\, a + ib \mid a, b \in \mathbb{R} \,\}$; each complex number $a + ib$ is associated with the point (a, b) in the complex plane.

ϕ — the golden ratio; $\phi = (1 + \sqrt{5})/2$; a fixed point of the map $F(x) = x^2 - 1$; arises naturally in conjunction with the Fibonacci sequence.

e — the base of the natural logarithm; it is shown in calculus that

$$\left(1 + \frac{1}{n}\right)^n \to e \text{ as } n \to \infty,$$

and that $e = \sum_{k=0}^{\infty} 1/k! = 2.71828\ldots$.

π — the ratio of a circle's circumference to its diameter; the constant $\pi = 3.14159\ldots$.

i — the imaginary unit; $i = \sqrt{-1}$; i and its negation $-i$ have squares which are negative unity.

Real Variables and Special Functions:

x — a real variable; often used as a function argument (as in $F(x)$, for example) or the real part of a complex number such as $x + iy$; the subscripted variable x_0 usually denotes the first in a sequence of real numbers sometimes obtained by iteration.

y — usually denotes a real variable; often used as a function value as in $y = F(x)$, or the imaginary part of a complex number like $x + iy$.

$|x|$ — the absolute value of x is defined to be $-x$ if x is negative, and x otherwise; formally,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} ;$$

the magnitude of x; the positive distance between x and the origin on the real line.

$\sqrt{x}$ — the square root of x; defined to be that number whose square is x.

Complex Variables and Special Functions:

z — a complex variable; $z = x + iy$ where $x, y \in \mathbb{R}$; z_0 usually denotes the first in a sequence of complex numbers obtained by iteration.

$\bar{z}$ — the conjugate of z; if $z = x + iy$, then $\bar{z} = x - iy$.

$\operatorname{re} z$ — the real part of z; if $z = x + iy$, then $\operatorname{re} z = x$;

$\operatorname{im} z$ — the imaginary part of z; if $z = x + iy$, then $\operatorname{im} z = y$; taken together, the real and imaginary parts completely specify the complex number z in what is known as **rectangular coordinates**.

$|z|$ — the absolute value or modulus of z, often denoted by the letter r; defined to be the real quantity $\sqrt{(\operatorname{re} z)^2 + (\operatorname{im} z)^2}$; the Euclidean, or straight-line, distance between z and the origin in $\mathbb{R} \times \mathbb{R}$.

$\arg z$ — the argument of z, oftentimes denoted by θ; taken together, the modulus and argument completely specify z in what is known as **polar coordinates**.

$\sqrt{z}$ — the square root of z; defined to be that number whose square is z; for an arbitrary complex number z,

$$\sqrt{z} = \pm\sqrt{r}\,(\cos\theta + i\sin\theta)$$

where $r = |z|$ and $\tan\theta = \operatorname{im} z / \operatorname{re} z$.

Functions and Functional Notation:

F — a function, or mapping between two sets; sometimes written as $F\colon A \to B$; the lone symbol F refers to an *object* of great conceptual power.

$F(x)$ — a function value, sometimes written as $y = F(x)$; this notation suggests a *process* whereby x is mapped to y via F; for example, if $F(x) = x^3 + x$, then $y = F(1) = 2$; that is, $(1, 2)$ is a point on the graph of F.

$\operatorname{dom} F$ — the domain of F; the set of all possible inputs; if $F\colon A \to B$, then $\operatorname{dom} F = A$.

$\operatorname{range} F$ — the range of F; any set containing all possible outputs; if $F\colon A \to B$, then $\operatorname{range} F = B$.

$\operatorname{image} F$ — the image of the function F; the set of all possible outputs; if $F\colon A \to B$, then $\operatorname{image} F = F(A) \subseteq B$; if $\operatorname{range} F = \operatorname{image} F$, then we say that F is **onto** B.

$\mapsto$ — maps to; specifies a function with no name; for example, when we write $x \mapsto 2x^2 - x + 1$, we mean that each x in the function's domain gets mapped to a corresponding y in the function's range via the rule $y = 2x^2 - x + 1$; a convenient notation in those situations where a function name is unnecessary or awkward.

F_c — a family of functions dependent upon a parameter c; for example, $T_c(x) = x^3 + cx$ defines a family of cubics that includes the member T_{-1} with rule $T_{-1}(x) = x^3 - x$.

$F_c(x)$ — a value of the function F_c; in the previous example, $T_{-1}(1) = 0$.

F^0 — the identity function; $F^0(x) = x$ for all x; we sometimes write $F^0 = \text{id}$.

F^{-1} — the inverse of F; if $F\colon A \to B$ and F is one-to-one and onto B, then $F^{-1}\colon B \to A$ exists; moreover, $F(F^{-1}(x)) = F^{-1}(F(x)) = x$, that is, composing a function with its inverse (or vice versa) gives rise to the identity map; for example, let $F\colon [-1/2, \infty) \to [-17/4, \infty)$ with $F(x) = x^2+x-4$ so that $F^{-1}\colon [-17/4, \infty) \to [-1/2, \infty)$ with $F^{-1}(x) = \sqrt{x + 17/4} - 1/2$ is the inverse of F (in this example, the reader should verify that $F(F^{-1}(x)) = F^{-1}(F(x)) = x$ before graphing F and F^{-1} on the same set of coordinate axes); the graph of a function and its inverse are symmetric to one another with respect to the diagonal $y = x$.

$F^{-1}(x)$ — a value of the function F^{-1}; in the previous example, the rule which produces this value is $F^{-1}(x) = \sqrt{x + 17/4} - 1/2$.

Composition of Functions:

$\circ$ — composed with; the process $F \circ G(x)$ is just another notation for $F(G(x))$; for example, $F \circ F^{-1} = F^{-1} \circ F$ is the identity function since

$$F \circ F^{-1}(x) = F^{-1} \circ F(x) = x$$

by definition; note that composition of functions is not generally commutative, that is, $F \circ G \neq G \circ F$ (exercise: find a counterexample).

F^n — the n-fold composition of a function with itself; the iterative definition is simply

$$F^n = \underbrace{F \circ F \circ \cdots \circ F}_{n \text{ times}}$$

while the recursive definition is given by

$$F^n = \begin{cases} F \circ F^{n-1} & \text{for } n > 0 \\ F^0 & \text{for } n = 0 \end{cases}$$

where F^0 is the identity transformation; the alternative notation $F^{\circ n}$ is sometimes used in lieu of the potentially ambiguous F^n.

$F^n(x)$ — a particular value of the composite function F^n; also thought of as the process of iterating F n times on x, thereby producing the forward **orbit** of x comprised of the sequence of values

$$x, F(x), F^2(x), \ldots, F^n(x);$$

the forward orbit of x is sometimes denoted by $O^+(x)$.

F^{-n} — the n-fold composition of F^{-1} with itself; like the above definition of F^n, the iterative definition of F^{-n} is

$$F^{-n} = \underbrace{F^{-1} \circ F^{-1} \circ \cdots \circ F^{-1}}_{n \text{ times}}$$

while the recursive definition is given by

$$F^{-n} = \begin{cases} F^{-1} \circ F^{-(n-1)} & \text{for } n > 0 \\ F^0 & \text{for } n = 0 \end{cases}$$

where once again F^0 is the identity transformation.

$F^{-n}(x)$ — a value of the function F^{-n}; also seen as the process of iterating F^{-1} n times on x; this gives the backward (or inverse) orbit of x consisting of the sequence of values

$$x, F^{-1}(x), F^{-2}(x), \ldots, F^{-n}(x);$$

the backward orbit of x is sometimes written as $O^-(x)$.

Fixed and Eventually Fixed Points:

$\text{fix}\, F$ — the set of fixed points of the function F; $x \in \text{fix}\, F$ if $F(x) = x$; in terms of the graph of F, a fixed point occurs at every intersection of the graph with the diagonal.

$\overline{\text{fix}\, F}$ — the set of eventually fixed points of the function F; $x \in \overline{\text{fix}\, F}$ if there exists an $m \geq 0$ such that $F^m(x)$ is fixed, that is, if $F^{m+1}(x) = F(F^m(x)) = F^m(x)$; note that all the fixed points of F are eventually fixed with $m = 0$, that is, $\text{fix}\, F \subseteq \overline{\text{fix}\, F}$.

Periodic and Eventually Periodic Points:

$\mathrm{per}_n\, F$ the set of periodic points of period n of F; $x \in \mathrm{per}_n\, F$ if $F^n(x) = x$, that is, periodic points are fixed under F^n; alternatively, think of iterating F n times before returning to x as in the orbit

$$x, F(x), F^2(x), \ldots, F^n(x) = x;$$

we say x is of prime period n if n is the smallest integer such that $F^n(x) = x$; note that $\mathrm{per}_1\, F = \mathrm{fix}\, F$ by definition.

$\mathrm{per}_n^m\, F$ a subset of the eventually periodic points of period n of F; $x \in \mathrm{per}_n^m\, F$ if $F^{m+n}(x) = F^m(x)$, that is, eventually periodic points of F are eventually fixed under F^n; in this case, think of iterating F m times before entering the periodic part of the orbit of x, and then n more times to traverse the loop as in

$$x, F(x), F^2(x), \ldots, F^m(x), F^{m+1}(x), \ldots, F^{m+n-1}(x), F^{m+n}(x),$$

where $F^{m+n}(x) = F^m(x)$; the integer m is called the **preperiod** of the orbit of x.

$\overline{\mathrm{per}_n\, F}$ the set of eventually periodic points of period n of F; $x \in \overline{\mathrm{per}_n\, F}$ if there exists an $m \geq 0$ such that $F^m(x)$ is periodic, that is, if $F^{m+n}(x) = F^n(F^m(x)) = F^m(x)$; note that all periodic points are eventually periodic by taking $m = 0$, and that $\overline{\mathrm{per}_1\, F} = \overline{\mathrm{fix}\, F}$; also note that

$$\overline{\mathrm{per}_n\, F} = \bigcup_{m \in \mathbb{N}} \mathrm{per}_n^m\, F,$$

that is, the set of eventually periodic points may include points of arbitrary preperiod.

Basins of Attraction:

$W^s(p)$ the stable set of the periodic point p; $x \in W^s(p)$ if there exists an n such that $F^n(p) = p$ and $F^{kn}(x) \to p$ as $k \to \infty$.

$W^u(p)$ the unstable set of the periodic point p; $x \in W^u(p)$ if there exists an n such that $F^n(p) = p$ and $F^{-kn}(x) \to p$ as $k \to \infty$.

Miscellaneous:

$\to$ approaches; for example, when we write $F^n(x_0) \to 2$ as $n \to \infty$, we mean that the orbit of x_0 gets arbitrarily close to 2 as the number of iterations n gets large.

∞ infinity; a set is countably infinite if it can be put into one-to-one correspondence with the natural numbers, otherwise it is uncountable; the integers and the rationals are countably infinite, but the reals are uncountable, which is meant to suggest that there are more irrational numbers than there are rationals.

$\Rightarrow$ implies; often used in a mathematical derivation to indicate that one step logically follows another.

$:=$ assign to; denotes a process of substitution; used in programming languages and pseudocode algorithms to denote the assignment of a value to a variable; for example, $y := y + 3$ adds 3 to the old value of y and then assigns this sum to the variable y; the old value of y is lost as a result.

Appendix B

Map Index

This is an index to all of the iterated mappings referred to in the text *A First Course in Chaotic Dynamical Systems* by Robert L. Devaney. A **boldfaced** entry is the page number for a figure or table, while an *italicized* number is the page of an exercise. Otherwise the entry is a textual reference to that particular mapping. Superscripts indicate multiple references on the same page or set of pages.

Linear Functions:

Quadratic Functions:

Cubic Mappings:

Complex Functions

Rational Maps:

Exponential Functions:

Trigonometric Functions:

Non-analytic complex maps: