Final Exam; Math 561
December 18, 2000

This exam is a closed book, no notes, no "crib sheets" exam. Calculators are permitted. There are 6 problems on the exam – don't overlook those on the back of the page. The highest possible number of points for each part of each problem is indicated. Good luck!

1. Find the solution of the initial value problem: (14 points)

$$2\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} = 0\ , \quad u(x,0) = \sin x\ .$$

Please show your work – don't just write down the answer.

2. List at least two ways in which solutions of the wave equation and the heat equation are qualitatively different from each other. (16 points)

3. State the maximum principle for solutions of Laplace's equation. (8 pts.)

4. Consider the function

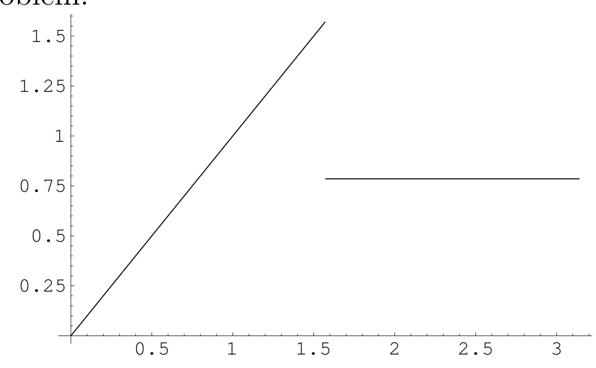
$$f(x) = \begin{cases} x & 0 \le x < \pi/2 \\ \pi/4 & \pi/2 \le x \le \pi \end{cases}$$

whose graph is shown below.

(a) At what points in the interval $0 \le x \le \pi$ does the Fourier **sine** series of f converge to $f(x)$? To what value does the Fourier **sine** series converge at those points where it fails to converge to $f(x)$? (8 pts.)

(b) Repeat part (a) using the Fourier **cosine** series instead of the Fourier sine series. (8 pts.)

Hint: Do not try to compute the Fourier series of f. That is not necessary to solve the problem.



5. Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u = 0 \quad , \quad 0 < x < \pi \ , \ 0 < y < \pi$$
$$u(0, y) = u(\pi, y) = u(x, 0) = 0 \quad , \quad u(x, \pi) = f(x) \ .$$

(a) Assume $u(x, y) = X(x)Y(y)$. Separate variables and derive the differential equations satisfied by X and Y. What are the boundary conditions on the X equation? (10 pts.)

(b) What are the eigenvalues and eigenfunctions of the X equation? (8 pts.)

(c) What is the general form of the solution if you ignore the boundary condition $u(x, \pi) = f(x)$? (8 pts.)

(d) Use the condition $u(x, \pi) = f(x)$ to write down a formula for the constants in the general solution. (8 pts.)

6. Suppose $u(x, r)$, $0 \le x \le \pi$, $r \ge 0$ satisfies the partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} = -r^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \quad , \quad u(0, r) = u(\pi, r) = 0 \ .$$

Use the method of separation of variables to find the general form of the solution of this problem, assuming that the solution is finite as r approaches zero. **You do not have to solve for the constants in the general solution.** (12 points)