Mathematics of Financial Derivatives

Lecture 9

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1. Forward rates

2. Forward rate agreements

3. Mechanics of interest rate swaps
Forward rates
Forward rates

**Definition: forward rates**

Forward rates are the future rates of interest implied by current zero rates for periods of time in the future.

**Example**

Suppose the zero curve is as follows.

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>Zero rate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.0</td>
</tr>
<tr>
<td>2</td>
<td>4.0</td>
</tr>
</tbody>
</table>

If we invest $100 for 2 years, we get $100e^{0.04 \times 2} = $108.33. But we could equivalently invest $100 for 1 year, and reinvest the proceeds for 1 more year.
Example (continued)

In the absence of arbitrage opportunities, these two approaches have to yield the same result. So we can compute the 1-year forward rate in 1 year by solving

\[100e^{0.03 \times 1} e^{R_F \times 1} = 100e^{0.04 \times 2},\]

which yields \(R_F = 5\%\).
Forward rates

**Remark**
Observe that the overall rate (4%) is just the average of the intermediate rates over the whole period (only valid is the compounding is continuous).

**General case**
In general, if $R_1$ and $R_2$ are the zero rates for maturities $T_1$ and $T_2$, respectively, and $R_F$ is the forward rate for the period of time between $T_1$ and $T_2$, then it must hold that

$$e^{R_1 \times T_1} e^{R_F \times (T_2 - T_1)} = e^{R_2 \times T_2},$$

so that

$$R_F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}.$$
General case (continued)

The general situation corresponds to the following diagram.

Observe that we can rewrite the equation for $R_F$ as

$$R_F = R_2 + (R_2 - R_1) \frac{T_1}{T_2 - T_1}.$$
Forward rates

General case (continued)

Taking the limit as $T_2$ goes to $T_1$, we get

$$R_F = R + T \frac{\partial R}{\partial T},$$

where $R$ is the zero rate for maturity $T$. The value of $R_F$ obtained this way is known as the instantaneous forward rate for a maturity $T$. This is the forward rate that is applicable to a very short future time period that begins at time $T$.

Recall that $P(0, T) = e^{-RT}$, so that we also have

$$R_F = - \frac{\partial}{\partial T} \ln P(0, T).$$
**Locking forward rates**

If a financial institution can borrow or lend at the zero rates, it can lock in the forward rates. Suppose once again that we have the following zero curve.

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For example, it can borrow $100 at 3\% for 1 year and invest these $100 at 4\% for 2 years, the result is a cash outflow of $100e^{0.03\times1} = $103.05 at the end of year 1 and an inflow of $100e^{0.04\times2} = $108.33 at the end of year 2.
Locking forward rates (continued)

Since $108.33 = 103.05e^{0.05\times1}$, a return equal to the forward rate (5%) is earned on $103.05$ during the second year.

If an investor thinks that rates in the future will be different from today’s forward rates, there are many trading strategies that the investor will find attractive.

One of these involves entering into a contract known as a forward rate agreement.
Forward rate agreements
Definition: forward rate agreement (FRA)

A forward rate agreement (FRA) is an over-the-counter transaction designed to fix the interest rate that will apply to either borrowing or lending a certain principal during a specified future period of time.

The usual assumption underlying the contract is that the borrowing or lending would normally be done at LIBOR.

If the agreed fixed rate is greater than the actual LIBOR rate for the period, the borrower pays the lender the difference between the two applied to the principal. If the reverse is true, the lender pays the borrower the difference applied to the principal.
Example

Suppose that a company enters into a FRA that is designed to ensure it will receive a fixed rate of 4% on a principal of $100 million for a 3-month period starting in 3 years. The FRA is an exchange where LIBOR is paid and 4% is received for the 3 months period. If the 3-month LIBOR proves to be 4.5% for the 3-month period, the cash flow to the lender will be

\[
100,000,000 \left[ \left( 1 + \frac{0.04}{4} \right)^{\frac{4\frac{1}{4}}{4}} - \left( 1 + \frac{0.045}{4} \right)^{\frac{4\frac{1}{4}}{4}} \right]
\]

\[
= 100,000,000 \times (0.04 - 0.045) \times 0.25
\]

\[
= -$125,000
\]

at the 3.25-year point.
**Example (continued)**

This is equivalent to a cash flow of

\[
- \frac{125,000}{1 + 0.045 \times 0.25} = -$123,609
\]

at the 3-year point. The cash flow to the party on the opposite side of the transaction will be +$125,000 at the 3.25-year point or +$123,609 at the 3-year point.

**Remark**

Observe that all interest rates in this example are expressed with quarterly compounding. For simplifying calculations, we will always assume the compounding is at the same frequency as the specified future period of time for which the FRA applies.
Forward rates agreements

General case

Consider a FRA where company X is agreeing to lend money to company Y for the period of time between \( T_1 \) and \( T_2 \). Define:

- \( R_K \): The fixed rate of interest agreed to in the FRA.
- \( R_F \): The forward LIBOR interest rate for the period between times \( T_1 \) and \( T_2 \), calculated today.
- \( R_M \): The actual LIBOR rate observed in the market at time \( T_1 \) for the period between times \( T_1 \) and \( T_2 \).
- \( L \): The principal underlying the contract.

We will leave aside our usual assumption of continuous compounding and assume the rates \( R_K \), \( R_F \) and \( R_M \) are all measured with a compounding frequency reflecting the length of the period to which they apply. This means that compounding frequency \( = T_2 - T_1 \). This is the usual market practice for FRA).
Forward rates agreements

Remark

- We will leave aside our usual assumption of continuous compounding and assume the rates $R_K$, $R_F$ and $R_M$ are all measured with a compounding frequency reflecting the length of the period to which they apply.

- This means that if $T_2 - T_1 = 0.5$, they are expressed with semiannual compounding, if $T_2 - T_1 = 0.25$, they are expressed with quarterly compounding, and so on. This is the usual market practice for FRA.

- This allows to simplify formulas and calculations, and is not really restrictive as we know that we can always compute an equivalent rate from any frequency to any other frequency.
General case (continued)

Normally, company X would earn $R_M$ from the LIBOR loan. The FRA means that it will earn $R_K$. The extra interest rate (which may be negative) that it earns as a result of entering into the FRA is $R_K - R_M$.

The interest rate is set at time $T_1$ and paid at time $T_2$. The extra interest rate therefore leads to a cash flow to company X at time $T_2$ of

$$L(R_K - R_M)(T_2 - T_1).$$

Similarly, there is a cash flow to company Y at time $T_2$ of

$$L(R_M - R_K)(T_2 - T_1).$$
**Remark**

We see that there is another interpretation of the FRA. It is an agreement where company X will receive interest on the principal between $T_1$ and $T_2$ at the fixed rate $R_K$ and pay interest at the realized LIBOR rate of $R_M$.

Company Y will pay interest on the principal between $T_1$ and $T_2$ at the fixed rate of $R_K$ and receive interest at $R_M$.

This interpretation of a FRA will be very important when we talk about interest rate swaps later.
General case (continued)

FRA are usually settled at time $T_1$, even though interest rate payment normally occur at time $T_2$. This means that we must discount the payoff from time $T_2$ to time $T_1$.

So, for company X, the payoff at time $T_1$ is

$$\frac{L(R_K - R_M)(T_2 - T_1)}{1 + R_M(T_2 - T_1)}$$

and, for company Y, the payoff at time $T_1$ is

$$\frac{L(R_M - R_K)(T_2 - T_1)}{1 + R_M(T_2 - T_1)}.$$
Valuation of FRA

A FRA is worth 0 when the fixed rate $R_K$ equals the forward rate $R_F$.

When it is first entered into, $R_K$ is set equal to the current value of $R_F$, so that the value of the contract to each side is 0.

As time passes, interest rates change, so that the value is no longer 0.

Remark

The market value of a derivative at a particular time is referred to as its mark-to-market, or MTM, value.
Valuation of FRA (continued)

To calculate the MTM value of a FRA where the fixed rate of interest is being received, we imagine a portfolio consisting of 2 FRAs.

- The first FRA states that $R_K$ will be received on a principal of $L$ between times $T_1$ and $T_2$.
- The second FRA states that $R_F$ will be paid on a principal of $L$ between times $T_1$ and $T_2$.

The payoff from the first FRA at time $T_2$ is

$$L(R_K - R_M)(T_2 - T_1)$$

and the payoff of the second FRA at time $T_2$ is

$$L(R_M - R_F)(T_2 - T_1).$$
Valuation of FRA (continued)

The total payoff is

\[ L(R_K - R_F)(T_2 - T_1) \]

and is known for certain today. The portfolio is therefore a risk-free investment and its value today is the payoff at time \( T_2 \) discounted at the risk-free rate, or

\[ L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2}, \]

where \( R_2 \) is the continuously compounded riskless zero rate for a maturity \( T_2 \).

Because the value of the second FRA, where \( R_F \) is paid, is zero, the value of the first FRA, where \( R_K \) is received, must be

\[ V_{FRA} = L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2}. \]
Valuation of FRA (continued)

Similarly, the value of a FRA where $R_K$ is paid is

$$V_{FRA} = L(R_F - R_K)(T_2 - T_1)e^{-R_2 T_2}.$$ 

General FRA valuation procedure

To sum up, we have seen that a FRA can be valued by

- Calculate the payoff on the assumption that forward rates are realized (that is, on the assumption that $R_M = R_F$).
- Discount this payoff at the risk-free rate.
Example
Suppose the forward LIBOR rate for the period between time 1.5 years and time 2 years in the future is 5% (with semiannual compounding) and that some time ago a company entered into a FRA where it will receive 5.8% (with semiannual compounding) and pay LIBOR on a principal of $100 million for the period.

The 2-year risk-free interest rate is 4% (with continuous compounding).

The value of the FRA is hence

\[100,000,000 \times (0.058 - 0.050) \times 0.5e^{-0.04 \times 2} = $369,200.\]
Mechanics of interest rate swaps
**Definition: interest rate swap**

In an interest rate swap, one company agrees to pay to another company cash flows equal to interest at a predetermined fixed rate on a notional principal for a predetermined number of years. In return, it receives interest at a floating rate on the same notional principal for the same period of time from the other company.

**Remark**

The floating rate in most interest rate swaps is the LIBOR. As is often done in practice, we will also use the LIBOR as being the risk-free rate with which we discount certain cash flows.

We will refer to a swap where LIBOR is exchanged for a fixed rate of interest as a **LIBOR-for-fixed** swap.
Example
Consider a 3-year swap initiated on March 5, 2014, between Microsoft and Intel. We suppose Microsoft agrees to pay Intel an interest rate of 5% per annum on a principal of $100 million, and in return Intel agrees to pay Microsoft the 6-month LIBOR rate on the same principal.

We assume the agreement specifies that payments are to be exchanged every 6 months and that the 5% interest rate is quoted with semiannual compounding.
Example (continued)
The first exchange of payments would take place on September 5, 2014 (6 months after the start date).

Microsoft would pay Intel $2.5 million (the interest on the $100 million principal for 6 months at 5%).

Intel would pay Microsoft interest on the $100 million principal at the 6-month LIBOR rate prevailing 6 months prior to September 5, 2014 – that is, on March 5, 2014. Suppose it was 4.2%. Intel pays Microsoft
\[0.5 \times 0.042 \times \$100 = \$2.1 \text{ million}.\]

Remark
Note that there is no uncertainty about this first exchange of payments because it is determined by the LIBOR rate at the time the swap begins.
Example (continued)

The second exchange of payments would take place on March 5, 2015 (1 year after the start date).

Microsoft would pay Intel $2.5 million (the interest on the $100 million principal for 6 months at 5%).

Intel would pay Microsoft interest on the $100 million principal at the 6-month LIBOR rate prevailing 6 months prior to March 5, 2015 – that is, on September 5, 2014. Suppose it was 4.8%. Intel pays Microsoft $2.4 million.

\[ 0.5 \times 0.048 \times 100 = 2.4 \text{ million}. \]
Remarks

- In total, there are six exchanges of payments on the swap. The fixed payments are always $2.5 million. The floating-rate payments on a given payment date are calculated using the 6-month LIBOR rate prevailing 6 months before the payment date.

- A swap is generally structured so that one side remits the difference between the two payments to the other side. In our example, Microsoft would pay Intel $0.4 million on September 5, 2014, and $0.1 million on March 5, 2015.
Example (continued)

Cash flows (millions of dollars) to Microsoft in a $100 million 3-year interest rate swap when a fixed rate of 5% is paid and LIBOR is received.

<table>
<thead>
<tr>
<th>Date</th>
<th>6-months LIBOR rate (%)</th>
<th>Floating cash flow received</th>
<th>Fixed cash flow paid</th>
<th>Net cash flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mar. 5, 2014</td>
<td>4.20</td>
<td></td>
<td></td>
<td>-0.40</td>
</tr>
<tr>
<td>Sep. 5, 2014</td>
<td>4.80</td>
<td>+2.10</td>
<td>-2.50</td>
<td>-0.10</td>
</tr>
<tr>
<td>Mar. 5, 2015</td>
<td>5.30</td>
<td>+2.40</td>
<td>-2.50</td>
<td>+0.15</td>
</tr>
<tr>
<td>Sep. 5, 2015</td>
<td>5.50</td>
<td>+2.65</td>
<td>-2.50</td>
<td>+0.25</td>
</tr>
<tr>
<td>Mar. 5, 2016</td>
<td>5.60</td>
<td>+2.75</td>
<td>-2.50</td>
<td>+0.30</td>
</tr>
<tr>
<td>Sep. 5, 2016</td>
<td>5.90</td>
<td>+2.80</td>
<td>-2.50</td>
<td>+0.45</td>
</tr>
<tr>
<td>Mar. 5, 2017</td>
<td>5.90</td>
<td>+2.95</td>
<td>-2.50</td>
<td></td>
</tr>
</tbody>
</table>
Example (continued)

Observe that the principal is not exchange, and only used for calculating interest payments. For this reason, it is called the **notional principal**, or just **notional**.

If the notional were exchanged at the end of the life of the swap, the nature of the deal would not be changed in any way.

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<tr>
<td>Mar. 5, 2017</td>
<td><strong>+102.95</strong></td>
<td>-102.50</td>
<td></td>
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</table>
Mechanics of interest rate swaps

Some remarks

When we assume that the notional is exchanged at the end of the life of the swap, it provides an interesting way of viewing the swap.

- The cash flows in the third column of the previous table are the cash flows from a long floating position in a floating-rate bond.
- The cash flows in the fourth column of the table are the cash flows from a short position in a fixed-rate bond.

The table shows that the swap can be regarded as the exchange of a fixed-rate bond for a floating-rate bond.

In that case, Microsoft is long a floating-rate bond and short a fixed-rate bond. Intel is long a fixed-rate bond and short a floating-rate bond.
Using the swap to transform a liability

For Microsoft, the swap could be used to transform a floating-rate loan into a fixed-rate loan. Suppose that Microsoft has arranged to borrow $100 million at LIBOR + 0.1%. After Microsoft has entered into the swap, it has the following 3 sets of cash flows:

- It pays LIBOR + 0.1% to its outside lenders.
- It receives LIBOR under the terms of the swap.
- It pays 5% under the terms of the swap.

These 3 sets of cash flows net out to an interest rate payment of 5.1%. Thus, for Microsoft, the swap could have the effect of transforming borrowings at a floating rate of LIBOR + 0.1% into borrowings at a fixed rate of 5.1%.
Using the swap to transform a liability (continued)

For Intel, the swap could have the effect of transforming a fixed-rate loan into a floating-rate loan. Suppose that Intel has a 3-year $100 million loan outstanding on which it pays 5.2%. After it has entered into the swap, it has the following 3 sets of cash flows:

- It pays 5.2% to its outside lenders.
- It pays LIBOR under the terms of the swap.
- It receives 5% under the terms of the swap.

These 3 sets of cash flows net out to an interest rate payment of LIBOR + 0.2%. Thus, for Intel, the swap could have the effect of transforming borrowings at a fixed rate of 5.2% into borrowings at a floating rate of LIBOR + 0.2%.
Using the swap to transform a liability (continued)

We can summarize this situation with the following diagram.