Hedging using futures

Hedging aims at eliminate the risk as much as possible. Perfect hedges are rare, so the strategy is to be as close to perfect as possible. Here, we only look at hedge-and-forget strategies, not dynamical ones.

**Short hedges**
A short hedge involves a short position in futures contracts

**Remark**
- A short hedge is appropriate when the hedger already owns an asset and expects to sell it at some time in the future.
- The strategy is that if the asset looses value, the futures position gains exactly the opposite value, and if the asset gains value, the futures position looses exactly the opposite value, offsetting the variation risk completely.
Hedging using futures

Example
An oil producer has just negotiated to sell 1M barrels in 3 months at the market price at that time. So in these 3 months, he will lose $1,000 for every 1 cent decrease of the price of oil and gain $1,000 for each 1 cent increase of the price of oil.

Suppose today’s price is $80 per barrel and the futures prices for oil in 3 months are $79 per barrel. Each futures is for 1,000 barrels delivery.

He hedges his position by shorting 1,000 futures contracts. 3 months later, the effect is that the price is locked at $79 (or close).

Example (continued)
Assume the price in 3 months is $75. The company gets $75M for the oil and $(79 - 75) \times 1M = 4M$, so in total $79M$.

Assume the price in 3 months is $85. The company gets $85M for the oil and $(79 - 85) \times 1M = -6M$, so in total $79M$.

In all cases, the company receives $79M: there is no uncertainty (i.e. risk) anymore.

Long hedges
A long hedge involves a long position in futures contracts

Remark

• A long hedge is appropriate when a company knows it will have to buy an asset in the future and wants to lock a price now.

• Same principle and effect as the short hedge, but reversed.
Hedging using futures

Arguments for hedging
Most companies are in the business of manufacturing, retailing or providing a service and are not experts in how stocks, interest rates, etc. will evolve. So they want to protect themselves against these risks to have a clear view of what they will be paid: no surprises.

Remark
In practice, many risks are left unhedged.

Hedging using futures

Arguments against hedging
- **Shareholders could do it themselves**: in practice, not really possible, but diversified shareholders might not need it (if for example they have a share of a copper user and a share of a copper producer).
- **Hedging can lead to a worse outcome**: Assume prices of what you sell increase and you hedged: your profit goes up and your shareholders are happy. But if the prices decrease, your profit goes down and shareholders are not happy at all. For those who don’t hedge, their profit stays about constant (think of the case where the price of gold goes up, then the price of gold jewelry will go up as well and the profit stays constant).

Basis risk
In what we saw so far, the hedges were perfect because the delivery dates and the companies dates matched perfectly with no uncertainty, the asset being hedged and the asset underlying the futures are exactly the same, etc...

In practice, it’s most of the time not like this. Some reasons are listed below.
- The asset hedged is different from the futures underlying asset.
- Uncertainty about the dates of buying, selling assets.
- The hedge might require the futures to be closed before maturity.
**Hedging using futures**

**Definition**
The basis is the difference between the spot price of the asset to be hedged and the futures prices of the contracts used to hedge.

![Basis Diagram](attachment:image.png)

**Same assets**

<table>
<thead>
<tr>
<th>Time</th>
<th>Futures ($)</th>
<th>Spot ($)</th>
<th>Basis ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before maturity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maturity</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Different assets**

<table>
<thead>
<tr>
<th>Time</th>
<th>Futures ($)</th>
<th>Spot ($)</th>
<th>Basis ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before maturity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maturity</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Cross hedging**

This is the situation where the asset being hedged is different from the asset underlying the futures contracts.

**Example**

Assume an airline wants to hedge the price of jet fuel. There are no futures for jet fuel, but there are futures for oil. The correlation between the prices of jet fuel and oil is not 100%; there is basis.

**Hedge ratio**

The hedge ratio is the ratio of the size of the position taken in futures contracts to the size of the exposure. When the assets are the same, it is always equal to 1.

**Remark**

When cross hedging, setting the ratio to 1 is not always optimal: a value for the hedge ratio that minimizes the variance of the hedge position should be chosen.
Hedging using futures

Calculating the minimum variance hedge ratio

The MVHR depends on the relationship between changes in the spot and futures prices. Define

\[ \Delta S = \text{change in spot price, } S, \text{ during a period of time equal to the life of the hedge; } \]
\[ \Delta F = \text{change in futures price, } F, \text{ during a period of time equal to the life of the hedge. } \]

Then, the MVHR denoted \( h^* \) is the slope of the best fit line from a linear regression of \( \Delta S \) against \( \Delta F \).

\[ h^* = \frac{\sigma_S}{\sigma_F} \rho \]

where \( \rho \) is the correlation between \( \Delta S \) and \( \Delta F \), \( \sigma_S \) is the standard deviation of \( \Delta S \) and \( \sigma_F \) is the standard deviation of \( \Delta F \).

Example

- If \( \rho = 1 \) and \( \sigma_F = \sigma_S \), \( h^* = 1 \): perfect hedge.
- If \( \rho = 1 \) and \( \sigma_F = 2\sigma_S \), \( h^* = 1/2 \).

Optimal number of contracts to use

\[ N^* = h^* \frac{\text{Size of position being hedged (units)}}{\text{Size of one futures contract (units)}} \]
Interest rates (Part 1)

Types of rates

- Interest rates depend on the credit risk of the borrower.
- Sometimes, IR are expressed in basis points: 1bp = 0.01%.

Two important rates

- Treasury rates: rate at which the government borrows its own currency (treasury bills and bonds). Usually, it is assumed that a government has no credit risk. Hence, these rates can be viewed as risk-free.
- **LIBOR**: London Interbank Offered Rate. Short term borrowing rates (one day to one year). Every business day, the British Banker Association asks different banks to provide quotes of IR at which they could borrow. The top and bottom quarters of the quotes are discarded and the rest is averaged to form the LIBOR.

The risk-free rate

- Derivatives are valued using a portfolio with no risk. The return of that portfolio is called the risk-free rate and plays a key role.
- In practice, this risk-free rate can refer to different rates. Traditionally, the LIBOR is used even if it's not risk-free. Sometimes, government bonds are used.
- In this course, we will just refer to risk-free rate without specifying.
Measuring interest rates: discrete compounding

If the IR offered by a bank is 10% per annum, this statement is ambiguous. How are the interests computed? Are they reinvested or not?

**Annual compounding**

If the IR is measured with annual compounding, the interest is added (and reinvested) after 1 year.

**Example**

IR = 10% per annum compounded annually. $100 grows to $100 \times (1 + 10\%) = $100 \times 1.1 = $110 at the end of 1 year.

Measuring interest rates: discrete compounding

**Semiannual compounding**

If the IR is measured with semiannual compounding, it means that IR/2 is earned every 6 months and reinvested for the remaining 6 months.

**Example**

IR = 10% per annum compounded semiannually. $100 grows to $100 \times (1 + 10\%/2) \times (1 + 10\%/2) = $100 \times 1.05 \times 1.05 = $100 \times 1.05^2 = $110.25 at the end of 1 year.

Measuring interest rates: discrete compounding

**Compounding quarterly**

If the IR is compounded quarterly, it means that IR/4 is earned every 3 months and reinvested.

**Example**

IR = 10% per annum compounded quarterly. $100 grows to $100 \times 1.025 \times 1.025 \times 1.025 \times 1.025 = $100 \times 1.025^4 = $110.38 at the end of 1 year.
Measuring interest rates: discrete compounding

Compounding $m$ times per annum

In general, assume $A$ is invested for $n$ years, with an interest rate $R$ compounded $m$ times per annum. The terminal value of the investment is

$$A \left( \left(1 + \frac{R}{m} \right)^m \right)^n = A \left(1 + \frac{R}{m}\right)^{mn}.$$

Measuring interest rates: continuous compounding

What if we let $m \to \infty$?

Observe that

$$A \left(1 + \frac{R}{m}\right)^{mn} = Ae^{mn \left(1 + \frac{R}{m} \right)} = Ae^{\frac{R}{m} \left(1 + \frac{R}{m} \right)}.$$

As

$$\lim_{m \to \infty} \ln \left(1 + \frac{R}{m} \right) = 1,$$

we get

$$\lim_{m \to \infty} A \left(1 + \frac{R}{m}\right)^{mn} = Ae^{R}.$$

Continuous compounding

Compounding a sum of money continuously for $n$ years at the rate $R$ involves multiplying by $e^{Rn}$. In this course, we compound continuously unless otherwise specified.

Example

IR = 10% per annum compounded continuously. $100 grows to $100 \times e^{0.1} = $110.52 at the end of 1 year.

Calculating equivalent rates

Assume that $R_c$ is a rate with continuous compounding and $R_m$ is an equivalent rate with compounding $m$ times per annum. What is the relation between $R_c$ and $R_m$? We have to solve

$$A e^{R_c m} = A \left(1 + \frac{R}{m}\right)^{mn},$$

which gives

$$R_c = m \ln \left(1 + \frac{R}{m}\right) \quad \text{and} \quad R_m = m \left(e^{\frac{R}{m}} - 1 \right).$$

Equivalent rates

We can always use continuous compounding even if we are in a discrete compounding framework by computing the equivalent continuously compounded rate.
Determination of forward and futures prices

Short selling

Short selling
Shorting an asset means selling an asset that is not owned.

Example
Suppose a trader wants to short 500 shares of a stock. The steps to do so are
1. borrowing the shares from someone who owns them and sell them.
2. pay the lender any income that the stock pays (e.g. dividends).
3. close out the position by buying 500 shares of the stock later and return them to the lender.

Remark
Sometimes a fee is charged for lending the shares to the party doing the shorting.

Forward price of investment assets

Consider a no income investment asset. Denote its today price by $S_0$, $F_0$ the forward price today on this asset, $T$ the maturity of the forward and $r$ the risk-free rate.

Forward price of a no income asset
The relationship between $F_0$ and $S_0$ is
\[ F_0 = S_0 e^{rT}. \]

Remark
This conclusion is obtained by the arbitrage argument we have seen during Lecture 1.
Forward price of investment assets

Consider an investment asset that will provide an income with present value \( I \) during the life of the forward contract.

**Forward price of a known income asset**

The relationship between \( F_0 \) and \( S_0 \) is then

\[
F_0 = (S_0 - I)e^{rT}.
\]

**Arbitrage argument**

Consider a forward contract with maturity 9 months on a stock that will pay $40 after 4 months (dividend) and is currently worth $900. Suppose the 4 months and 9 months risk-free rates are 3\% and 4\%, respectively. We know that the forward price should be

\[
(S_0 - Fe^{-rT})e^{rT} = 886.60.
\]

Forward price of investment assets

<table>
<thead>
<tr>
<th>Forward Price = $910</th>
<th>Forward Price = $870</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Action now</strong></td>
<td><strong>Action now</strong></td>
</tr>
<tr>
<td>Borrow $900, $39.60 for 4 months and $860.40 for 9 months</td>
<td>Short 1 unit of asset to get $900</td>
</tr>
<tr>
<td>Buy 1 unit of asset</td>
<td>Invest $39.60 for 4 months and $860.40 for 9 months</td>
</tr>
<tr>
<td>Enter into forward contract to sell asset in 9 months for $910</td>
<td>Enter into a forward contract to buy asset in 9 months for $870</td>
</tr>
<tr>
<td><strong>Action in 4 months</strong></td>
<td><strong>Action in 4 months</strong></td>
</tr>
<tr>
<td>Receive $40 of income on asset</td>
<td>Receive $40 from 4 months investment</td>
</tr>
<tr>
<td>Use $40 to repay first loan with interest</td>
<td>Pay income of $40 on asset</td>
</tr>
<tr>
<td><strong>Action in 9 months</strong></td>
<td><strong>Action in 9 months</strong></td>
</tr>
<tr>
<td>Sell asset for $910</td>
<td>Receive $886.60 from 9 months investment</td>
</tr>
<tr>
<td>Use $886.60 to repay second loan with interest</td>
<td>Buy asset for $870 and close out short position</td>
</tr>
<tr>
<td><strong>Profit realized</strong> = $23.40</td>
<td><strong>Profit realized</strong> = $16.60</td>
</tr>
</tbody>
</table>

Forward price of investment assets

Consider the case where the investment asset provides a known yield rather than a known income during the life of the forward contract. Denote this yield by \( q \).

**Forward price of a known yield asset**

The relationship between \( F_0 \) and \( S_0 \) is then

\[
F_0 = S_0 e^{(r - q)T}.
\]

**Remark**

This formula can be deduced from the known income case, as the income from a yield \( q \) on \( S_0 \) is worth \( I = S_0 (1 - e^{-qT}) \) today. By applying the known income formula, we get

\[
(S_0 - S_0 (1 - e^{-qT})) e^{rT} = S_0 e^{(r - q)T}.
\]
Valuing forward contracts

The value of a forward contract at the time it is first entered into is zero. At a later stage, it may have a positive or negative value. It is important to know how to value the contract to keep track every day of the intrinsic value of a position (marking to market).

Suppose that $K$ is the delivery price for a forward that was negotiated some time ago, the delivery date is $T$ years from today, and $r$ is the $T$ risk-free rate. $F_0$ denotes the forward price that would be applicable if negotiated today. Let $f$ be the value of the forward contract today.

### Value of a forward contract

Let the above notation prevail. Then, for a long position,

$$f = (F_0 - K)e^{-rT}$$

and for a short position,

$$f = (K - F_0)e^{-rT}.$$
Are forward prices and futures prices equal?

The case of deterministic interest rates
If the risk-free interest rate is a known function of time, then an arbitrage argument can be used to show that forward and futures price are the same (provided their delivery dates are the same).

The case of random interest rates
When interest rates vary unpredictably (as they do in the real world), forward and futures prices are in theory no longer the same. Consider a situation where the price of the underlying asset $S$ is positively correlated with interest rates. When $S$ increases, a long position in futures makes an immediate gain due to the daily settlement procedure. The gain can then be reinvested in a likely higher interest rate due to correlation. A long position in forwards is not affected daily by these variations and cannot benefit from reinvesting using the higher interest rate, and should hence have a different price.

Properties of stock options

Factors affecting options prices

There are 6 factors affecting the price of a stock option:

- The current stock price, $S_0$
- The strike price $K$
- The maturity $T$
- The volatility of the stock price $\sigma$
- The risk-free interest rate $r$
- The dividends that are expected to be paid.

We will consider what happens to option prices when there is a change in one of these factors, with all the others remaining fixed.
Factors affecting options prices

**Stock price and strike price**
Recall that the payoff of a call option is \((S_T - K)^+\). Hence, a call option becomes more valuable if the stock price increases and less valuable if the strike price increases.

Similarly, as that the payoff of a put option is \((K - S_T)^+\), it becomes less valuable if the stock price increases and more valuable if the strike price increases.

**Maturity**
American options become more valuable as \(T\) increases: consider two investor holding 2 options that only differ by their maturity. Then the holder of the long-life option has the same exercise opportunities as the holder of the short-life option, and more. Hence, the price of the long-life option is necessarily higher.

For European options, it is also usually the case, unless a situation where a big dividend is paid in between two close enough maturities, making the stock price decreases and in that case, call options could be worth less with a longer maturity.

**Volatility**
The volatility of a stock price is a measure of how uncertain we are about future stock price movements. As volatility increases, the chance that the stock will do very well or very poorly increases. For the owner of the stock, these two outcomes tend to offset each other, but not for the owner of a call or put option.

The owner of a call benefits from price increases, but has limited exposure to price decrease. Similarly, the owner of a put benefits from price decreases, but has limited exposure to price increases. So the price of the options increase as the volatility increases.
Factors affecting options prices

Risk-free interest rate
As interest rates increase, the expected return from the stock impact of these two effects is to increase the price of calls and decrease the price of puts (less intuitive).

Amount of future dividends
Dividends have the effect of reducing the stock price on the dividend payment day. This decreases the value of calls and increases the value of puts.

Summary

<table>
<thead>
<tr>
<th>Variable</th>
<th>European call</th>
<th>European put</th>
<th>American call</th>
<th>American put</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current stock price</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Strike price</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Maturity</td>
<td>?</td>
<td>?</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Volatility</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Risk-free rate</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Amount of dividends</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

Assumptions and notation

Assumptions
• No transaction costs.
• Borrowing and lending are possible at the risk-free rate.
• No arbitrage opportunities.

Notation
• $S_0$: Current stock price.
• $K$: Strike price of the option.
• $T$: Maturity of the option.
• $r$: risk-free rate for maturity $T$.
• $C$: Value of American call option.
• $P$: Value of American put option.
• $c$: Value of European call option.
• $p$: Value of European call option.
Upper and lower bounds for option prices

Upper bounds

- A call option can never be worth more than the stock:
  \[ C \leq S_0 \quad \text{and} \quad c \leq S_0. \]
  Otherwise, an arbitrageur could make a riskless profit by buying the stock and selling the call option.
- An American put option can never be more than the strike and a European put option can not be more than the strike at maturity, so it cannot be more than the present value of the strike:
  \[ P \leq K \quad \text{and} \quad p \leq Ke^{-rT}. \]
  Otherwise, an arbitrageur could make a riskless profit by writing the put option and placing the proceeds at the risk-free rate.

Lower bound for European calls on no dividend stocks

A lower bound for the price of a European call on a no dividend stock is
\[ S_0 - Ke^{-rT}. \]

Arbitrage argument

Consider two portfolios:
- Portfolio A: a European call and \( Ke^{-rT} \) dollars placed at the risk-free rate until \( T \).
- Portfolio B: One share of the stock.

If \( S_T > K \), the value of portfolio A at \( T \) is
\[ (S_T - K) + K = S_T. \]

As option prices are always positive,
\[ c \geq \max(S_0 - Ke^{-rT}, 0). \]
Upper and lower bounds for option prices

Lower bound for European put on no dividend stocks

A lower bound for the price of a European call on a no dividend stock is

\[ Ke^{-rT} - S_0. \]

Arbitrage argument

Same reasoning as before with the portfolios

- Portfolio C: a European put and one share of the stock.
- Portfolio D: \( Ke^{-rT} \) dollars placed at the risk-free rate until \( T \).