Lecture 2: Lyapunov Exponents [1]

The goal of this lecture is to extend our stability considerations of lecture 1 to define the Lyapunov exponents which will allow us to determine whether two nearby orbits will converge toward each other or diverge from each other. This will lead us to a way of quantifying the *sensitive dependence on initial conditions* which we observed numerically in the Lorenz equations.

We showed in lecture 1 how to characterize the stability of fixed points for both continuous and discrete dynamical systems and also discussed the stability of periodic orbits of systems of differential equations. In this lecture we first discuss the stability of periodic orbits of discrete dynamical systems.

Consider the dynamical system

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n) \ , \quad \mathbf{x} \in \mathbf{R}^n \ . \tag{1}$$

A *periodic orbit of length* k for (1) is a set of k points $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$ such that $\mathbf{x}_1 = \mathbf{f}(\mathbf{x}_0), \mathbf{x}_2 = \mathbf{f}(\mathbf{x}_1), \ldots, \mathbf{x}_0 = \mathbf{f}(\mathbf{x}_{k-1})$. To determine whether or not such an orbit is stable, we note that

$$\mathbf{x}_0 = \mathbf{f}(\mathbf{x}_{k-1}) = \mathbf{f}(\mathbf{f}(\mathbf{x}_{k-2})) = \ldots = \mathbf{f}(\mathbf{f}(\ldots \mathbf{f}(\mathbf{x}_0))) = \mathbf{f}^{(k)}(\mathbf{x}_0) \ ,$$

that is, $\mathbf{x}_0$ is a *fixed point* of the k-fold composition of $\mathbf{f}$ with itself. Thus our previous results immediately yield

Theorem 2.1 *Suppose that $\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}$ is a periodic orbit of length k for (1). Then this orbit is stable if all the eigenvalues of $D\mathbf{f}^{(k)}(\mathbf{x}_0)$ lie inside the unit circle in the complex plane. If one or more of the eigenvalues of this matrix lie outside the unit circle, the periodic orbit is unstable.*

Remark 2.2 *The eigenvalues of $D\mathbf{f}^{(k)}(\mathbf{x}_0)$ are the same as those of $D\mathbf{f}^{(k)}(\mathbf{x}_j)$ for $j = 1, 2, \ldots, k-1$ (why?) so whether or not a periodic orbit is stable does not depend on which point in the orbit we choose to start with.*

Example 2.3 *Let $f(x) = 3.1x(1-x)$. ($x \in \mathbf{R}$.) What are the period-2 orbits for the dynamical system defined by f? Are they stable? If x_0, x_1 is a periodic orbit for this function, then*

$$x_0 = f(f(x_0)) = (3.1)^2 x_0(1-x_0)(1-3.1x_0(1-x_0))$$

This equation has four solutions: $x = 0$, $x = 0.677419...$, $x = 0.558014...$, and $x = 0.764567$. The first two of these aren't very interesting – they are just fixed points of the dynamical system (you should check this) which while technically period-2 orbits aren't really anything new. The points $x_0 = 0.558014...$, and $x_1 = 0.764567$ are a non-trivial period-2 orbit. To determine the stability of this orbit, we differentiate $f^{(2)}(x)$, finding

$$Df^{(2)}(x) = 9.61 - 2(39.401)x + 3(59.582)x^2 - 4(29.791)x^3$$

which, if we substitute in x_0 gives $Df^{(2)}(0.558014...) = .59...$, so this period-2 orbit is stable.

[1] These lecture notes are for the course MA 574, and are being prepared for the instructor's personal use.

Calculations for higher dimensional examples like the Henon map are similar, but computationally more complicated.

Before moving on, I want to talk a bit more about the meaning of these calculations since we will use the intuition gained here to decide how to treat more complicated situations.

Remark 2.4 *Be the chain rule, if* $\mathbf{x}_1 = \mathbf{f}(\mathbf{x}_0)$*,* $\mathbf{x}_2 = \mathbf{f}(\mathbf{x}_1)$*, etc, then*

$$D\mathbf{f}^{(k)}(x_0) = D\mathbf{f}(\mathbf{f}^{(k-1)}(\mathbf{x}_0))D\mathbf{f}(\mathbf{f}^{(k-2)}(\mathbf{x}_0))\dots D\mathbf{f}(\mathbf{x}_0) = D\mathbf{f}(\mathbf{x}_{k-1})D\mathbf{f}(\mathbf{x}_{k-2})\dots D\mathbf{f}(\mathbf{x}_0)$$

which means that the Jacobian matrix $D\mathbf{f}^{(k)}(x_0)$ *can be evaluated as the product of the Jacobian matrices of* $\mathbf{f}(\mathbf{x})$*, evaluated at the points of the orbit of* $\mathbf{x}_0$*.*

Now suppose that we consider an orbit whose initial point $\tilde{\mathbf{x}}_0$ is close to a point $\mathbf{x}_0$ in a period-k orbit. Applying Taylor's theorem as we did in our analysis of the stability of the fixed point of a map, we see that after k iterations of the map, we'll have

$$\tilde{\mathbf{x}}_k - \mathbf{x}_k = \mathbf{f}^{(k)}(\tilde{\mathbf{x}}_0) - \mathbf{f}^{(k)}(\mathbf{x}_0) \approx D\mathbf{f}^{(k)}(\mathbf{x}_0)(\tilde{\mathbf{x}}_0 - \mathbf{x}_0)$$

Thus, the eigenvalues of the Jacobian matrix $D\mathbf{f}^{(k)}(\mathbf{x}_0)$ tell us how an initially small deviation from the periodic orbit is stretched or shrunk as we go once around the orbit.

Let $\Gamma_1, \dots, \Gamma_n$ be the eigenvalues of $D\mathbf{f}^{(k)}(\mathbf{x}_0)$. We suppose for simplicity that these eigenvalues are simple and that we have numbered them so that $|\Gamma_1| \geq |\Gamma_2| \geq \dots \geq |\Gamma_n|$. If we pick $(\tilde{\mathbf{x}}_0 - \mathbf{x}_0)$ to be one of the eigenvectors of $D\mathbf{f}^{(k)}(\mathbf{x}_0)$ then

$$(\tilde{\mathbf{x}}_k - \mathbf{x}_k) = \Gamma_\ell(\tilde{\mathbf{x}}_0 - \mathbf{x}_0) \ ,$$

i.e. in going once around the periodic orbit, the initial displacement will be stretched (or shrunk) by an amount $|\Gamma_\ell|$. The maximum stretching (or an initial displacement in the direction of one of the eigenvectors) will correspond to $|\Gamma_1|$ and this accords with our conclusion that the fixed point is stable if all the eigenvalues of $D\mathbf{f}^{(k)}(\mathbf{x}_0)$ lie inside the circle of radius one in the complex plane.

Remark 2.5 *There is a slight complication which may arise if we choose initial displacements that are not aligned with the eigenvectors of* $D\mathbf{f}^{(k)}(\mathbf{x}_0)$ *– an initial displacement may start to grow, even though all the eigenvalues of* $D\mathbf{f}^{(k)}(\mathbf{x}_0)$ *have absolute value less than one. However, if we "go around" the periodic orbit several times then all initially small displacement will eventually begin to decay. As an example of this phenomenon, suppose that*

$$D\mathbf{f}^{(k)}(\mathbf{x}_0) = \begin{pmatrix} 0.5 & 1 \\ 0 & 0.45 \end{pmatrix}$$

Show that if one takes an initial displacement

$$(\tilde{\mathbf{x}}_0 - \mathbf{x}_0) = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$$

Then $(\tilde{\mathbf{x}}_k - \mathbf{x}_k)$ *has longer length than* $(\tilde{\mathbf{x}}_0 - \mathbf{x}_0)$ *but if we continue to iterate this procedure, the resulting vectors eventually go to zero.*

Note that since Γ_1 represents the amount by which a small deviation from the periodic orbit is stretched after going around the periodic orbit once, it is the total stretching that occurs in k-steps. The average amount of stretching that occurs in each step is $\Gamma_1^{1/k}$, and the periodic orbit is stable if this quantity is smaller than one. If $\Gamma_1^{1/k} < 1$, then $\gamma_1 = \log(\Gamma_1^{1/k}) = \frac{1}{k}\log(\Gamma_1) < 0$, and we have the following corollary of our stability theorem.

Corollary 2.6 *The periodic orbit* $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$ *is stable if* $\gamma_1 = \frac{1}{k}\log(|\Gamma_1|) < 0$*, and unstable if* $\gamma_1 = \frac{1}{k}\log(|\Gamma_1|) > 0$*.*

2.1 Lyapunov Exponents

We now turn to a general orbit, remaining for the moment with discrete dynamical systems. We want to determine how the orbits of two initial conditions $\mathbf{x}_0$ and $\tilde{\mathbf{x}}_0$ which are very close to each other evolve over time. Let $\{\tilde{\mathbf{x}}_k\}_{k=0}^{\infty}$, and $\{\mathbf{x}_k\}_{k=0}^{\infty}$ be the corresponding orbits. Then, as usual

$$\tilde{\mathbf{x}}_k - \mathbf{x}_k = \mathbf{f}(\tilde{\mathbf{x}}_0) - \mathbf{f}(\mathbf{x}_0) \approx D\mathbf{f}^{(k)}(\mathbf{x}_0)(\tilde{\mathbf{x}}_0 - \mathbf{x}_0) \; . \tag{2}$$

We are interested in knowing if the two orbits approach each other or diverge from one another. Let $\delta\mathbf{x} = \tilde{\mathbf{x}}_0 - \mathbf{x}_0$ be the initial separation. We would like to find a number γ which measures the average stretching or shrinking of $\|\delta\mathbf{x}\|$ with each iteration as we did in the case of periodic orbits. That is, we would like to find a real number γ, such that

$$\|D\mathbf{f}^{(k)}(\mathbf{x}_0)\delta\mathbf{x}\| \approx e^{\gamma k} \; . \tag{3}$$

Note that *if* such a number exists, it may depend on both $\mathbf{x}_0$ and $\delta\mathbf{x}$.

Suppose for the moment that such a number does exist. How could we calculate it? Taking logarithms in (3), we see that

$$k\gamma \approx \log\|D\mathbf{f}^{(k)}(\mathbf{x}_0)\delta\mathbf{x}\| \; .$$

Since we are interested in the asymptotic behavior of the system, we divide by k and take the limit as k tends to infinity. This leads to the following definition.

Definition 2.7 *If the limit*

$$\gamma = \lim_{k\to\infty}\frac{1}{k}\log\|D\mathbf{f}^{(k)}(\mathbf{x}_0)\delta\mathbf{x}\| \; ,$$

exists, μ *is called the* Lyapunov exponent *(associated with* $\delta\mathbf{x}$ *and* $\mathbf{x}_0$*).*

Remark 2.8 *Note the similarity of this definition with the discussion of Corollary 2.6. Thus, if this number exists and is negative we will say that the orbit with initial condition* x_0 *is stable and if the limit is positive we will say that the orbit is unstable.*

Remarkably, this limit usually does exist!

Theorem 2.9 *(Oseledec) If* $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ *is "nice enough", then for "almost every"* $\mathbf{x}_0$*, there exists a set of subspaces of* $\mathbf{R}^n$*,*

$$\mathbf{R}^n = E^1 \supset E^2 \supset \ldots \supset E^m \supset E^{m+1} = \emptyset \; , \quad (m \le n)$$

and a set of real number $\gamma_1 > \gamma_2 > \ldots > \gamma_m$ *such that for* $\delta\mathbf{x} \in E^j \backslash E^{j+1}$*,*

$$\lim_{k\to\infty}\frac{1}{k}\log\|D\mathbf{f}^{(k)}(\mathbf{x}_0)\delta\mathbf{x}\| = \gamma_j \; , \quad j = 1, 2, \ldots, m \; .$$

Remark 2.10 *For a discussion of exactly what "nice enough" means see [1]. It is a quite weak restriction – the examples one encounters in actual applications almost always satisfy these conditions. More relevant will be the meaning of "almost every". I will return to this when we discuss invariant measures for dynamical systems in lecture 3.*

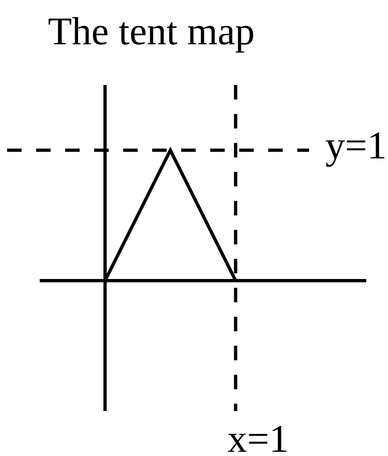
Remark 2.11 *The proof of this theorem is quite difficult. If you want to see what's involved, consult [2].*

Note that referring to (3), we see that if one or more of the Lyapunov exponents are positive, we expect the orbits of the dynamical system to be very unstable. Even orbits corresponding to initial conditions which are very close to one another will separate exponentially rapidly.

We'll now look at a pair of examples in which we can actually compute the Lyapunov exponents.

Example 2.12 *The tent map. Let*

$$f(x) = \begin{cases} 2x & ; \quad 0 \leq x \leq 1/2 \\ 2 - 2x & ; \quad 1/2 < x \leq 1 \end{cases}$$



Note that if $x_0 \in [0,1]$, then the orbit $\{x_k\}_{k=0}^{\infty}$ of x_0 remains in $[0,1]$ for all k. Pick any x_0 whose orbit does not land on $x = 1/2$. (Why do we have to avoid $x = 1/2$?) Then

$$Df^{(k)}(x_0) = Df(x_{k-1})Df(x_{k-2})\ldots Df(x_0)$$

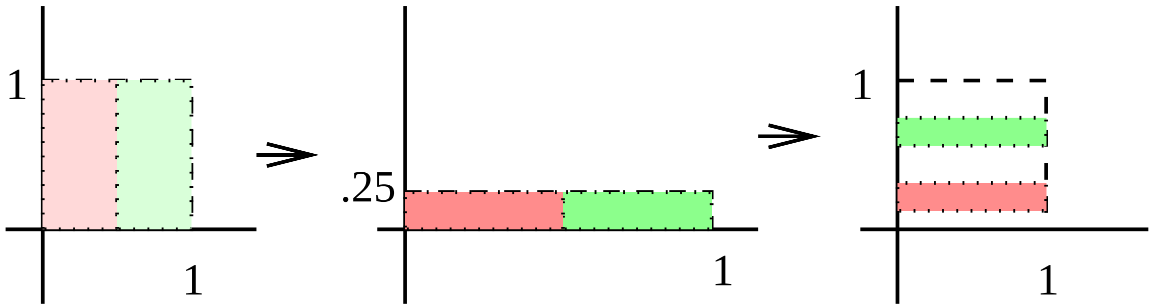
by the chain rule. Since $|Df(x)| = 2$ for all $x \neq 1/2$, we have $|Df^{(k)}(x_0)| = 2^k$, and hence

$$\gamma = \lim_{k\to\infty} \frac{1}{k} \log |Df^{(k)}(x_0)| = \log 2 \ .$$

Since $\gamma > 1$, we expect that the orbits of this dynamical system are unstable, and indeed they are. If, for instance, one computes the orbits of $x_0 = 0.12345$ and $\tilde{x}_0 = 0.12346$, one finds that they separate rapidly, and after about 12 iterations their orbits become completely uncorrelated with one another.

Example 2.13 *Baker's transformation. This is a much for interesting example for a couple of reasons.*

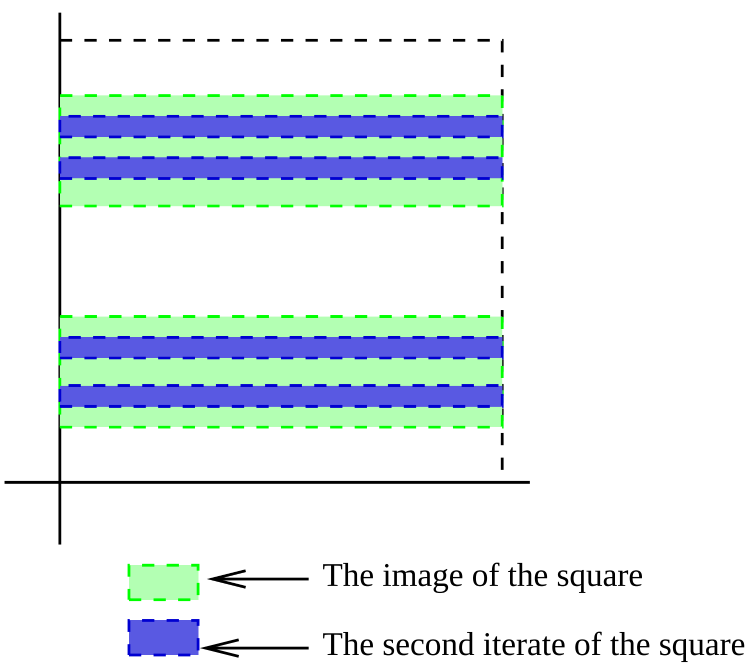
- *It exhibits the "stretching and folding" that is characteristic of chaotic systems.*
- *It has a non-trivial attractor (whose dimension will turn out to be fractional).*



$$B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \begin{pmatrix} 2x \\ \frac{1}{4}y + \frac{1}{8} \end{pmatrix} & ; \quad 0 \le x < 1/2 \\ \begin{pmatrix} 2x - 1 \\ \frac{1}{4}y + \frac{5}{8} \end{pmatrix} & ; \quad 1/2 \le x \le 1 \end{cases}$$

First let's look at the attractor. Note that B maps the square $[0,1]\times[0,1]$ onto the pair of strips $[0,1]\times[\frac{1}{8},\frac{3}{8}]\cup[0,1]\times[\frac{5}{8},\frac{7}{8}]$ If we now apply the map again, we see that the image of these two strips will be four strips. More explicitly,

$$B^2([0,1]\times[0,1]) = [0,1]\times[\frac{5}{32},\frac{7}{32}]\cup[0,1]\times[\frac{9}{32},\frac{11}{32}]\cup[0,1]\times[\frac{21}{32},\frac{23}{32}]\cup[0,1]\times[\frac{25}{32},\frac{27}{32}]\ .$$



*Continuing in this way, we see that the attracting set $\mathcal{A}$ will be the union of infinitely many horizontal lines. As a useful exercise, try to describe the attractor explicitly. (*Hint: *Try to express the y-coordinates of the lines in the attractor in binary notation.*

Let's now turn to the Lyapunov exponents. Pick any (x_0, y_0) whose orbit doesn't land on the boundary of the square. (There are problems with defining the derivative there.) Then we need *to find*

$$DB^{(k)}(x_0, y_0) = DB(x_{k-1}, y_{k-1})DB(x_{k-2}, y_{k-2}) \ldots DB(x_0, y_0) \ .$$

Here, DB is the Jacobian matrix, which we easily compute to be

$$DB(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

Thus the Jacobian is independent *of the point (x, y) at which we evaluate it! This is a* tremendous *simplification. Let $E^1 = \mathbf{R}^2$, and $E^2 = \{(0, y)\}$. Then if $\delta\mathbf{x} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ with $\xi \neq 0$ we see that $\delta\mathbf{x} \in E^1 \backslash E^2$, and so that*

$$DB^{(k)}(x_0, y_0)\delta\mathbf{x} = \begin{pmatrix} 2^k & 0 \\ 0 & \frac{1}{4^k} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 2^k\xi \\ \frac{1}{4^k}\eta \end{pmatrix}$$

from which it easily follows that

$$\gamma_1 = \lim_{k\to\infty} \frac{1}{k} \log \|DB^{(k)}(x_0, y_0)\delta\mathbf{x}\| \lim_{k\to\infty} \frac{1}{k} \log \sqrt{2^{2k}\xi^2 + 4^{-2k}\eta^2} = \log 2 \ .$$

Similarly, if $\delta\mathbf{x} = \begin{pmatrix} 0 \\ \eta \end{pmatrix} \in E^2$,

$$\gamma_2 = -\log 4 \ .$$

Remark 2.14 *The attractor in the Baker's map is quite different than the stable stationary points or limit cycles we examined earlier. If one takes two initial conditions close to each other, both will approach the attractor, but they will nonetheless separate rapidly (along the attractor) due to the positivity of γ_1. If a system has one or more positive Lyapunov exponents, we will say it exhibits* sensitive dependence on initial conditions. *An attractor with one or more positive Lyapunov exponents will be called a strange attractor. It will turn out that not only do strange attractors exhibit sensitive dependence on initial conditions, but in addition they have fractional dimensions. This is also true of the attractor in the Baker's transformation as we will see in lecture 4.*

References

[1] J.-P. Eckmann and D. Ruelle. Ergodic theory of chaos and strange attractors. *Rev. Modern Phys.*, 57(3, part 1):617–656, 1985.

[2] Russell A. Johnson, Kenneth J. Palmer, and George R. Sell. Ergodic properties of linear dynamical systems. *SIAM J. Math. Anal.*, 18(1):1–33, 1987.