Midterm 1; Math 561
October 22, 2004

This exam is a closed book, no notes, no "crib sheets" exam. Calculators are permitted. The number of points that each problem is worth is printed next to the problem. There are 5 problems on the exam – **don't overlook the problems on the back of the page.** Good luck!

1. Solve the first order equation

$$3\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

subject to the initial condition

$$u(x,0) = \frac{1}{1+x^2} \ .$$

You must show your work for this problem - don't just write down your answer. (18 points)

2. Consider the partial differential equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - t^2 u \ ; \quad 0 < x < \pi \ , \quad t > 0 \\ &\frac{\partial u}{\partial x}(0,t) = 0 \ , \ u(\pi,t) = 0 \ , \\ &u(x,0) = \phi(x) \end{aligned}$$

(a) Use the method of separation of variables to derive the equations and boundary conditions satisfied by the x and t dependent parts of the solution. **You do not have to solve these equations!** (18 points)

(b) Is $\lambda = 0$ an eigenvalue of this problem? More precisely, is there a non-zero solution of the equations you derived in part (a) of this problem if $\lambda = 0$? (10 points)

3. In one of the homework problems you proved that the energy of a solution of the damped wave equation decreases. Consider a solution of the heat equation with Dirichlet boundary conditions:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \ , \quad 0 < x < \ell \\ &u(0,t) = u(\ell,t) = 0 \ . \end{aligned}$$

Define the energy of a solution to be

$$\mathcal{E}[u](t) = \frac{1}{2}\int_0^\ell (u(x,t))^2 dx \ .$$

Show that $\mathcal{E}[u](t)$ is a non-increasing function of time. (**Hint:** Show that the time derivative of $\mathcal{E}[u](t)$ is always less than or equal to zero.) (18 points)

4. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \ , \quad 0 < x < \ell$$
$$\frac{\partial u}{\partial x}(0,t) = 0 \ , \quad \frac{\partial u}{\partial x}(\ell,t) = 1 \ .$$

In light of our discussion of the physical meaning of these boundary conditions, would you expect the solution $u(x,t)$ of this problem to approach a steady state as t tends toward infinity? Why or why not? You don't need to actually compute the steady state – I'm interested in knowing whether or not you think it exists and why. (18 points)

5. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} \ , \ -\infty < x < \infty \ ,$$

with initial conditions $u(x,0) = \phi(x)$, and $\frac{\partial u}{\partial t}(x,0) = \psi(x)$ where,

$$\phi(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq 1 \\ 2 \text{ if } 5 \leq x \leq 6 \\ 0 \text{ otherwise} \end{cases}$$

and

$$\psi(x) = 0 \ .$$

Is $u(3,1/2) = 0$? That is, is the value of the solution $u(x,t)$ with the given initial conditions zero at the point $(x,t) = (3,1/2)$? Please explain your answer. (**Hint:** You should be able to answer this question without long and involved calculations.) (18 points)