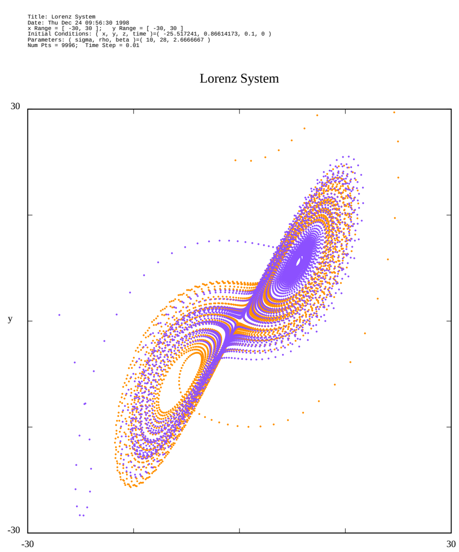
Lecture 1; Attractors[1]

Much of this course will be devoted to understanding the behavior of solutions of differential equations. If one asks "how do solutions of differential equations behave", the sad answer is, "to a first approximation, in every possible way". This doesn't bode well for the task of making sense of solutions of such equations, but there is a saving grace. For many differential equations, after an initial period of more or less arbitrary behavior (a transient) the equation settles down to a behavior which is, if not necessarily "predictable" in the classical sense, is at least "describable". This motion may still be much more complicated that the fixed points and limit cycles we have studied so far in this course but nonetheless we'll see in this module that one can still develop tools to describe these very chaotic motions.

As the following figure shows, even the very simple set of equations

$$\begin{aligned} \dot{x} &= \sigma(x-y) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta z + xy \end{aligned}$$

known as the Lorentz equations can exhibit very complicated behavior. (ρ, β, and σ are all



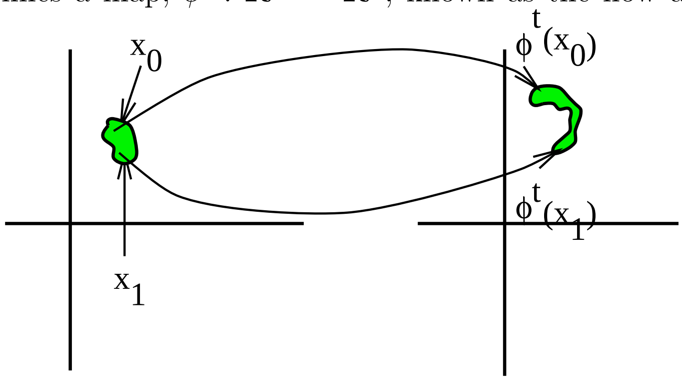
parameters.)

From the pictures of trajectories of the Lorenz equation, we see that if we are only concerned with the long-time behavior of the system, it suffices to restrict our attention to a small part of the entire phase space. This leads us to the notion of an attracting set. To define what we mean by an attracting set, I first introduce the notion of a flow. Given a differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x}(0) = \mathbf{x}_0 \ , \quad \mathbf{x} \in \mathbf{R}^n \tag{1}$$

[1]These lecture notes are for the course MA 574, and are being prepared for the instructor's personal use. They are not for public distribution.

the solution $\mathbf{x}(t)$ tells us the location at time t of the point initially at $\mathbf{x}_0$. – *i.e.* (1) defines a map, $\phi^t : \mathbf{R}^n \to \mathbf{R}^n$, known as the flow associated with the differential equation (1).



For each time "t", the differential equation defines a mapping of the phase space to itself -- the flow, ϕ^t.

Definition 1.1 *We say that $\mathcal{A}$ is an attracting set for the flow ϕ^t, if*

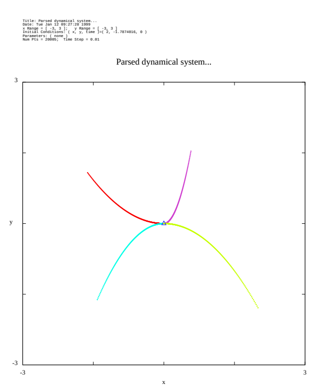
- $\phi^t(\mathcal{A}) = \mathcal{A}$ *for all* $t \geq 0$,
- *There exists an open set $\mathcal{O} \supset \mathcal{A}$, such that $\phi^t(\mathcal{O}) \subset \mathcal{O}$ for t sufficiently large.*

Example 1.2 *Consider the system of equations*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x \\ -2y \end{pmatrix} \tag{2}$$

In this case, the flow is

$$\phi^t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-t}x \\ e^{-2t}y \end{pmatrix}$$



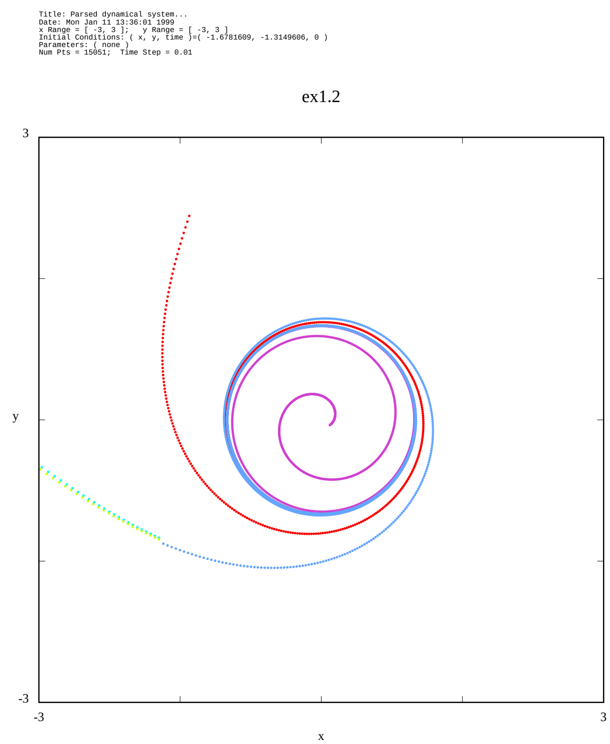
and hence the origin is an attracting set.

Example 1.3 *Consider the system of equations*

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} r(1-r) \\ 2 \end{pmatrix} \tag{3}$$

In this case, the flow is

$$\phi^t \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r/(r - e^{-t}(r-1)) \\ \theta + 2t \end{pmatrix}$$



In this case, the attracting set is the circle $r = 1$.

Remark 1.4 *Although we can derive explicit formulas for the flows in these examples, those of you who have taken MA 573 probably recognize that it would be easier to analyze these examples qualitatively.*

One thing which will be important for discussing actual applications is to understand which initial conditions lead to solutions that actually approach the attractor. This is called the basin of attraction of the attracting set.

Definition 1.5 *The basin of attraction,* $\mathcal{B}$*, of an attracting set* $\mathcal{A}$ *is the set of all points which approach* $\mathcal{A}$ *under the action of the flow. More precisely,* $x \in \mathcal{B}$ *if* $dist(\phi^t(x), \mathcal{A}) \to 0$ *as* $t \to \infty$.

Remark 1.6 *What is the basin of attraction for the limit cycle in Example 1.3?*

1.1 Attractors vs. Attracting Sets

From our definition we see that for any initial condition x_0 in the basin of attraction of an attracting set $\mathcal{A}$, $\phi^t(x_0)$ will eventually lie on, or at least near, $\mathcal{A}$. However, there *may* be some smaller subset of $\mathcal{A}$ to which "most" trajectories are attracted. Following [1] we will say that the *attractor*, $\tilde{\mathcal{A}} \subset \mathcal{A}$ is the set which "typical" trajectory approaches as $t \to \infty$.

In our two previous examples, the attractor and the attracting set coincided. However, the following example from [1] shows that this need not be the case.

Example 1.7 *Consider the pair of differential equations*

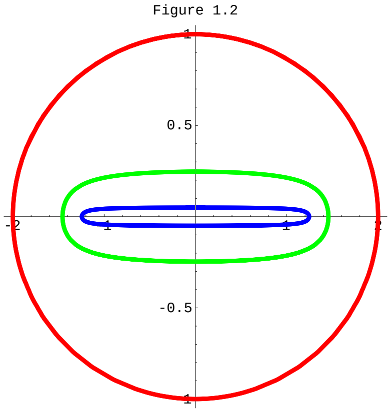
$$\begin{aligned} \dot{x} &= x(1-x^2) && (4) \\ \dot{y} &= -y \ . && (5) \end{aligned}$$

In this case we can solve the equations explicitly and we find

$$x(t) = 1/\sqrt{1 - c_1 e^{-t/2}} \ , \quad y(t) = c_2 e^{-t} \ ,$$

where c_1 and c_2 are constants determined by the initial conditions. As the diagram below shows, the attracting set is the entire interval $[-1, 1]$, but "most" trajectories go to either ± 1.

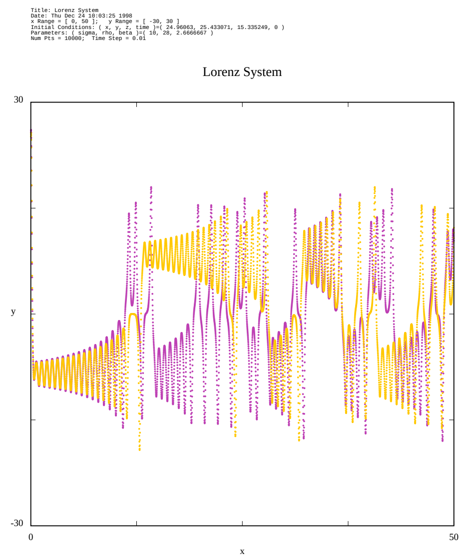
Figure 1.2



It turns out that it's relatively difficult to give a mathematically precise definition of an attractor (see [2], for instance), so we'll adopt the more heuristic approach that an attractor is that part of the attracting set toward "most" trajectories tend.

While the attractors we looked at above were very simple objects, the pictures of the attractor in the Lorenz equations look as if very complicated sets can arise as attractors of even simple systems of equations. Part of our goal in these lectures is to develop tools to describe the motion of dynamical systems on such complicated sets.

Remark 1.8 *There is an important distinction to keep in mind here (or a pitfall to avoid). Although most trajectories in the Lorenz equations seem to approach the same set in phase space (the attractor) this does not mean that the solutions themselves will approach each other as the*



figures below show.

Even solutions whose initial conditions are quite close to one another will typically separate rapidly with time. This is one of the hallmarks of "chaotic" behavior of differential equations and is known as sensitive dependence on initial conditions.

In order to understand better the behavior of complicated attractors (or attracting sets) we begin by looking at what determines the stability, or instability, of simpler attracting sets.

Case 1: Attracting set is a fixed point:

Let's start with an exactly solvable situation.

$$\dot{\mathbf{x}} = A\mathbf{x} \ , \tag{6}$$

where $\mathbf{x} \in \mathbf{R}^n$ and A is an $n \times n$ matrix. The origin $\mathbf{x} = 0$ is a fixed point for this differential equation (or for the flow defined by this differential equation) and we want to determine if it is an attractor. For simplicity, assume that A has n distinct eigenvalues (this is the *generic* situation). Let $\lambda_1, \ldots, \lambda_n$ be these eigenvalues, and $\mathbf{v}^1, \ldots, \mathbf{v}^n$ the corresponding eigenvectors. Then the general solution of (6) is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}^1 + c_2 e^{\lambda_2 t}\mathbf{v}^2 + \ldots + c_n e^{\lambda_n t}\mathbf{v}^n \ . \tag{7}$$

Thus if $\mathrm{Re}(\lambda_j) < 0$ for $j = 1, \ldots, n$, the origin will be *stable* – all nearby orbits will attract it exponentially fast, and it will be the attractor of (6). Conversely if one or more of the eigenvalues has positive real part, the origin will be unstable. (In fact, if even one eigenvalue has positive real part, *almost every* solution will move away from the origin.)

Case 1b: Now suppose that we have a nonlinear system of equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{8}$$

(for example, the Lorenz equations.) Suppose that $\mathbf{x} = \mathbf{x}^*$ is a fixed point, or stationary point of this system of equations $-i.e.$, $\mathbf{f}(\mathbf{x}^*) = 0$, so that $\mathbf{x}(t) = \mathbf{x}^*$ is a solution of (8). Is it an attractor?

For $\mathbf{x}$ close to $\mathbf{x}^*$, Taylor's theorem tells us that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^*) + D\mathbf{f}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) + \dots$$

and the higher order terms should be negligible if we remain close to $\mathbf{x}^*$. Letting $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$, and ignoring the higher order terms in (1.1), (8) becomes

$$\dot{\mathbf{y}} = D\mathbf{f}(\mathbf{x}^*)\mathbf{y} \ .$$

But this is of the same form as (6) and so we have the following important theorem.

Theorem 1.9 *A fixed point $\mathbf{x}^*$ of a system of ordinary differential equations is* stable *(and hence an attractor) if all the eigenvalues of $D\mathbf{f}(\mathbf{x}^*)$ have negative real part. If one or more of the eigenvalues of $D\mathbf{f}(\mathbf{x}^*)$ has positive real part, then $\mathbf{x}^*$ is unstable.*

Remark 1.10 *Although we've been quite cavalier in throwing away the higher order terms in (8), this can be mathematically justified without too much effort.*

Remark 1.11 *As those who have taken MA 573 know, the situation is much more subtle if the eigenvalues of $D\mathbf{f}(\mathbf{x}^*)$ lie on the imaginary axis.*

1.2 Maps

Differential equations tell us how things change with time. They assume that time is continuous. In other circumstances, it may be more appropriate to think of time as discrete, *i.e.* in biological models in which the appropriate meaning of "time" is the number of generations since some specified event. If x_n is the population in generation n, we might be able to express x_{n+1} as some function of x_n–*i.e*

$$x_{n+1} = f(x_n)$$

(One common (though probably not very realistic) model is to take $f(x) = \lambda x(1 - x)$, with λ a parameter.) In this case, given an initial population x_0, we can follow how the population evolves in time through the sequence of points $x_0, x_1, \ldots$. This solution is analogous to the orbit $\mathbf{x}(t)$ of a system of differential equations with initial condition $\mathbf{x}_0$, so we will call it the orbit of the discrete dynamical system defined by (1.2).

Remark 1.12 *We will see below that another context in which discrete dynamical systems arise is in the study of the stability of periodic orbits of systems of differential equations.*

As a first step in understanding the behavior of dynamical systems defined by maps, let's suppose that x^* is a fixed point of the dynamical system defined by (1.2) where $f(x)$ is a function of one variable, like $\lambda x(1-x)$, and ask what determines the stability of x^* in this case.

The first difference we see is that if x^* is a fixed point of (1.2), then $f(x^*) = x^*$, not $f(x^*) = 0$.

Now suppose that $x \approx x^*$. Then by Taylor's theorem,

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*) + \ldots = x^* + f'(x^*)(x - x^*) + \ldots .$$

Thus, if $x_n \approx x^*$, then

$$x_{n+1} \approx x^* + f'(x^*)(x_n - x^*)$$

or

$$x_{n+1} - x^* \approx f'(x^*)(x_n - x^*)$$

and we conclude

Theorem 1.13 *If $|f'(x^*)| < 1$, then $|x_{n+1} - x^*| < |x_n - x^*|$, so that successive points in an orbit near x^* get closer and closer to it as time goes on – i.e., x^* is an attractor. Conversely, if $|f(x^*)| > 1$, then $|x_{n+1} - x^*| > |x_n - x^*|$ –i.e. x^* is unstable.*

Remark 1.14 *Note here the important distinction between dynamical systems defined by differential equations and those defined by maps. In the case of differential equations, stability depends on whether or not (the real part of) some quantity is greater or less than zero, but in the discrete time case – i.e. the case of maps, it is determined by whether or not some quantity is of absolute value greater or less than one.*

For higher dimensional dynamical systems defined by maps, the stability or instability of a fixed point again depends on the eigenvalues of the Jacobian matrix.

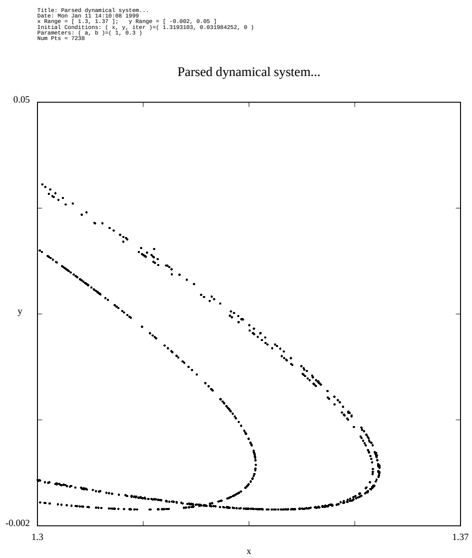
Suppose that $\mathbf{x} \in \mathbf{R}^2$, and we have a dynamical system defined by

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n) . \tag{9}$$

Example 1.15 *An example to which we will frequently return is the Henon mapping;*

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 + y_n - ax_n^2 \\ bx_n \end{pmatrix} .$$

Here a and b are real parameters. Note that if $b = 0$, this reduces to a one-dimensional map of the sort that we discussed above. Henon discovered that for many values of the parameters, the orbits of the Henon map display a very complicated behavior, as in the figure below.



Returning to our general example (9), suppose that $\mathbf{x}^*$ is a fixed point, so that $\mathbf{f}(\mathbf{x}^*) = \mathbf{x}^*$. If we mimic the argument with Taylor's theorem which we used to study the stability of fixed points for systems of differential equations we find that for discrete dynamical systems the relevant criterion is whether or not the absolute value of the eigenvalues of the Jacobian matrix are less than one or greater than one – i.e. if all the eigenvalues have absolute value less than one the fixed point is stable, but if one or more of the eigenvalues has absolute value greater than one the fixed point is unstable.

Suppose that $\mathbf{f}$ defines a discrete dynamical system in the plane. Let λ_1 and λ_2 be the eigenvalues of the Jacobian $\mathbf{f}(\mathbf{x}^*)$. Then if both $|\lambda_1|$ and $|\lambda_2|$ are less than 1, the fixed point is stable, or in other words, and attractor. If, on the other hand, one or both of the eigenvalues has absolute value greater than one, the fixed point is unstable.

Example 1.16 *Let's find the fixed points of the Henon map, with $a = 1$ and $b = 0.3$, and determine their stability. The fixed points will be the solutions of the equation $\mathbf{f}(\mathbf{x}^*) = \mathbf{x}^*$, or more explicitly,*

$$\begin{pmatrix} 1 + y - x^2 \\ 0.3x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

substituting $y = 0.3x$ into the first of these equations, we find that there are two possible x values for the fixed points– $x^*_\pm \frac{-0.7 \pm \sqrt{0.49+4}}{2}$, with corresponding values $y^*_\pm = (0.3)(\frac{-0.7 \pm \sqrt{0.49+4}}{2})$. The Jacobian matrix for the Henon map is

$$Df(x, y) = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$$

or substituting in the values of our fixed points, and $a = 1$, $b = 0.3$,

$$Df(x^*_\pm, y^*_\pm) = \begin{pmatrix} -2(\frac{-0.7 \pm \sqrt{0.49+4}}{2}) & 1 \\ 0.3 & 0 \end{pmatrix}$$

For $Df(x_+^*, y_+^*)$, the eigenvalues are -1.60579 and 0.186824, while for $Df(x_-^*, y_-^*)$ the eigenvalues are 2.92164 and -0.102682. Note that in both cases, at least one of the eigenvalues has absolute value greater than 1, and hence both of these fixed points are unstable.

1.3 Poincaré maps and the stability of periodic orbits

We next develop a natural connection between flows and maps. Suppose that the flow, ϕ^t, associated with

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \ , \quad \mathbf{x} \in \mathbf{R}^n \ ,$$

has a periodic orbit, *i.e.* there exists an initial condition $\mathbf{x}^0$ such that $\phi^T(\mathbf{x}^0) = \mathbf{x}^0$ for some $\mathbf{x}^0$. Note that the orbit $\{\phi^t(\mathbf{x}^0)\}_{t=0}^T$ forms a closed loop in the phase space. As we saw in Example 1.3 such a periodic orbit may be an attractor, and we want to determine analytically whether or not a given periodic orbit is stable. Note that if $\mathbf{x}^*$ is a point on the periodic orbit, just considering $D\mathbf{f}(\mathbf{x}^*)$ won't do for at least a couple of reasons:

- First or all, this will in general depend on $\mathbf{x}^*$ – what point on the periodic orbit should one choose?
- One won't have $f(\mathbf{x}^*) = 0$ which we used in an essential way in our discussion of the stability of fixed points of flows.

To circumvent these difficulties, we take to course of action that mathematicians always resort to – we reduce it to a problem we have already solved! In the present case, we do so by constructing the *Poincaré map.* Let Σ be a hyperplane in the phase space of our differential equation that is transverse to the periodic orbit at a point $\mathbf{x}^*$. We can define a map $P : \Sigma \to \Sigma$ by defining $P(\mathbf{x})$ for any point $\mathbf{x} \in \Sigma$ to be the point at which the orbit $\{\phi^t(\mathbf{x})\}_{t\geq 0}$ through the point $\mathbf{x}$ cuts back through Σ. To be definite, one should choose an orientation in which one passes through Σ – say "front-to-back" or "left-to-right". Alternatively, if one restricts attention to a sufficiently small neighborhood of $\mathbf{x}^*$ in Σ, the first time one returns to this neighborhood is unambiguously defined. (Though then one runs into the problem that one may never return to such a neighborhood.) Note that the Poincaré map P has a number of important properties:

- If $\mathbf{x}^*$ is a point on the periodic orbit of the flow, then $P(\mathbf{x}^*) = \mathbf{x}^*$. Thus, a periodic orbit of the flow becomes a *fixed point* of the Poincaré map.
- If the periodic orbit of the flow is stable, then for any initial condition $\mathbf{x}_0$ near the periodic orbit, the orbit $\{\phi^t(\mathbf{x}^*)\}_{t\geq 0}$ will approach the periodic orbit as t goes to infinity. However, thinking of the definition of the Poincaré map, we see that this implies that the points $\mathbf{x}_1 = P(\mathbf{x}_0)$, $\mathbf{x}_2 = P(\mathbf{x}_1)$, ... approach $\mathbf{x}^*$ – *i.e.* $\mathbf{x}^*$ is a stable fixed point of P.
- The converse of this is also true – if $\mathbf{x}^*$ is a stable fixed point of P, then the periodic orbit of the flow is also stable, and hence an attractor.

Thus we have

Theorem 1.17 *Suppose that*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

has a periodic orbit passing through the point $\mathbf{x}^*$*. Let* $DP(\mathbf{x}^*)$ *be the Jacobian matrix of the Poincaré map evaluated at* $\mathbf{x}^*$*. Let* $\Lambda_1, \ldots, \Lambda_{n-1}$ *be the eigenvalues of* $DP(\mathbf{x}^*)$*. If* $|\Lambda_j| < 1$ *for all* j*, then the periodic orbit of (1.17) is stable.*

Remark 1.18 *The numbers* $\Lambda_1, \ldots, \Lambda_{n-1}$ *are called the characteristic multipliers, or Floquet multipliers of the periodic orbit.*

Example 1.19 *Let's look again at Example 1.3 and compute the Poincaré map of the limit cycle we found there. In this case we know from explicit computation that the limit cycle is stable, but we will confirm that with the Poincaré map. The first thing we must do is pick a section. We'll take the x-axis. Then the Poincaré map will be a map from the x-axis to itself. Note that the Poincaré map is a 1-dimensional map whereas our original differential equation was two dimensional. This is an additional advantage to working with the Poincaré map – it reduces the dimension of the system that one must consider. If we take a point on the x-axis near* $x = 1$*, we must follow the flow with this initial condition until it returns to the x-axis. This will be when* θ *has gone from* 0 *to* 2π*. Since* $\theta(t) = \theta_0 + 2t$*, that means that* $P(r) = r(t = \pi)$*. Using our formula for the flow from (1.3), we see that*

$$P(r) = \frac{r}{r - e^{-\pi}(r-1)} \ .$$

This is the Poincaré map for our limit cycle. Note that $r = 1$ *is a fixed point for* P*, as the general theory tells us it ought to be. To compute the stability of the periodic orbit, we evaluate the derivative of* P *at the fixed point. Since* $P'[r] = \frac{e^{\pi}}{(1+(e^{\pi}-1)r)^2}$*, we see that*

$$P'(1) = e^{-\pi} < 1$$

Thus, the limit cycle for our original system of differential equations is stable.

References

[1] J.-P. Eckmann and D. Ruelle. Ergodic theory of chaos and strange attractors. *Rev. Modern Phys.*, 57(3, part 1):617–656, 1985.

[2] David Ruelle. Small random perturbations of dynamical systems and the definition of attractors. *Comm. Math. Phys.*, 82(1):137–151, 1981/82.