Inverse spectral theory for uniformly open gaps in a weighted Sturm-Liouville problem

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Abstract

Motivated by a PDE existence problem, we study the inverse problem for a weighted Sturm-Liouville operator $\mathcal{L}_s$ associated with the eigenvalue problem $y'' + \lambda\, s(x)\, y = 0$, where s is a real-valued, periodic, even function that is bounded from below by a positive constant and belongs to the L^2-based Sobolev space $H^r[0,1], r \geq 1$. Choosing gap lengths and gap midpoints as coordinates, we define a spectral map $\mathcal{G}$, that assigns to a coefficient s the structure of the spectrum of $\mathcal{L}_s$. We find that $\mathcal{G}$ is a real-analytic isomorphism locally around $s = 1$, which, in particular, implies the existence of coefficients $s \in H^r[0,1], r < 3/2$, whose spectrum features band structure with all gaps uniformly open around the gap midpoints. This result paves the way for the construction of so-called breathers in nonlinear wave equations with such coefficients s. Apart from the novelty of treating the inverse spectral problem for the full Banach scale $H^r[0,1], r \geq 1$, the local nature of our result allows more concise and transparent proofs. In particular, instead of using any preliminary transformations, we treat the weighted problem directly by adapting techniques used for Schrödinger operators with distribution potentials.

Keywords: Inverse Sturm-Liouville Theory, Partial Differential Equations

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1. Introduction

Consider a function $s \in L^2[0,1]$ (the Hilbert space of square integrable functions on the unit interval) that is real-valued, periodic, even and bounded from below by a positive constant. Extend s to a 1-periodic function on the real line $\mathbb{R}$ by $s(x+1) = s(x), x \in \mathbb{R}$, and define the Sturm-Liouville operator $\mathcal{L}_s = -\frac{1}{s(x)} \frac{d^2}{dx^2}$ acting on $L^2(\mathbb{R}, s(x)dx)$. Its spectrum $\sigma(\mathcal{L}_s)$ is the set of values λ such that the equation for $y = y(x)$ given by

$$y'' + \lambda\, s(x)\, y = 0\,, \tag{1}$$

has only (nontrivial) solutions that are bounded on $\mathbb{R}$. The set $\sigma(\mathcal{L}_s)$ is contained in $\mathbb{R}$ and is the union of a sequence of closed intervals

$$B_n = [\lambda_{2n}, \lambda_{2n+1}]\,, \quad n \geq 0\,, \tag{2}$$

often referred to as *bands*, satisfying

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots .$$

In other words, the spectrum $\sigma(\mathcal{L}_s)$ is the union of *bands*. The intervening, possibly void, open intervals $(\lambda_{2n-1}, \lambda_{2n}), n \geq 1$, are called *gaps*. The subject of the present work is the discussion of the dependence of the length and position of the gaps on the coefficient s. To be more precise, our work is motivated by a PDE existence problem which involves the characterization of coefficients s that give rise to spectrum with all gaps uniformly open around $\omega_*^2 n^2, n \geq 1$, for some $\omega_* \in \mathbb{R}$ (as illustrated in Figure 1, see Corollary 3 for the precise statement).

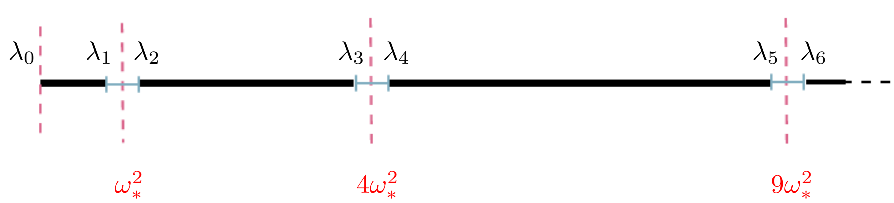

Figure 1: Structure of $\sigma(\mathcal{L}_s)$ with s as given in Corollary 3.

The length of gaps is related to the location of eigenvalues of special boundary value problems. In particular, for even coefficients s it holds true (see, for instance, [21]) that

$$\{\lambda_{2n-1}(s), \lambda_{2n}(s)\} = \{\mu_n(s), \nu_n(s)\}\,, \quad n \geq 1\,,$$

where $\mu_n(s), n \geq 1$, is the Dirichlet spectrum associated with (1), that is, values of λ for which $y'' + \lambda s(x) y = 0, y(0) = 0, y(1) = 0$, has a nontrivial solution, and $\nu_n(s), n \geq 0$, is the Neumann spectrum associated with (1), that is, values of λ for which $y'' + \lambda s(x) y = 0, y'(0) = 0, y'(1) = 0$, has a nontrivial solution. The *signed gap lengths* G_n can therefore be computed as

$$G_n(s) = \mu_n(s) - \nu_n(s)\,.$$

It is well known that the asymptotics of Dirichlet and Neumann eigenvalues (and, hence, also of the gap lengths G_n) is decided by the smoothness of s (see, for instance, [10] or [11] and [12]). Therefore, we introduce the following function spaces. Denote by l^2 the space of square summable sequences and let

$$h^r = \left\{ (a_n)_{n\in\mathbb{N}} \;\middle|\; ((n^2+1)^{r/2} a_n)_{n\in\mathbb{Z}} \in l^2 \right\} \quad (r \in \mathbb{R})$$

be weighted spaces of sequences, to which the Sobolev spaces

$$H^r[0,1] = \left\{ s \;\middle|\; \left((n^2+1)^{r/2}\, \hat{s}(n) \right)_{n\in\mathbb{Z}} \in l^2 \right\} \quad (r \in \mathbb{R})$$

are naturally linked through the Fourier transform

$$\hat{f}(n) = \int_0^1 f(x)\, e^{-2\pi i n x}\, dx\,.$$

Hence, $h^0 = l^2, H^0[0,1] = L^2[0,1]$, $s \in H^1[0,1] \Leftrightarrow s' \in L^2[0,1]$, etc. and we have the compact Sobolev embedding $H^{j+m}[0,1] \subset C^j[0,1]\,, m > 1/2\,$, where $C^j[0,1]$ consists of functions whose $j-th$ derivative is continuous (see, for instance, [1]).

If $s = 1$, then $\mu_n(s) = \nu_n(s) = n^2\pi^2, n \in \mathbb{N}$, and, hence, $G_n(s) = 0$, that is, there are no gaps in the spectrum of $\mathcal{L}_s$ and $\sigma\left(\mathcal{L}_s\right) = [0, \infty)$. Our main result sheds light on the structure of the spectrum for varying coefficients $s = s(x)$

that are H^r-close to $s = 1$. In particular, we find that (cf. Proposition 11), locally around $s = 1$, if $s \in H^r[0,1], r \geq 1$, we have

$$\mu_n = \frac{n^2\pi^2}{\left(\int_0^1 \sqrt{s(\xi)}\,d\xi\right)^2} + \gamma_n^{\mathrm{Dir}}, \qquad \nu_n = \frac{n^2\pi^2}{\left(\int_0^1 \sqrt{s(\xi)}\,d\xi\right)^2} + \gamma_n^{\mathrm{Neu}}, \tag{3}$$

with $(\gamma_n^{\mathrm{Dir}})_{n\in\mathbb{N}}, (\gamma_n^{\mathrm{Neu}})_{n\in\mathbb{N}} \in h^{r-2}$. Motivated by this, we construct a map that encodes the dependence of the band structure on the coefficient s by using the signed gap lengths $G_n = \mu_n - \nu_n$ and the scaling of the gap midpoints $(\int_0^1 \sqrt{s(\xi)}\,d\xi)^{-2}$ as coordinates. The following result is local in nature, i.e. it sheds light on how the quantities

$$\left(\int_0^1 \sqrt{s(\xi)}\,d\xi\right)^{-2}\Bigg|_{s=1} = 1\,, \quad (G_n(s)|_{s=1})_{n\in\mathbb{N}} = (0)_{n\in\mathbb{N}}\,, \tag{4}$$

are perturbed for $s = 1 + \widetilde{s}$ with "small" $\widetilde{s}$.

Theorem 1 (Inverse problem). *Fix $r \geq 1$ and set*

$$E = \left\{ s \in L^2[0,1] \;\mid \text{even, periodic and } s(x) \geq s_0 > 0 \text{ for some } s_0 \in \mathbb{R} \right\}.$$

The gap structure mapping

$$\mathcal{G} : E \cap H^r[0,1] \longrightarrow \mathbb{R} \times h^{r-2} \tag{5}$$

$$s \longmapsto \left[\frac{1}{\left(\int_0^1 \sqrt{s(\xi)}\,d\xi\right)^2}\;,\; (G_n(s))_{n\in\mathbb{N}} \right]$$

is a real-analytic isomorphism locally around $s = 1$, that is, there is a neighborhood $U \subset E \cap H^r[0,1]$ of $s = 1$ and a neighborhood $V \in \mathbb{R} \times h^{r-2}$ of $[1, (0)_{n\in\mathbb{N}}]$ such that $\mathcal{G} : U \longrightarrow V$ is bijective and both, $\mathcal{G}$ and $\mathcal{G}^{-1}$ are real-analytic.

Remark 2. *Since we will be working with functions s in a neighborhood of the constant function 1, it will be convenient to write $s(x) = 1 + \tilde{s}(x)$, and we will assume without (further) loss of generality that $\|\tilde{s}\|_{H^r} \leq 1/2$. Note that since we will always assume that $r \geq 1$, this implies that $s(x) \geq 1/2$ for all x.*

The proof of Theorem 1 is executed in Section 2. Its methodology is inspired by [22], where the inverse problem for Dirichlet eigenvalues of a Schrödinger type equation

$$\ddot{v}(t) + \,(q(t) - \lambda)\, v(t) = 0\,, \tag{6}$$

for $v = v(t)$ is treated. If the coefficient s in the weighted equation (1) is smooth enough, one can immediately transfer results for (6) to (1) via the Liouville transformation

$$t = \int_0^x \sqrt{s(\xi)}\, d\xi\,, \quad v(t) = s(x(t))^{\frac{1}{4}}\, y(x(t))\,, \tag{7}$$

posed on the interval $t \in [0, I]$, $I = I(s) = \int_0^1 \sqrt{s(\xi)}\, d\xi$, and with the potential

$$q(t) = s(x(t))^{-\frac{3}{4}}\, \frac{d^2}{dx^2} \left(s(x(t))^{-\frac{1}{4}} \right)\,. \tag{8}$$

The importance of the Liouville transformation lies in the fact that the eigenvalues associated with (1) and (6) coincide. Unfortunately, it will turn out that the inverse problem for uniformly open gaps naturally requires non-smooth coefficients $s \in H^r[0,1], r < 3/2$ (which (3) already foreshadows). Hence, we execute the proof without the Liouville transformation, but would like to stress that the formal correspondence of $s \in H^r[0,1]$ and $q \in H^{r-2}[0,I]$, which is implied by the transformation, motivated us to adapt techniques for Schrödinger equations with distribution potentials as described in [14] (see also Remark 7).

Corollary 3 (Uniformly open gaps). *Fix $r \in [1, 3/2)$. There exists $\delta_0 > 0$ so that for any $\delta \in (0, \delta_0)$ there exists an even function $s_\delta \in H^r[0,1]$ such that the spectrum of the operator $\mathcal{L}_s$ with $s = 1 + s_\delta$ (extended periodically to the entire line) has a gap structure satisfying*

$$\omega_*^2 n^2 - \lambda_{2n-1} \geq \delta\,, \quad \lambda_{2n} - \omega_*^2 n^2 \geq \delta\,, \quad n \geq 1\,, \tag{9}$$

with $\omega_* = \pi / I(s), I(s) = \int_0^1 \sqrt{s(\xi)}\, d\xi$.

A detailed discussion of Corollary 3 is given in Section 3 where we explain why our findings imply the existence of $s \in H^r[0,1]$ with $r < 3/2$ giving rise to uniformly open gaps and how this result can be used to construct special solutions for nonlinear wave equations. In fact, our interest in the structure of the essential spectrum of $\mathcal{L}_s$ stems from an effort to extend the results from [4] where breathers – time-periodic, spatially localized solutions – in nonlinear wave equations with spatially periodic coefficients were constructed via center manifold reduction in infinite dimensions. The novelty of [4] was to tailor the spectrum of $\mathcal{L}_s$ via s to fulfill (9). However, in [4] the setup (9) was obtained through direct, elementary calculations of the so-called discriminant (see Section 3) for (1) resulting in the special choice

$$s(x) = 1 + 15\chi_{[6/13,7/13)}\,(\,x \bmod 1\,)\,, \quad \omega_* = \frac{13}{16}\pi\,.$$

It is the purpose of this article to explore the existence of a class of coefficients s that give rise to a band structure with the features described in Corollary 3 beyond this special choice, which in turn would pave the way to construct breathers for more general nonlinear wave equations. In particular, we were interested in whether or not there were continuous, (or even Hölder) coefficient functions which gave rise to operators with uniformly open gaps.

Remark 4. *Note that, although Corollary 3 seems like an abstract existence result, our proof technique actually delivers an approximation procedure for such coefficients as a byproduct. To be more precise, we find that the leading order approximation in s of the gap lengths $G_n = G_n(s)$ is given by the Fourier cosine coefficients of s (see Lemma 14): the n-th Fourier cosine coefficient controls the n-th gap. This might lead to an approximatin algorithm that manipulates each gap successively. Another option would be to follow the explicit construction procedure from [4] which relies on explicit formulas for the fundamental system of* (1)*. This, however, leads inevitably to the use of the theory of special functions, which is both, technically cumbersome and less likely to allow the specification of a large class of coefficients.*

1.1. **Related work**

The body of literature on direct and inverse Sturm-Liouville problems is enormous. The particular problem we treat in the present work has, however, not been addressed yet. Most literature deals with Schrödinger equations (6) which we refrain to survey here. We would like to point out, however, that there is a significant difference between inverse problems for Schrödinger equations (with regular potential) (6) and the weighted type treated here. The core parts of the proof of Theorem 1 and Corollary 3 rely on rather detailed estimates of eigenvalue asymptotics as given in (3). These are usually obtained from corresponding estimates on a fundamental system for the ODE. These two steps turn out to be much more intricate for our setting than for Schrödinger equations (see Remark 7). More relevant for our work are results that were achieved for the so-called impedance type Sturm-Liouville problem

$$\frac{d}{dt}\left(p^2(t)\frac{d}{dt}w\right) + \lambda\, p^2(t)\, w = 0\,, \tag{10}$$

for $w = w(t)$,that can be transformed into (1) by a Liouville change of variables $s(x) = p(x(t)), t = \int_0^x \sqrt{s(\xi)}d\xi$. Closest to our setting is a series of articles [3], [2] by Andersson where the direct and inverse eigenvalue problem for (10) with discontinuous coefficients $(\ln p)' \in L^m(0,l), 1 \leq m \leq \infty$, (such that a transformation to (6) is not possible) is treated. Similar to our approach, he focuses on coefficients that are not too far away from $p(x) \equiv 1$, i.e. his results are local in nature. However, in contrast to the present work, he uses Dirichlet and Dirichet-Neumann eigenvalues as spectral data to recover the coefficient for the inverse problem. In [5], [6] Coleman and McLaughlin generalized the findings of Andersson and solved the inverse problem globally, that is, not only for p in a neighborhood of $p(x) \equiv 1$, although they restricted to functions $(\ln p)' \in L^2(0,1)$. More recently, Korotyaev examined the same problem in a series of articles ([19], [18]) but with different tools: He chooses a different spectral map and gives estimates on eigenvalue asymptotics in $l^m, m \geq 1$. Perhaps closest to the inverse problem treated here is the work by Klein and Korotyaev [17] where the spectral map resembles our choice very much, that is, gap lengths and gap midpoints are chosen as coordinates. They solve the global inverse problem, but restrict themselves to the function space H^1. Most recently, an impedance type problem was studied by Albeverio, Hryniv and

MYKYTYUK using very similar tools as in the present work, but, again, a different spectral map (Neumann and Neumann-Dirichlet eigenvalues) and the L^m-based Sobolev spaces $W^{1,m}(0,1), m \in [1,\infty)$. To summarize, the main differences between our work and the existing literature are the choice of function space $H^r[0,1], r \geq 1$, and the coordinates for the spectral map, both of which are heavily motivated by [4].

1.2. **Outline of the proof**

In order to retrieve information about the spectral map (5) we need to examine the asymptotics of Dirichlet and Neumann eigenvalues. It is standard to use the fundamental system $\{\widetilde{y}_1, \widetilde{y}_2\}$ that solves (1) subject to the initial conditions

$$\tilde{y}_1'' + \lambda s \tilde{y}_1 = 0, \quad \tilde{y}_1(0) = 1,\ \tilde{y}_1'(0) = 0\,, \tag{11}$$

and, respectively,

$$\tilde{y}_2'' + \lambda s \tilde{y}_2 = 0, \quad \tilde{y}_2(0) = 0,\ \tilde{y}_2'(0) = 1\,, \tag{12}$$

as characteristic functions for Dirichlet and Neumann eigenvalues, since

$$\widetilde{y}_1(1;\lambda) = 0 \quad \text{iff} \quad \lambda = \nu_n, \quad \widetilde{y}_2(1;\lambda) = 0 \quad \text{iff} \quad \lambda = \mu_n\,, \tag{13}$$

where we use the notation $\widetilde{y}_j = \widetilde{y}_j(x;\lambda)$ to stress the dependence of the fundamental system on the spectral parameter λ. In fact, the main challenge and novelty of our work consists in deriving estimates for $\widetilde{y}_{1,2}(1;\lambda)$ (see Proposition 5) which we then translate into estimates for Dirichlet and Neumann eigenvalue asymptotics (see Proposition 11) by employing the implicit function theorem for (13). The analysis of eigenvalue asymptotics is carried out such that local boundedness and the range of the spectral map (5) immediately follow (see Proposition 11). Combining the analyticity of each coordinate function G_n (see Lemma 13 and Appendix A for the proof) and the local boundedness of the full map $\mathcal{G}$, the analyticity of $\mathcal{G}$ follows (see Lemma 14). We conclude by computing the Gâteaux derivative of $\mathcal{G}$ at $s = 1$ in direction $\widetilde{s}$ and find that it is given by the Fourier cosine transform of $\widetilde{s}$ which is an isomorphism from $E \cap H^m[0,1]$ to h^m (see Lemma 14). Hence, the statement of Theorem 1 follows by the inverse function theorem as in [22].

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2. Proof of Theorem 1

Since Theorem 1 is stated for $H^r[0,1], r \geq 1$, which, in particular, contains non-smooth coefficients, the main challenge of the proof is to find estimation techniques to prove eigenvalue asymptotics without using classic techniques such as the Prüfer or Liouville transformation (see, for instance, [25]). In fact, our estimation technique is inspired by [14] where Schrödinger equations (6) with singular potential $q \in H^m, m \geq -1$, are treated, which, in the spirit of the Liouville transformation (although not well-defined here), corresponds to $s \in H^r[0,1], r = m+2 \geq 1$.

Proposition 5 (**Characteristic functions**). *Fix $r \geq 1$. Let $s \in H^r_{\mathbb{C}}[0,1]$. For the solutions $\tilde{y}_1$ and $\tilde{y}_2$ of* (11) *and* (12) *we have the representation*

$$\tilde{y}_1(1;\lambda) = \left(\frac{s(0)}{s(1)}\right)^{1/4} \left(\cos\left(\sqrt{\lambda} I\right) + \int_0^I \cos\left(\sqrt{\lambda}[I-2t]\right) f^+(t)\, dt\right), \tag{14}$$

$$\tilde{y}_2(1;\lambda) = \left(\frac{s(0)}{s(1)}\right)^{1/4} \left(\frac{\sin\left(\sqrt{\lambda} I\right)}{\sqrt{\lambda}} + \int_0^I \frac{\sin\left(\sqrt{\lambda}[I-2t]\right)}{\sqrt{\lambda}} f^-(t)\, dt\right), \tag{15}$$

with some functions $f^\pm \in H^{r-1}(0,I)$ and $I = \int_0^1 \sqrt{s(t)}\, dt$.

Remark 6. *We would like to emphasize that asymptotic estimates as implied by Proposition 5 are usually obtained from arguments involving integration-by-parts which extracts a decay factor $\frac{1}{\sqrt{\lambda}}$ from the sine/cosine in the integral, whereas in our case of non-smooth coefficients (to ensure open gaps) the estimation technique is more subtle.*

Remark 7. *Let us remark on the difference between the Schrödinger equations* (6) *and weighted equations* (1)*. The basic estimates for the fundamental system for* (6) *are of the form (see, for instance, [26])*

$$v_1(1,\lambda) = \cos(\sqrt{\lambda}) + \frac{\sin\sqrt{\lambda}}{2\sqrt{\lambda}} \int_0^1 q(\tau)\, d\tau + \mathcal{O}(\lambda^{-1})$$

$$v_2(1,\lambda) = \frac{\sin(\sqrt{\lambda})}{\sqrt{\lambda}} - \frac{\cos\sqrt{\lambda}}{2\lambda} \int_0^1 q(\tau)\, d\tau + \mathcal{O}(\lambda^{-3/2})\,.$$

Loosely speaking, the extra $1/\sqrt{\lambda}$ factor in these representations allows a more immediate derivation of asymptotics of eigenvalues, and, hence, of gap lengths. In more detail, it is known that (see, for instance, [20], [11], [12], [13])

$$q \in H^m \Longleftrightarrow (G_n)_{n\in\mathbb{N}} \in h^m \qquad (m \geq 0)\,. \tag{16}$$

Moreover, it is known that finite gap potentials q (that is, potentials that leave all but a finite number of gaps closed) are norm dense in L^2 (cf. [20]). In particular, a band structure with the properties (9) *from Corollary 3 cannot be obtained for Hill operators with regular potentials. However, one anticipates from* (16) *that distribution potentials $q \in H^m, m < 0$, do yield the possibility that all gaps are uniformly open. Among the first who studied spectral problems for Hill operators with singular potentials were* Kappeler *and* Möhr *([16]),* Savchuk *and* Shkalikov *([23], [24]),* Djakov *and* Mityagin *([8], [9]) and* Hryniv *and* Mykytyuk *([14], [15]). We refrain from attempting to give an exhaustive list of contributions to the field, but rather highlight [14], which gave the most relevant input for our work. They prove the following asymptotics for the Dirichlet eigenvalues μ_n (Theorem 1.1 in [14], restated): Assume that $q \in H^{\alpha-1}(0,1)$ for some $\alpha \in [0,1]$ and fix an arbitrary distributional primitive*

$\rho \in H^{\alpha}(0,1)$ of q. Then there exists a function $\widetilde{\rho} \in H^{2\alpha}(0,1)$ such that $\sqrt{\mu_n} = n\pi - \mathcal{F}_{\sin}[\rho](2n) - \mathcal{F}_{\sin}[\widetilde{\rho}](2n)$, where $\mathcal{F}_{\sin}[f](n) = \int_0^1 f(x)\sin(\pi n x)$ is the Fourier sine transform .

Proof. It is convenient to first write the differential equation in first order form, setting $z = y'$ and $Y = (y,z)^T$. Then

$$\frac{d}{dx}Y = \begin{pmatrix} 0 & 1 \\ -\lambda s(x) & 0 \end{pmatrix} Y \, .$$

Inspired by [14], we now diagonalize the "main" part of the equation by setting $Y = PU$, where

$$P(x) = \begin{pmatrix} 1 & 1 \\ i\sqrt{\lambda s(x)} & -i\sqrt{\lambda s(x)} \end{pmatrix} \; ; \; P^{-1}(x) = \begin{pmatrix} 1/2 & \frac{-i}{2\sqrt{\lambda s(x)}} \\ 1/2 & \frac{i}{2\sqrt{\lambda s(x)}} \end{pmatrix} .$$

Then

$$\frac{d}{dx}U = D(x)U + \frac{1}{4}\sigma(x)JU \, , \tag{17}$$

where

$$D(x) = \begin{pmatrix} i\sqrt{\lambda s(x)} - \frac{1}{4}\sigma(x) & 0 \\ 0 & -i\sqrt{\lambda s(x)} - \frac{1}{4}\sigma(x) \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

and $\sigma(x) = \frac{s'(x)}{s(x)}$. The advantage of this representation of the solution is that the principal part of the solution is diagonal and the off-diagonal part no longer depends on λ. The (initial value problem for the) non-autonomous system of equations

$$U' = D(x)U \; , \quad U(y) = U_y$$

has solution

$$U(x) = \Lambda(x,y)S(x,y)U_y$$

with

$$\Lambda(x,y) := \left(\frac{s(y)}{s(x)}\right)^{1/4}, \quad S(x,y) := \begin{pmatrix} e^{i\sqrt{\lambda}\Delta(x,y)} & 0 \\ 0 & e^{-i\sqrt{\lambda}\Delta(x,y)} \end{pmatrix} ,$$

and $\Delta(x,y)=\int_y^x \sqrt{s(t)}dt$. We can then write the solution to the full problem (17) as

$$U(x)=\Lambda(x,0)S(x,0)U_0+\frac{1}{4}\int_0^x \Lambda(x,t)S(x,t)\sigma(t)JU(t)dt\ .$$

We employ successive approximations introducing the recursion

$$\begin{aligned}\Sigma_0(x,0)&=\Lambda(x,0)S(x,0),\\ \Sigma_{n+1}(x,0)&=\frac{1}{4}\int_0^x \Lambda(x,t)S(x,t)\sigma(t)J\Sigma_n(t,0)dt\ .\end{aligned}$$

In order to ensure regularity for the corresponding solution we develop a relatively simple formula for Σ_n using

$$JS(x,y)=(S(x,y))^{-1}J=S(y,x)J\ .$$

Combined with basic properties of the solution operators Λ and S, we see that we can write

$$\begin{aligned}&\Sigma_{n+1}(x,0)\\ &=\frac{1}{4}\int_0^x \Lambda(x,x_n)S(x,x_n)\sigma(x_n)J\Sigma_n(x_n,0)dx_n\\ &=\frac{1}{4^2}\int_0^x\int_0^{x_n}\Lambda(x,x_{n-1})\sigma(x_n)\sigma(x_{n-1})S(x,x_n)JS(x_n,x_{n-1})J\Sigma_{n-1}(x_{n-1},0)dx_{n-1}dx_n\\ &=\frac{1}{4^{n+1}}\int_0^x\cdots\int_0^{x_1}\Lambda(x,0)\left(\prod_{k=0}^n\sigma(x_k)\right)S(x,x_n)JS(x_n,x_{n-1})\cdots JS(x_1,x_0)JS(x_0,0)d\underline{x}\\ &=\frac{1}{4^{n+1}}\int_0^x\cdots\int_0^{x_1}\Lambda(x,0)\left(\prod_{k=0}^n\sigma(x_k)\right)S(x,0)\mathcal{S}(\underline{x})J^{n+1}d\underline{x}\end{aligned}$$

with $\underline{x}=(x_0,\dots,x_n)$ (suppressing the n-dependence for readability) and

$$\begin{aligned}\mathcal{S}(\underline{x})&=S(x_n,0)^{-2}S(x_{n-1},0)^2\cdots S(x_0,0)^{2(-1)^{n+1}}\\ &=\begin{pmatrix}\exp\left(2i\sqrt{\lambda}r_n(\underline{x})\right) & 0\\ 0 & \exp\left(-2i\sqrt{\lambda}r_n(\underline{x})\right)\end{pmatrix}\end{aligned}$$

where

$$r_n(\underline{x})=\sum_{k=0}^n(-1)^{k+1}\Delta(x_{n-k},0).$$

Recall that we are interested in the special initial value problems (11) and (12). The corresponding solutions are given by

$$U(x) = \sum_{n=0}^{\infty} U_n(x), \quad U_0(x) = \Sigma_0(x,0)U(0), \quad U_{n+1}(x) = \Sigma_{n+1}(x,0)U(0).$$

with $U(0)$ accordingly given by

$$U_{\mathrm{Neu}}(0) = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad U_{\mathrm{Dir}}(0) = \frac{1}{2i\sqrt{\lambda s(0)}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

To show convergence of the series and prove the claimed representation it is convenient to introduce the following Lemma for the quantities $u_{n+1}^{\pm}$ which directly allow a representation for the components of the diagonal matrix Σ_{n+1}.

Lemma 8. *Define for $n \geq 1$ the auxiliary terms*

$$u_{n+1}^{\pm}(x_{n+1}) = \int_0^{x_{n+1}} \cdots \int_0^{x_1} \exp\left[\pm i\sqrt{\lambda}\left(\Delta(x_{n+1},0) + 2r_n(\underline{x})\right)\right] \left(\prod_{k=0}^{n} \sigma(x_k)\right) d\underline{x}.$$

Then

- *$u_{n+1}^{\pm}$ can be bounded point-wise, i.e.*

$$\left|u_{n+1}^{\pm}(x)\right| \leq \exp\left(\left|\mathrm{Im}(\sqrt{\lambda})\right| \Delta(x,0)\right) \frac{1}{n!} \left(\int_0^x |\sigma(t)| dt\right)^n,$$

- *and we have the representation*

$$u_{n+1}^{\pm}(1) = \int_0^I \exp(\pm i\sqrt{\lambda}(I-2t)) h_{n+1}(t)\, dt$$

with

$$h_{n+1}(t) = (-1)^{n+1} \int_{\Gamma_n(t)} \tilde{\sigma}\left(t + R_{n-1}(\xi_1,\ldots,\xi_n)\right) \tilde{\sigma}(\xi_1)\ldots\tilde{\sigma}(\xi_n)\, \xi_1 \ldots d\xi_n, \tag{18}$$

where

$$R_n(\xi_0,\ldots,\xi_n) = \sum_{k=0}^{n} (-1)^{k+1}\xi_{n-k}, \quad \widetilde{\sigma}(\cdot) = \frac{s'}{s^{3/2}}\left(\Delta^{-1}(\cdot)\right) \tag{19}$$

and the domain is given by

$$\Gamma_n(t) = \left\{(\xi_1,\ldots,\xi_n) \mid 0 \leq \xi_1 \leq \ldots \leq \xi_n \leq t + R_{n-1}(\xi_1,\ldots,\xi_n) \leq I\right\}. \tag{20}$$

Proof. The pointwise bound can be essentially read off directly noting that

$$|\Delta(x_{n+1},0)+2r_n(\underline{x})| \le |\Delta(x_{n+1},0)|$$

since

$$x_1 \le x_2 \le \ldots \le x_{n+1}.$$

The representation in terms of h_n is obtained by the change of variables $\xi_n = \Delta(x_n,0)$ (which is invertible since $\sqrt{s(x)} \ge 0$) for $n \ge 1$ and $t = r_n(x_0,\ldots,x_n)$.

■

Noting that

$$J^{n+1}U_{\mathrm{Neu}}(0) = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad J^{n+1}U_{\mathrm{Dir}}(0) = (-1)^{n+1}\frac{1}{2i\sqrt{\lambda s(0)}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and transforming back to the original variables via $Y = PU$ we get using Lemma 8 the Basic Estimate

$$\begin{aligned}
&|\widetilde{y}_j(x)| \\
&\le |C_s(x)|\exp\left(|\mathrm{Im}(\sqrt{\lambda})\Delta(x,0)|\right) + \frac{|C_s(x)|}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n+1}|u^+_{n+1}(x)| + |u^-_{n+1}(x)| \\
&\le |C_s(x)|\exp\left(|\mathrm{Im}(\sqrt{\lambda})\Delta(x,0)|\right) + \frac{|C_s(x)|}{2}\exp\left(|\mathrm{Im}(\sqrt{\lambda})\Delta(x,0)| + \frac{1}{4}\|\sigma\|_{L^2}\sqrt{x}\right)
\end{aligned} \tag{21}$$

where $C_s(x) := s(0)^{1/4}s(x)^{-1/4} = \Lambda(x,0)$ and

$$|\widetilde{y}_j'(x)| \le \frac{|\widetilde{y}_j(x)|}{|\sqrt{\lambda s(x)}|}.$$

This demonstrates convergence in $H^1[0,1]$. It remains to demonstrate the specific representation at $x = 1$. Again using the representation in Lemma 8 we have

$$\begin{aligned}
\widetilde{y}_1(1) &= C_s(1)\cos\left(\sqrt{\lambda}I\right) + C_s(1)\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n+1}\left(u^+_{n+1}(1) + u^-_{n+1}(1)\right) \\
&= C_s(1)\cos\left(\sqrt{\lambda}I\right) + C_s(1)\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n+1}\int_0^I \cos(\sqrt{\lambda}(I-2t))h_{n+1}(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}\widetilde{y}_2(1) &= C_s(1)\frac{\sin\left(\sqrt{\lambda}I\right)}{\sqrt{\lambda}} + C_s(1)\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^{n+1}\left(u_{n+1}^{+}(1)+u_{n+1}^{-}(1)\right) \\ &= C_s(1)\frac{\sin\left(\sqrt{\lambda}I\right)}{\sqrt{\lambda}} + C_s(1)\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^{n+1}\int_0^I \frac{\sin(\sqrt{\lambda}(I-2t))}{\sqrt{\lambda}}h_{n+1}(t)\,dt\end{aligned}$$

Interlude.. We make use of a blend of results from [14]. Define

$$I_n(f_0,\dots,f_n)(t) = \int_{\Gamma_n(t)} f_0(t+R_{n-1}(\xi_1,\dots,\xi_n))f_1(\xi_1)\cdots f_n(\xi_n)\,d\xi_1\cdots d\xi_n$$

with R_n as in (19) and the domain Γ_n as in (20). Furthermore, consider

$$\alpha\in[0,1],\quad \gamma=\min\{3\alpha,1+\alpha\},\quad \tau\in H^{\alpha}.$$

Then by [14], Lemma A.4, Theorem 4.3 and Corollary 4.5, it holds true that

$$\|I_n(\tau,\dots,\tau)\|_{H^{\gamma}} \le C\|\tau\|_{H^{\alpha}}^n,\quad n\in\mathbb{N},$$

and, consequently,

$$\|\sigma\|_{H^{r-1}}\le 1 \Rightarrow \|f^{\pm}\|_{H^{r-1}} \le C\|\sigma\|_{H^{r-1}}. \tag{22}$$

From this one can immediately conclude the following result.

Lemma 9 (from [14]). *Using the notation* $\Delta(1,0)=I$*, let* $\tilde{\sigma}\in H^{r-1}(0,I), r\in[1,2]$*. Define* $h_1=\tilde{\sigma}$ *and* h_{n+1} *as in* (18) *and define*

$$f^{\pm}=\sum_{n=1}^{\infty}\left(\pm\frac{1}{4}\right)^n h_n. \tag{23}$$

Then $f^{\pm}\in H^{r-1}(0,I)$*.*

This gives the desired statement about the regularity of $f^{\pm}$ and concludes the proof.

■

Note that from (22), the definition of $\tilde{\sigma}$, and the definition of the function $\tilde{s}$ in Remark 2, we have

Corollary 10. *There exists a constant $C > 0$ such that*

$$\|f^{\pm}\|_{H^{r-1}} \leq C\|\tilde{s}\|_{H^r} .$$

Proposition 11 (Dirichet and Neumann asymptotics, local boundedness and range of $\mathcal{G}$). *Let $s \in E \cap H^r_{\mathbb{C}}(0,1), r \geq 1$, be sufficiently close to 1. Then the* Dirichlet *and* Neumann *eigenvalues are of the form*

$$\mu_n = \left(\frac{n\pi}{I}\right)^2 + \gamma^{\mathrm{Dir}}, \quad \nu_n = \left(\frac{n\pi}{I}\right)^2 + \gamma^{\mathrm{Neu}}, \tag{24}$$

with $I = I(s) = \int_0^1 \sqrt{s(\xi)}\, d\xi$, and the remainders

$$(\gamma^{\mathrm{Dir}})_{n\in\mathbb{N}}, (\gamma^{\mathrm{Neu}})_{n\in\mathbb{N}} \in h^{r-2}.$$

Furthermore, the mapping $\mathcal{G}$ as defined in Theorem 1 has range contained in $\mathbb{C} \times h^{r-2}$ and $\mathcal{G}$ is locally bounded.

Remark 12. *Even though one could use Theorem 5.1 in [14] to conclude the eigenvalue asymptotics, we prefer to give yet another proof which directly paves the way for proving the local boundedness.*

Proof. By the substitutions

$$\rho := \sqrt{\lambda} I, \quad \rho\tau := \sqrt{\lambda}(I - 2t), \quad F(\tau) := \frac{I}{2} f^{-}\left(\frac{I}{2}(1-\tau)\right), \tag{25}$$

the characteristic equation $\widetilde{y}_2(1,\lambda) = 0$ (with $\widetilde{y}_2$ as in Lemma 5) can be recast as

$$0 = \sin(\rho) + \int_{-1}^{1} \sin(\rho\tau) F(\tau)\, d\tau =: \mathcal{G}(\rho). \tag{26}$$

Using the Riemann-Lebesgue lemma one can readily conclude that the integral approaches zero for large ρ. We would like to use Rouché's theorem on regions $|\rho - n\pi| = \frac{\pi}{2}, n \in \mathbb{N}, \rho \in \mathbb{C}$, for n large to gain a more precise insight into this convergence. First consider the case $r > 1$

$$
\begin{aligned}
|\mathcal{G}(\rho)-\sin(\rho)| = \left|\int_{-1}^{1}\sin(\rho y)F(y)\,dy\right| &= \left|\int_{-1}^{1}\sin(\rho y)\left(\sum_{k\in\mathbb{Z}}\hat{F}_k e^{ik\pi y}\right)\,dy\right| \\
&\le 2\sum_{k\in\mathbb{N}}|\hat{F}_k||\sin(\rho)|\left|\frac{k\pi}{k^2\pi^2-\rho^2}\right| \\
&\le 2\exp(|\mathrm{Im}(\rho)|)\left(\sum_{k\in\mathbb{N}}|\hat{F}_k|k^{r-1}k^{-(r-1)}\left|\frac{1}{k\pi-\rho}\right|\right) \\
&\le 2C\exp(|\mathrm{Im}(\rho)|)\ \frac{||F||_{H^{r-1}[-1,1]}}{\rho^{r-1}},
\end{aligned}
$$

where we used the Cauchy-Schwarz inequality and the fact that

$$
\left(\sum_{k\in\mathbb{N}}\frac{1}{|k|^{2(r-1)}\,|k\pi-\rho|^2}\right)^{1/2} \le \frac{C}{\rho^{r-1}},
$$

for $C > 0$ independent of F and ρ. Furthermore, since for $|\rho - n\pi| \ge \pi/4$ we have $\exp(|\mathrm{Im}(\rho)|) < 4|\sin(\rho)|$ (which is proven in [22], Lemma 2.1), it follows that

$$
|\mathcal{G}(\rho)-\sin(\rho)| \le 4C|\sin(\rho)|\ \frac{||F||_{H^{r-1}[-1,1]}}{\rho^{r-1}}.
$$

Consequently, if $\rho^{r-1} > 4C||F||_{H^{r-1}[-1,1]}$ we can conclude that

$$
|\mathcal{G}(\rho)-\sin(\rho)| < |\sin(\rho)|,
$$

and, hence, by Rouché's theorem that $\mathcal{G}$ has exactly one root inside the region $|\rho - n\pi| = \frac{\pi}{2}$. In other words, we can guarantee that

$$
|\sqrt{\mu_n}I - n\pi| < \frac{\pi}{2}, \quad n > N,
$$

for some $N \in \mathbb{N}$ with

$$
N > \left(4C||F||_{H^{r-1}[-1,1]}\right)^{1/(r-1)}. \tag{27}
$$

For $r = 1$

$$
\begin{aligned}
|\mathcal{G}(\rho) - \sin(\rho)| = \left|\int_{-1}^{1} \sin(\rho y) F(y)\, dy\right| &= \left|\int_{-1}^{1} \sin(\rho y) \left(\sum_{k\in\mathbb{Z}} \hat{F}_k e^{ik\pi y}\right)\, dy\right| \\
&\leq 2\sum_{k\in\mathbb{N}} |\hat{F}_k||\sin(\rho)| \left|\frac{k\pi}{k^2\pi^2 - \rho^2}\right| \\
&\leq 2\widetilde{C}\exp(|\mathrm{Im}(\rho)|)\ ||F||_{L^2[-1,1]},
\end{aligned}
$$

for $\widetilde{C} > 0$ independent of F and ρ. As before, since for $|\rho - n\pi| \geq \pi/4$ we have $\exp(|\mathrm{Im}(\rho)|) < 4|\sin(\rho)|$, it follows that

$$
|\mathcal{G}(\rho) - \sin(\rho)| \leq 4\widetilde{C}|\sin(\rho)|||F||_{L^2[-1,1]}.
$$

Consequently, choosing $\tilde{s}$ small enough to ensure $1 > 4\widetilde{C}||F||_{L^2[-1,1]}$ we can conclude that

$$
|\mathcal{G}(\rho) - \sin(\rho)| < |\sin(\rho)|\,,
$$

and, hence, by Rouché's theorem that $\mathcal{G}$ has exactly one root inside the region $|\rho - n\pi| = \frac{\pi}{2}$. In other words, we can guarantee that

$$
|\sqrt{\mu_n}I - n\pi| < \frac{\pi}{2}, \quad n \in N\,.
$$

In order to get more precise estimates on the eigenvalue asymptotics, we replace ρ in (26) by the representation

$$
\sqrt{\mu_n}I = n\pi + l_n\,, \quad |l_n| < \pi/2\,,
$$

to get

$$
\begin{aligned}
&\sin(n\pi + l_n) = -\int_{-1}^{1} \sin([n\pi + l_n]\tau)\, F(\tau) d\tau \\
&= -\int_{-1}^{1} \sin(n\pi\tau)\cos(l_n\tau)\, F(\tau)d\tau - \int_{-1}^{1} \cos(n\pi\tau)\sin(l_n\tau)\, F(\tau)d\tau\,. \qquad (28)
\end{aligned}
$$

We will make use of a fixed point argument to derive the desired estimate on

l_n. Define $\eta_n = \sin(l_n)$ such that

$$
\begin{aligned}
\eta_n &= (-1)^{n+1}\left[\int_{-1}^{1}\sin(n\pi\tau)\cos\left(\sin^{-1}(\eta_n)\,\tau\right)F(\tau)d\tau\right.\\
&\quad\left.+\int_{-1}^{1}\cos(n\pi\tau)\sin\left(\sin^{-1}(\eta_n)\,\tau\right)F(\tau)d\tau\right]\\
&=\Phi_n(\eta)
\end{aligned}
$$

with $\eta = (\eta_n)_{n\in\mathbb{N}}$. Note that $(\Phi_n(\eta))_{n\in\mathbb{N}} = \Phi(\eta) : h^{r-1} \to h^{r-1}$, since, by integration by parts,

$$
\begin{aligned}
&\left|\int_{-1}^{1}\sin(n\pi\tau)\cos\left(\sin^{-1}(\eta_n)\,\tau\right)F(\tau)d\tau\right|\\
&\quad\leq\left|\int_{-1}^{1}\sin(n\pi\tau)\,F(\tau)d\tau\right|\,\left|\cos\left(\sin^{-1}(\eta_n)\right)\right|+\left|\sin^{-1}(\eta_n)\right|\|F\|_{L^2[-1,1]}\,,
\end{aligned}
$$

and the analogous estimate is true for the second term in $\Phi_n(\eta)$. Furthermore, Φ is a contraction, since

$$
\begin{aligned}
&|\Phi_n(\eta)-\Phi_n(\widetilde{\eta})|\\
&\quad\leq\left|\int_{-1}^{1}\left\{\cos\left(\sin^{-1}(\eta_n)\,\tau\right)-\cos\left(\sin^{-1}(\widetilde{\eta}_n)\,\tau\right)\right\}\sin(n\pi\tau)\,F(\tau)d\tau\right|\\
&\quad\quad+\left|\int_{-1}^{1}\left\{\sin\left(\sin^{-1}(\eta_n)\,\tau\right)-\sin\left(\sin^{-1}(\widetilde{\eta}_n)\,\tau\right)\right\}\cos(n\pi\tau)\,F(\tau)d\tau\right|\\
&\quad\leq C_0\|F\|_{L^2[-1,1]}\,|\eta-\widetilde{\eta}|\ ,
\end{aligned}
$$

for $C\|F\|_{L^2[-1,1]} < 1$ which can be achieved by choosing $s = 1+\widetilde{s}$ with $\widetilde{s}$ sufficiently small (recall that F depends on s). This guarantees that the iteration $\eta^{(k+1)} = \Phi(\eta^{(k)})$ has a unique fixed point η^*. Choosing, for instance, $\eta^{(0)} = 0$ gives $\eta^{(1)} = \Phi(\eta^{(0)}) = \int_{-1}^{1}\sin(n\pi\tau)\,F(\tau)d\tau$ and, by standard estimates we have

$$
\|\eta^*\|_{h^{r-1}} \leq \frac{\|F\|_{H^{r-1}[-1,1]}}{1-C_0\|F\|_{L^2[-1,1]}}\,,
$$

which, expressed in our quantity of interest becomes

$$
\|l^*\|_{h^{r-1}} \leq \tilde{C}_0\|F\|_{H^{r-1}[-1,1]} \leq \check{C}_0\|\tilde{s}\|_{H^r[0,1]}\,,\quad \sin(l^*) = \eta^*\,,
$$

where we made use of (22) and (25). Since the fixed point is unique, we know that it must coincide with the Dirichlet remainders that solve (28). The analysis for the Neumann eigenvalues is analogous. This concludes the proof, since local boundedness and the range of $\mathcal{G}$ are given as a byproduct of our analysis of asymptotics.

∎

Lemma 13 (**Analyticity of the coordinate functions of** $\mathcal{G}$**).** *Let the conditions in Proposition 11 be fulfilled. Then each coordinate function of* $\mathcal{G}$ *in* (5) *is analytic.*

Proof. The proof can be adapted essentially line-by-line from [22]. For completeness, we execute it in Appendix Appendix A.

∎

Lemma 14. *Fix* $r \geq 1$*. The mapping*

$$\mathcal{G} : E \cap H^r[0,1] \longrightarrow \mathbb{R} \times h^{r-2} \tag{29}$$

$$s \longmapsto \left[\frac{1}{\left(\int_0^1 \sqrt{s(\xi)}\, d\xi\right)^2} \;,\; (G_n(s))_{n\in\mathbb{N}} \right]$$

is real-analytic in a neighborhood of $s = 1$*. Moreover, its directional derivative is given by*

$$D_1[\mathcal{G}](\tilde{s}) = \left[-\int_0^1 \tilde{s}(x)\, dx \,,\, \left(2n^2\pi^2 \int_0^1 \cos(2n\pi x)\tilde{s}(x) \right)_{n\in\mathbb{N}} \right] \tag{30}$$

Proof. The gradient $\frac{\partial \mu_n}{\partial s}$: Let $s \in C^0([0,1]) \subset L^2([0,1])$ (noting that $C^0([0,1])$ is dense in $L^2([0,1])$) and y_n be the solution of the Dirichlet eigenvalue problem

$$y_n'' + \mu_n s y_n = 0, \quad y_n(0) = 0, y_n(1) = 0, \tag{31}$$

Note that $y_n = y_n(x,s)$ and $\mu_n = \mu_n(s)$. Taking a directional derivative at a point s_0 in direction $\tilde{s}$ ($s_0, \tilde{s} \in C^0([0,1])$) of equation (31) yields

$$\underbrace{D_{s_0}[y_n](\tilde{s})'' + \mu_n(s_0)s_0 D_{s_0}[y_n](\tilde{s})}_{=:Q_{s_0}[D_{s_0}[y_n](\tilde{s})]} = -\left(\mu_n(s_0)\tilde{s} + D_{s_0}[\mu_n](\tilde{s})s_0\right)y_n(s_0)\,.$$

Projection on $y_n(s_0)$ via the L^2 scalar product gives

$$\langle Q_{s_0}[D_{s_0}[y_n](\tilde{s})], y_n(s_0)\rangle = -\langle(\mu_n(s_0)\tilde{s} + D_{s_0}[\mu_n](\tilde{s})s_0)y_n(s_0), y_n(s_0)\rangle.$$

But since $\langle Q_{s_0}[D_{s_0}[y_n](\tilde{s})], y_n(s_0)\rangle = \langle D_{s_0}[y_n](\tilde{s})], Q_{s_0}[y_n(s_0)]\rangle = 0$[1], we immediately get

$$D_{s_0}[\mu_n](\tilde{s}) = \langle -\frac{\mu_n(s_0)}{\langle y_n(s_0)^2, s_0\rangle} y_n(s_0)^2, \tilde{s}\rangle.$$

Hence, the gradient of μ_n is given by

$$\frac{\partial \mu_n(s)}{\partial s} = -\frac{\mu_n(s)}{\langle y_n(s)^2, s\rangle} y_n(s)^2.$$

The gradient of the DIRICHLET remainder $\frac{\partial \gamma^{\mathrm{Dir}}}{\partial s}$ at $s(x) = 1$: Taking the directional derivative of μ_n at $s_0 = 1$ in direction $\tilde{s} \in L^2([0,1])$ gives

$$D_1[\mu_n](\tilde{s}) = \langle -2n^2\pi^2 \sin^2(n\pi\cdot), \tilde{s}\rangle = n^2\pi^2(\langle \cos(2n\pi\cdot), \tilde{s}\rangle - \langle 1, \tilde{s}\rangle).$$

Since

$$D_1\left[\left(\frac{n\pi}{I}\right)^2\right](\tilde{s}) = -\langle 1, \tilde{s}\rangle,$$

we get the directional derivative of the DIRICHLET remainders to be simply

$$D_1[\gamma^{\mathrm{Dir}}](\tilde{s}) = n^2\pi^2\langle \cos(2n\pi\cdot), \tilde{s}\rangle = n^2\pi^2 \int_0^1 \tilde{s}(t)\cos(2\pi n t)\,dt = \frac{n^2\pi^2}{2}\hat{f}_n(\tilde{s}).$$

The analogous procedure for the NEUMANN eigenvalues gives

$$D_1[\nu_n](\tilde{s}) = \langle -2n^2\pi^2 \cos^2(n\pi\cdot), \tilde{s}\rangle = n^2\pi^2(-\langle \cos(2n\pi\cdot), \tilde{s}\rangle - \langle 1, \tilde{s}\rangle),$$

so

$$D_1[\gamma^{\mathrm{Neu}}](\tilde{s}) = -n^2\pi^2\langle \cos(2n\pi\cdot), \tilde{s}\rangle = -n^2\pi^2 \int_0^1 \tilde{s}(t)\cos(2\pi n t)\,dt = -\frac{n^2\pi^2}{2}\hat{f}_n(\tilde{s}).$$

[1]Note that we integrated by parts twice. The boundary terms vanish, since $y_n(0) = y_n(1)$.

$\blacksquare$

By the analytic inverse function theorem we can readily conclude the statement of Theorem 1 from Lemma 14.

3. Breathers and uniformly open gap potentials

Motivated by the construction of breather solutions for nonlinear wave equations via dynamical systems tools, we posed the particular inverse problem of finding coefficients s such that the spectrum of the corresponding weighted Sturm-Liouville operator $\mathcal{L}_s = -\frac{1}{s(x)}\frac{d^2}{dx^2}$ has all gaps uniformly open around $\omega_*^2 n^2, n \geq 1$, for some $\omega_* \in \mathbb{R}$. In order to solve this problem we formalized the dependence of the (signed) gap lengths and gap midpoints (to first order) on the coefficient s as the mapping $\mathcal{G}$ from (5). We find that, for $s \in H^r[0,1], r \geq 1$, the gap lengths form a sequence in h^{r-2} and established that $\mathcal{G}$ is a bijection locally around $s = 1$. Since constant sequences $(C)_{n\in\mathbb{N}}$ (for some $C > 0$) are in h^{r-2} if $r < 3/2$, we know (by Theorem 1) that there must be coefficients $s \in H^r[0,1], r < 3/2$, that give rise to uniformly open gaps. What remains unclear is the exact location of the gaps. This is the subject of this section.

Let us consider (1) on the real line, that is,

$$y'' + \lambda\, s(x)\, y = 0\,, \quad x \in \mathbb{R}\,. \tag{32}$$

We have that $\lambda \in \sigma(\mathcal{L}_s)$ if and only if (32) has all (nontrivial) solutions bounded. The qualitative behavior of solutions of (32) is decided by the Floquet exponents. The proof of Floquet's theorem via consideration of the discriminant (the trace of the monodromy matrix)

$$\mathcal{D}(\lambda) = \widetilde{y}_1(1;\lambda) + \widetilde{y}_2'(1;\lambda), \tag{33}$$

where $\{\widetilde{y}_1, \widetilde{y}_2\}$ is the fundamental system solving (11) and (12), gives the well-known relation

$$e^{l_\pm(\lambda)} = \frac{1}{2}\left(\mathcal{D}(\lambda) \pm \sqrt{\mathcal{D}(\lambda)^2 - 4}\right) \tag{34}$$

for the Floquet exponents $l_{\pm}$. This representation suggests that the derivation of explicit formulas for the Floquet exponents is only possible if one can write down explicit expressions for $\{\tilde{y}_1, \tilde{y}_2\}$. Since our equation is non-autonomous, this case is rather exceptional and the computation of Floquet exponents is usually carried out numerically. From a qualitative point of view, further elementary considerations along (34) allow to conclude that bounded solutions are only possible for $|\mathcal{D}(\lambda)| \leq 2$. By [10] we know that

$$\{\lambda \mid \ |\mathcal{D}(\lambda)| \leq 2\} = \cup_{n \in \mathbb{N}} B_n = \sigma(\mathcal{L}_s) \tag{35}$$

where B_n are the intervals from (2), that is, the bands.

Let us now turn to the connection between the Dirichlet and Neumann eigenvalues and the structure of the spectrum of $\mathcal{L}_s$. By Abel's theorem,

$$\det \begin{pmatrix} \tilde{y}_1(x) & \tilde{y}_2(x) \\ \tilde{y}_1'(x) & \tilde{y}_2'(x) \end{pmatrix} = \det \begin{pmatrix} \tilde{y}_1(1) & \tilde{y}_2(1) \\ \tilde{y}_1'(1) & \tilde{y}_2'(1) \end{pmatrix} \tag{36}$$

so, if μ_n is a Dirichlet eigenvalue, $1 = \tilde{y}_1(1)\tilde{y}_2'(1)$, and therefore $|\mathcal{D}(\mu_n)| \geq 2$. The same is true for Neumann eigenvalues. In other words, Dirichlet and Neumann eigenvalues always lie in gaps. As a consequence,

$$|G_n| \geq |\mu_n - \nu_n|,$$

so the distance between the these eigenvalues gives a lower bound for the gap lengths. Further considerations (as, for instance, in [21]) show that, if s is even,

$$\{\lambda_{2n-1}(s), \lambda_{2n}(s)\} = \{\mu_n(s), \nu_n(s)\}\,, \quad n \geq 1\,.$$

so, in fact, $|G_n| = |\mu_n - \nu_n|$. From this statement, it is not clear *a priori* which of the eigenvalues μ_n and ν_n is the lower and upper gap edge. However, taking a closer look at the computation of the directional derivative of Dirichlet and Neumann eigenvalues in the proof of Lemma 14, it becomes clear that in a neighborhood of $s = 1$, the Dirichlet and Neumann eigenvalues are arranged on opposite sides of $n^2\omega_*^2, n \in \mathbb{N}$, for $\omega_* = 1/I(s)$. In view of this, Corollary 3 is a direct consequence of Theorem 1.

4. Summary and future work

In [4], a novel construction procedure for breather solutions in nonlinear wave equations with spatially periodic coefficients was demonstrated. The crucial step for this construction was the tailoring of the spectrum of the linear part via the periodic coefficients. While [4] only gave the specific example

$$s(x) = 1 + 15\chi_{[6/13,7/13)}\,(\,x \bmod 1\,)\,, \tag{37}$$

which was achieved by elementary computations of the discriminant, the present investigation lays the groundwork for an extension of the construction procedure beyond this special coefficient by formulating the tailoring of the spectrum as inverse problem. To be more precise, we studied the inverse problem for a weighted Sturm-Liouville operator $\mathcal{L}_s$ associated with the eigenvalue problem $y'' + \lambda\, s(x)\, y = 0$, where s is a real-valued, periodic, even function that is bounded from below by a positive constant and belongs to the L^2-based Sobolev space $H^r[0,1], r \geq 1$. The spectrum of $\mathcal{L}_s$ is given by a countably infinite union of real intervals, called *bands*, that are separated by possibly void open intervals, called *gaps*. Hence, the spectrum of $\mathcal{L}_s$ is fully characterized by the length and location of gaps. Furthermore, in our setting, the edges of the n-th gap are given by the n-th Dirichlet eigenvalue, μ_n, and n-th Neumann eigenvalue, ν_n, and we established here that

$$\mu_n = \frac{n^2\pi^2}{\left(\int_0^1 \sqrt{s(\xi)}\, d\xi\right)^2} + \gamma_n^{\mathrm{Dir}}\,, \qquad \nu_n = \frac{n^2\pi^2}{\left(\int_0^1 \sqrt{s(\xi)}\, d\xi\right)^2} + \gamma_n^{\mathrm{Neu}}\,,$$

with $(\gamma_n^{\mathrm{Dir}})_{n\in\mathbb{N}}, (\gamma_n^{\mathrm{Neu}})_{n\in\mathbb{N}} \in h^{r-2}$ for $s \in E \cap H^r[0,1]$ (as in Theorem 1). Hence, it is convenient to define a spectral map $\mathcal{G}$, that assigns to a coefficient s the structure of the spectrum of $\mathcal{L}_s$ in terms of gap lengths and gap midpoints. We find that $\mathcal{G}$ is a real-analytic isomorphism locally around $s = 1$ (see Theorem 1). In particular, this implies the existence of coefficients $s \in H^r[0,1], r < 3/2$, giving rise to band structure with all gaps uniformly open around the gap midpoints (see Corollary 3), which in turn paves the way for breather construction in nonlinear wave equations with such coefficients s.

We would like to stress that this inverse spectral problem for the weighted Sturm-Liouville operator has not heretofore been treated for the full Banach scale $H^r[0,1], r \geq 1$, which includes the more challenging range of non-smooth potentials. Moreover, the local nature of our result allows more concise and transparent proofs. In particular, instead of using any preliminary transformations, we treat the weighted problem directly by adapting techniques used for Schrödinger operators with distribution potentials.

The contribution of the present work is, therefore, at least twofold: It addresses a new inverse problem and simultaneously sets the course for further developing a new tool for PDE existence problems. Apart from applying our results to PDE problems, we plan to further extend it in at least two ways. The next natural step would be to achieve a global result in $H^r[0,1]$, that is, away from a neighborhood of $s = 1$. Moreover, motivated by the special coefficient in (37) that is known to yield uniformly open gaps, we aim to extend our investigations to $s \in H^r[0,1], r \geq 0$. Such a case of discontinuous s in a weight type Sturm-Liouville problem will most likely need a completely new approach that could, however, be inspired by the methodology presented here and also contribute to a more general theory for Schrödinger operators with distribution potentials.

Appendix A. Proof of analyticity of the coordinate functions of $\mathcal{G}$

It is standard to use the fundamental system $\{\widetilde{y}_1, \widetilde{y}_2\}$ that solves (11) and (12) subject to the initial conditions

$$\tilde{y}_1'' + \lambda s \tilde{y}_1 = 0, \quad \tilde{y}_1(0) = 1\,,\ \tilde{y}_1'(0) = 0\,, \tag{A.1}$$

and, respectively,

$$\tilde{y}_2'' + \lambda s \tilde{y}_2 = 0, \quad \tilde{y}_2(0) = 0\,,\ \tilde{y}_2'(0) = 1\,, \tag{A.2}$$

as characteristic functions for Dirichlet and Neumann eigenvalues, since it holds true that

$$\widetilde{y}_1(1;\lambda) = 0 \quad \text{iff} \quad \lambda = \nu_n, \quad \widetilde{y}_2(1;\lambda) = 0 \quad \text{iff} \quad \lambda = \mu_n.$$

We used the notation $\widetilde{y}_j = \widetilde{y}_j(x;\lambda)$ to stress the dependence of the fundamental system on the spectral parameter λ.

Following [22] we carry out the proof of analyticity in several steps: First we prove analyticity of $\widetilde{y}_j$ in s and λ, that is, of the map $(s,\lambda) \mapsto \widetilde{y}_j(1,s,\lambda)$ on $L^2_{\mathbb{C}}[0,1] \times \mathbb{C}$. Via the implicit function theorem we can show that each coordinate function μ_n and ν_n is analytic on $L^2_{\mathbb{C}}[0,1]$, and, hence, so are the corresponding remainders $\gamma_n^{\mathrm{Dir}}, \gamma_n^{\mathrm{Neu}}$. By the local boundedness of this map one can finally conclude analyticity.

Lemma 15. *Fix $r \geq 0$. The solutions $\widetilde{y}_j(x;s,\lambda)$ (of (11) and (12)) can be represented by a power series in $\tilde{s} = s-1$, which converges uniformly on bounded subsets of $[0,1] \times L^2_{\mathbb{C}}(0,1) \times \mathbb{C}$. Moreover, for fixed $x \in [0,1], \lambda \in \mathbb{R}$, $\widetilde{y}_j(x;\cdot,\lambda)$ is a locally analytic function around $s = 1$.*

Proof. We will demonstrate the result for $\widetilde{y}_2$. An analogous argument can be employed for $\widetilde{y}_1$. Expanding $\widetilde{y}_2$ as a (formal) power series in $\tilde{s}$, i.e.

$$\widetilde{y}_2(x,\lambda,s) = \mathcal{Y}_0(x,\lambda) + \mathcal{Y}_1(x,\lambda,\tilde{s}) + \sum_{n\geq 2} \mathcal{Y}_n(x,\lambda,\tilde{s}),$$

where $\mathcal{Y}_1$ is a linear functional in $\tilde{s}$ while $\mathcal{Y}_n(x,\lambda,\tilde{s})$ is a short hand notation for an $n-$linear form with $\tilde{s}$ in each argument, the equation hierarchy for $\mathcal{Y}_n$ reads

$$\begin{aligned} &\mathcal{Y}_0'' + \lambda\mathcal{Y}_0 = 0, && \mathcal{Y}_0(0) = 0, \mathcal{Y}_0'(0) = 1, \\ &\mathcal{Y}_n'' + \lambda\mathcal{Y}_n = -\lambda\tilde{s}\mathcal{Y}_{n-1}, && \mathcal{Y}_n(0) = 0, \mathcal{Y}_n'(0) = 0, \quad n \geq 1. \end{aligned}$$

Hence, $\mathcal{Y}_0(x,\lambda) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}$ and by the variation of constant formula

$$\begin{aligned}
&\mathcal{Y}_n(x_n, \lambda, \tilde{s}) \\
&= \sqrt{\lambda} \int_0^{x_n} \sin(\sqrt{\lambda}(x_n - t))\tilde{s}(t)\mathcal{Y}_{n-1}(t)dt \\
&= \left(\sqrt{\lambda}\right)^{n-1} \int_0^{x_n} \cdots \int_0^{x_1} \prod_{j=1}^{n} \sin(\sqrt{\lambda}(x_j - x_{j-1}))\tilde{s}(x_{j-1}) \sin(\sqrt{\lambda}x_0)\, dx_0 \cdots dx_{n-1},
\end{aligned} \tag{A.3}$$

which gives

$$|\mathcal{Y}_n(x)| \leq \frac{1}{\sqrt{\lambda}} \left(\frac{1}{n!}\right) \left(\sqrt{\lambda}\sqrt{x} \int_0^1 |\tilde{s}(t)|\, dt\right)^n \exp(|\mathrm{Im}\sqrt{\lambda}|x).$$

This estimate immediately implies absolute and uniform convergence of the power series for $\widetilde{y}_2 = \widetilde{y}_2(x,\lambda,s)$ on bounded subsets of $[0,1] \times \mathbb{C} \times L^2$. Since $\mathcal{Y}_n(\tilde{s}) = A_n(\tilde{s},\ldots,\tilde{s})$ where A_n is a bounded, n-linear, symmetric map, the series

$$\widetilde{y}_2(\tilde{s}) = \sum_{n \geq 0} \mathcal{Y}_n(\tilde{s})$$

defines an analytic function on $L^2_{\mathbb{C}}$ (cf. [7]).

■

We are ready to prove Lemma 13. We will demonstrate the result for μ_n. An analogous argument can be employed for ν_n. Following [22], our strategy is to apply the implicit function theorem to the characteristic equation $\widetilde{y}_2(1,\mu_n,s) = 0$. To this end, we have to verify $\frac{\partial \widetilde{y}_2(1,\mu_n,s)}{\partial \lambda} \neq 0$. Assume first that $s \in C^0[0,1]$. The estimate can then be extended to $s \in L^2[0,1]$ by continuity. Let $\widetilde{y}_2$ solve the initial value problem (12). Taking the λ-derivative of $\widetilde{y}_2'' + \lambda s \widetilde{y}_2 = 0$ gives

$$(\widetilde{y}_2'')_\lambda + s\widetilde{y}_2 + s\lambda(\widetilde{y}_2)_\lambda = 0.$$

Hence,

$$0 = (\widetilde{y}_2'' + \lambda s\widetilde{y}_2)\,(\widetilde{y}_2)_\lambda - ((\widetilde{y}_2)''_\lambda + s\widetilde{y}_2 + s\lambda(\widetilde{y}_2)_\lambda)\,\widetilde{y}_2 = ((\widetilde{y}_2)_\lambda \widetilde{y}_2' - (\widetilde{y}_2)'_\lambda \widetilde{y}_2)' - s\widetilde{y}_2^2,$$

so

$$\int_0^1 ((\widetilde{y}_2)_\lambda(x)\widetilde{y}_2'(x) - (\widetilde{y}_2)_\lambda'(x)\widetilde{y}_2(x))' \, dx = \int_0^1 s(x)\widetilde{y}_2(x)^2 \, dx,$$

and

$$0 < \int_0^1 s(t)\widetilde{y}_2(t,\mu_n,s)^2 \, dt = \left(\frac{\partial \widetilde{y}_2(1,\mu_n,s)}{\partial\lambda}\right) \widetilde{y}_2'(1,\mu_n,s)\,,$$

since $\widetilde{y}_2(0,\lambda)$ and $(\widetilde{y}_2)_\lambda(0,\lambda)$ vanish for all λ and $\widetilde{y}_2(1,\mu_k) = 0$. We have that

$$\widetilde{y}_2(1;k^2\pi^2,1)|_{(\lambda,s)=(n^2\pi^2,1)} = 0\,, \quad \frac{\partial \widetilde{y}_2(1;\lambda,s)}{\partial\lambda}|_{(\lambda,s)=(n^2\pi^2,1)} \neq 0\,.$$

$$\widetilde{y}_2(1;k^2\pi^2,1) = 0, \quad \frac{\partial \widetilde{y}_2(1,\lambda,s)}{\partial\lambda}|_{\lambda=k^2\pi^2} \neq 0\,.$$

So there is a unique analytic function $\hat{\mu}_n = \hat{\mu}_n(s)$ defined in an H^r-neighborhood $\mathcal{U}$ of $s = 1$, for which

$$\widetilde{y}_2(1,\hat{\mu}_n(s),s) = 0.$$

It remains to show that μ_n is continuous on H^r, such that, by uniqueness, $\hat{\mu}_n(s) = \mu_n(s)$. To this end one can follow the proof of compactness of μ_n from [22], Theorem 2.5, line-by-line. We execute the details for completeness:
First, we show that $\widetilde{y}_1$ and $\widetilde{y}_2$ are uniformly compact on bounded subsets of $\{(\lambda,x) \in \mathbb{C}\times[0,1]\}$. Again, we will execute a detailed proof only for $\widetilde{y}_2$ and make use of the notation from the previous proof. Suppose the sequence s_m converges weakly to s. By the principle of uniform boundedness

$$\|s\| \le \sup_m \|s_m\| \le M < \infty\,.$$

If A is any bounded subset of $\{(\lambda,x) \in \mathbb{C}\times[0,1]\}$, then for $n \ge 1$ we have the estimate

$$|\mathcal{Y}_n(x;p,\lambda)| \le \left(\frac{1}{n!}\right)\left(\sqrt{\lambda}\right)^{n-1}\left(\sqrt{x}\int_0^1 |s(t)|\,dt\right)^n \exp(|\mathrm{Im}\sqrt{\lambda}|x) \le c\,\frac{M^n}{n!}$$

uniformly on A where

$$\mathcal{Y}_n(x_n,\lambda,s)$$
$$= \left(\sqrt{\lambda}\right)^{n-1}\int_0^{x_n}\cdots\int_0^{x_1}\prod_{j=1}^n \sin\left(\sqrt{\lambda}(x_j - x_{j-1})\right) s(x_{j-1}) \sin\left(\sqrt{\lambda}x_0\right)\,dx_0\cdots dx_{n-1}\,,$$

was already defined in (A.3) for real-valued s. Thus,

$$
\begin{aligned}
&|\widetilde{y}_1(x; s_m, \lambda) - \widetilde{y}_1(x; s, \lambda)| \\
&\quad \leq \sum_{n=1}^{N} |\mathcal{Y}_n(x; s_m, \lambda) - \mathcal{Y}_n(x; s, \lambda)| + \sum_{n=N+1}^{\infty} \left(|\mathcal{Y}_n(x; s_m, \lambda)| + |\mathcal{Y}_n(x; s, \lambda)|\right) \\
&\quad \leq \sum_{n=1}^{N} |\mathcal{Y}_n(x; s_m, \lambda) - \mathcal{Y}_n(x; s, \lambda)| + 2c \sum_{n=N+1}^{\infty} \frac{M^n}{n!}
\end{aligned}
$$

uniformly on A. The second sum converges to zero as N tends to infinity. Therefore, it is enough to show that each term $\mathcal{Y}_n(x; p_m, \lambda)$ converges to $\mathcal{Y}_n(x; p, \lambda)$ uniformly on A for fixed n. Note that one can express the difference of these quantities in terms of the L^2-scalar product

$$
\begin{aligned}
&Z_m(\lambda, x) := \mathcal{Y}_n(x; s_m, \lambda) - \mathcal{Y}_n(x; s, \lambda) \\
&= \left(\sqrt{\lambda}\right)^{n-1} \int_0^{x_n} \cdots \int_0^{x_1} \prod_{j=1}^{n} \sin\left(\sqrt{\lambda}(x_j - x_{j-1})\right) (s_m - s)(x_{j-1}) \sin\left(\sqrt{\lambda} x_0\right) \, dx_0 \cdots dx_{n-1} \\
&= \left\langle p_{\lambda,x}, \prod_{j=1}^{n} (\bar{s}_m - \bar{s})(x_{j-1}) \right\rangle_{L^2_{\mathbb{C}}([0,1]^n)}
\end{aligned}
$$

with

$$
p_{\lambda,x}(x_0, \ldots, x_n) = \left(\sqrt{\lambda}\right)^{n-1} \prod_{j=1}^{n} \sin\left(\sqrt{\lambda}(x_j - x_{j-1})\right) \sin\left(\sqrt{\lambda} x_0\right) \; \chi_{\{0 \leq t_1 \leq \cdots \leq t_n \leq x\}} \,.
$$

With this notation, we can reformulate our problem as proving

$$
\sup_A |Z_m(\lambda, x)| \longrightarrow 0, \quad m \longrightarrow \infty \,,
$$

which, by continuity properties in λ and x is equivalent to

$$
|Z_m(\lambda_*^m, x_*^m)| \longrightarrow 0, \quad m \longrightarrow \infty \,, \tag{A.4}
$$

where (λ_*^m, x_*^m) is a point (in the closure of A) at which the supremum of $|Z_m|$ is attained. Passing to a subsequence, we can assume that (λ_*^m, x_*^m) converges to (λ_*, x_*). We can now prove (A.4) by contradiction: So, assume that (A.4) is not true, so

$$
|Z_m(\lambda_*^m, x_*^m)| \geq \delta > 0 \,. \tag{A.5}
$$

Then by the bounded convergence theorem, we have the strong convergence

$$p_{\lambda_*^m, x_*^m} \longrightarrow p_{\lambda_*, x_*}$$

in $L^2_{\mathbb{C}}([0,1]^n)$. On the other hand, we have the weak convergence

$$\prod_{j=1}^{n} (s_m - s)(x_j) \longrightarrow 0$$

in $L^2_{\mathbb{C}}([0,1]^n)$. As a consequence, we have that

$$\begin{aligned}
&|Z_m(\lambda_*^m, x_*^m)| \\
&= \left| \left\langle p_{\lambda_*^m, x_*^m}, \prod_{j=1}^{n} (\bar{s}_m - \bar{s})(x_j) \right\rangle \right| \\
&= \left| \left\langle p_{\lambda_*^m, x_*^m} - p_{\lambda_*, x_*}, \prod_{j=1}^{n} (\bar{s}_m - \bar{s})(x_j) \right\rangle + \left\langle p_{\lambda_*, x_*}, \prod_{j=1}^{n} (\bar{s}_m - \bar{s})(x_j) \right\rangle \right| \longrightarrow 0 \,,
\end{aligned}$$

which is a contradiction to (A.5). The proof for $\widetilde{y}_1$ is analogous. The compactness of the Dirichlet eigenvalues follows from this result: Suppose the sequence s_m converges weakly to s. By the principle of uniform boundedness

$$\|s\| \leq \sup_m \|s_m\| \leq M < \infty \,.$$

Let $N > 2\exp(M), \varepsilon > 0$, and consider the intervals

$$I_n = \{\lambda \in \mathbb{R} : |\lambda - \mu_n(s)| < \varepsilon\}, \quad 1 \leq n \leq N \,.$$

For sufficiently small ε these intervals are all disjoint and contained in the half line $(-\infty, (N+1)^2\pi^2)$ by the proof of Proposition 11. Moreover, $\widetilde{y}_2(1, s, \lambda)$ changes sign on each of them, since $\mu_n(s)$ is a simple root.

As m tends to infinity, the functions $\widetilde{y}_2(1, s_m, \lambda)$ converge to $\widetilde{y}_2(1, s, \lambda)$ uniformly on $I_1 \cup \ldots \cup I_N$. Hence, for sufficiently high m, they also change sign on $I_1, \ldots, I_N$, so they must all have at least one root in each of these intervals. But there are only N roots on the whole half line $(-\infty, (N+1)^2\pi^2)$ by the proof of Proposition 11. Therefore, $\widetilde{y}_2(1, s_m, \lambda)$ has exactly one root in each interval I_n, which must be the $n-th$ Dirichlet eigenvalue for $s = s_m$. Consequently,

$$|\mu_n(s_m) - \mu_n(s)| < \varepsilon, \quad 1 \leq n \leq N \,,$$

for sufficiently large m. The proof for the Neumann eigenvalues ν_n is analogous and, hence, the analyticity of the differences $\mu_n - \nu_n$ follows.
The analyticity of $\frac{1}{I(s)^2}$ in a neighborhood of $s = 1$ is standard.

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